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Courant Brackets on Noncommutative Algebras and Omni-Lie Algebras

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Abstract. We define a Courant bracket on an associative algebra using the theory of Hochschild homology, and we introduce the notion of Dirac algebra. We show that the bracket of an omni-Lie algebra is quite a kind of Courant bracket.

1. Introduction

T. Courant [1] defines a skew-symmetric bracket (1) below on the set of sections $\Gamma(TM \oplus T^*M)$ on a smooth manifold M

$$[\![(X,\alpha),(Y,\beta)]\!]_{skew} := ([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}d(\langle Y,\alpha\rangle - \langle X,\beta\rangle)), \tag{1}$$

where $(X,\alpha),(Y,\beta) \in \Gamma(TM \oplus T^*M)$. The bracket is not a Lie bracket, but the modified bracket

$$[\![(X,\alpha),(Y,\beta)]\!] := ([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + d\langle Y,\alpha\rangle) \tag{2}$$

satisfies a Leibniz identity and the bracket (1) is given as the skew-symmetrization of (2). These brackets (1) and (2) are both called *Courant brackets*. In addition, he gives a smooth nondegenerate symmetric bilinear form on $TM \oplus T^*M$:

$$((x,a),(y,b)) := \frac{1}{2}(\langle y,a\rangle + \langle x,b\rangle), \tag{3}$$

where $(x,a),(y,b) \in TM \oplus T^*M$. The Courant bracket and the bilinear form are used to give a characterization of Poisson structure on M. Let π be a 2-vector field on M, and let L_π denote the graph of π, i.e., the set of elements $(\tilde{\pi}(a), a)$, where $a \in T^*M$ and $\tilde{\pi} : T^*M \to TM$ is the bundle map defined by $\pi(a_1,a_2) = \langle \tilde{\pi}(a_1), a_2\rangle$. π is a Poisson structure if and only if the Courant bracket is closed on the set of sections ΓL_π and L_π is maximally isotropic for the bilinear form (3). Such subbundles of $TM \oplus T^*M$ are called *Dirac structures* ([1]).

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DEFINITION 1.1. Let M be a smooth manifold. A subbundle L of $TM \oplus T^*M$ is called a Dirac structure, if the Courant bracket (2), or equivalently (1) is closed on the set of sections ΓL and L is maximally isotropic for the bilinear form (3).

A. Weinstein [8] gives a *linearization* of (1), or (2) motivated by an integrability problem of Courant brackets. We refer [3] for the study of the integrability problem of Courant brackets. Let V be a vector space. Weinstein's bracket is defined on the space $gl(V) \oplus V$:

$$[\![(\xi_1, v_1), (\xi_2, v_2)]\!]_{skew} := ([\xi_1, \xi_2], \frac{1}{2}(\xi_1(v_2) - \xi_2(v_1))), \tag{4}$$

where $(\xi_1, v_1), (\xi_2, v_2) \in gl(V) \oplus V$. This bracket is the skew-symmetrization of a Leibniz bracket:

$$[\![(\xi_1, v_1), (\xi_2, v_2)]\!] := ([\xi_1, \xi_2], \xi_1(v_2)). \tag{5}$$

The V-valued nondegenerate symmetric bilinear form is also defined by

$$((\xi_1, v_1), (\xi_2, v_2)) = \frac{1}{2}(\xi_2(v_1) + \xi_1(v_2)). \tag{6}$$

Similar to Poisson structures on a manifold, every Lie algebra structure on V is characterized as the graph. Let $\mu : V \otimes V \to V$ be a binary operation. Set the graph of μ: $L_\mu := \{(\tilde{\mu}(v), v) \mid v \in V\}$, where $\tilde{\mu} : V \to gl(V)$ is the map defined by $\tilde{\mu}(v)(u) = \mu(v, u)$. The operation μ is a Lie bracket if and only if Weinstein's bracket (4), or equivalently (5) is closed on L_μ and L_μ is maximally isotropic for the bilinear form (6). Such objects are called *D-structures* in [8]. He calls $gl(V) \oplus V$ an *omni-Lie algebra.* Here we consider relationships between Courant brackets (2) and Weinstein's brackets (4), or (5).

In [8] it is suggested that V is a non-unital algebra of linear functions on the dual space V^* with trivial multiplication. Then $gl(V)$ is the set of derivations of V. Furthermore $(0, v) \in gl(V) \oplus V$ is a certain derivative $D : v \mapsto (0, v)$, similar to manifolds cases $D : C^\infty(M) \to \Gamma(TM \oplus T^*M)$, $f \mapsto (0, df)$. So one can view omni-Lie algebras as geometrical "linearization" of Courant's original examples.

In this paper we construct an algebraic Courant bracket using Hochschild cohomology (resp. homology) groups. Let $\mathcal{A}$ be an associative and unital algebra, not necessarily commutative, and we set the Hochschild cohomology (resp. homology) group $H^1(\mathcal{A}, \mathcal{A})$ (resp. $H_1(\mathcal{A}, \mathcal{A})$). In Section 3, we define on the space $H^1(\mathcal{A}, \mathcal{A}) \oplus H_1(\mathcal{A}, \mathcal{A})$ a Leibniz bracket by the same formula as (2), using algebraic derivatives. We will call the bracket on $H^1(\mathcal{A}, \mathcal{A}) \oplus H_1(\mathcal{A}, \mathcal{A})$ a **Courant bracket** on $\mathcal{A}$. Denote $H^1(\mathcal{A}, \mathcal{A}) \oplus H_1(\mathcal{A}, \mathcal{A})$ by $E(\mathcal{A})$. Our motivation is given by the following example.

EXAMPLE 1.2. Let V be a finite-dimensional vector space. Set $V[1] := V \oplus \mathbf{R} \cdot 1$ as a unital algebra over the field $\mathbf{R}$, where the multiplication is almost trivial except the unit 1. Then $H^1(V[1], V[1])$ is just $gl(V)$ and the Courant bracket on $V[1]$ has the same formula as Weinstein's bracket (5):

$$[\![(\xi_1, dv_1), (\xi_2, dv_2)]\!] = ([\xi_1, \xi_2], d\xi_1(v_2)),$$

where $d : V[1] \to H_1(V[1], V[1])$ is an algebraic de Rham derivative. (See Section 5, for the detailed study.)

In addition, the symmetric bilinear form of $E(\mathcal{A})$ is also well-defined by the formula (3) without the factor $1/2$, by means of a duality between $H^1(\mathcal{A}, \mathcal{A})$ and $H_1(\mathcal{A}, \mathcal{A})$. We wish a *nondegenerate* symmetric bilinear form to define the notion of Dirac structure on $\mathcal{A}$. However, the bilinear form on $E(\mathcal{A})$ is degenerate in general. We notice that the kernel of the bilinear form becomes an ideal for the Courant bracket on $\mathcal{A}$. Thus we have the exact sequence of Leibniz algebras:

$$0 \to J \to E(\mathcal{A}) \to E(\mathcal{A})/J \to 0\,, \tag{7}$$

where J is the set of the kernel of the bilinear form. The quotient Leibniz algebra $\varepsilon(\mathcal{A}) := E(\mathcal{A})/J$ has an induced nondegenerate symmetric bilinear form and the induced Courant bracket. Even if $\mathcal{A}$ is noncommutative, thanks to the nondegeneracy of the bilinear form on $\varepsilon(\mathcal{A})$, the notion of **Dirac structure** is well-defined as a maximally isotropic submodule L of $\varepsilon(\mathcal{A})$ such that the induced Courant bracket is closed on the submodule. We call the pair $(\mathcal{A}, L)$ a (noncommutative) **Dirac algebra**. We will show that every Poisson bracket on a commutative algebra is characterized as the corresponding Dirac structure.

We denote the matrix algebra of an algebra $\mathcal{A}$ by $\mathcal{M}_r(\mathcal{A})$. In Proposition 3.3 we will show that the Courant bracket on $\mathcal{M}_r(\mathcal{A})$ is isomorphic to the one of $E(\mathcal{A})$ and the bilinear form is also preserved by the isomorphism. By this proposition, we obtain an Courant bracket isomorphism $\varepsilon(\mathcal{A}) \cong \varepsilon(\mathcal{M}_r(\mathcal{A}))$. The first main theorem of this paper is

THEOREM 1.3. *Let $\mathcal{A}$ be a unital and associative algebra. Then there exists a Courant bracket isomorphism $\varepsilon(\mathcal{A}) \cong \varepsilon(\mathcal{M}_r(\mathcal{A}))$ preserving the bilinear form. Thus Dirac structures on $\mathcal{A}$ and $\mathcal{M}_r(\mathcal{A})$ correspond bijectivelly.*

It is well-known that the dual bundle of a Lie algebroid $A \to M$ is a Poisson manifold with Lie-Poisson bracket. When M is a point, the Lie algebroid is a Lie algebra and the Lie-Poisson bracket is the ordinary one. One can view the algebra $V[1]$ of Example 1.2 as a linearization of the smooth functions on the vector bundle $V^* \to \{o\}$ on a point. In fact the part $\mathbf{R} \cdot 1$ is the set of functions on the base point. Thus $\varepsilon(V[1])$ is the linearization of Courant's original type example $TV^* \oplus T^*V^*$. The second main result of this paper is

THEOREM 1.4. *Let V be a vector space of finite dimension. Then $\varepsilon(V[1])$ is isomorphic to omni-Lie algebra $gl(V) \oplus V$, i.e., Weinstein's bracket on $gl(V) \oplus V$ is the (induced) Courant bracket on $\varepsilon(V[1])$.*

The paper organized as follows.

In Section 2 we recall some basic properties of Hochschild (co)homology theory and the algebraic operations corresponding to Lie derivative, interior product and exterior derivative.

In Section 3 we define the Courant bracket, the bilinear form on $E(\mathcal{A})$ and study the basic property. Especially we show that the algebraic Courant bracket on $\mathcal{A}$ satisfies the axioms of Courant algebroids.

In subsection 4.1 we study the bilinear form and introduce the quotient space $\varepsilon(\mathcal{A})$ with nondegenerate bilinear form. Algebraic Dirac structures are introduced (Definition 1.1). Theorem 1.3 is proved.

In subsection 4.2 we show that a Poisson algebra is a Dirac algebra and every Poisson bracket is characterized by the corresponding Dirac structure.

In Section 5 the second main theorem is shown.

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2. Preliminalies

In this section we recall Hochschild (co)homology groups of algebras and set an algebraic differential-calculus. We refer the book [6] for the detailed study of the theory of Hochschild (co)homology.

2.1. Hochschild homology. Let k be a commutative ring, $\mathcal{A}$ be an algebra over the ring k. The Hochschild n-complex is $C_n(\mathcal{A},\mathcal{A}) := \mathcal{A}\otimes\mathcal{A}^{\otimes n}$, where the tensor product is defined over k. The boundary map $b : C_n(\mathcal{A},\mathcal{A}) \to C_{n-1}(\mathcal{A},\mathcal{A})$ is defined by the rule below. Let $P_i : C_n(\mathcal{A},\mathcal{A}) \to C_{n-1}(\mathcal{A},\mathcal{A})$ be a k-homomorphism:

$$P_i(a_0\otimes ...\otimes a_n) := (-1)^i(a_0\otimes ...\otimes a_ia_{i+1}\otimes ...\otimes a_n),\ (0\le i\le n-1)$$
$$P_n(a_0\otimes ...\otimes a_n) := (-1)^n(a_na_0\otimes ...\otimes a_{n-1}),$$

where $a_0, ..., a_n \in \mathcal{A}$. The map b is defined by the formula:

$$b(a_0\otimes ...\otimes a_n) := \sum_{i=0}^{n} P_i(a_0\otimes ...\otimes a_n).$$

It holds that $b^2 = 0$, and thus the homology groups $H_n(\mathcal{A},\mathcal{A})$ are defined. For example, since $b(a_0\otimes a_1) = [a_0, a_1] = a_0a_1 - a_1a_0$, the 0-th Hochschild homology group is $H_0(\mathcal{A},\mathcal{A}) = \mathcal{A}/[\mathcal{A},\mathcal{A}]$, where $[\mathcal{A},\mathcal{A}]$ is a k-module generated by all $[a, a']$. We denote the center of $\mathcal{A}$ by $Z(\mathcal{A})$. One can check that by the action of $Z(\mathcal{A}) \to C_n(\mathcal{A},\mathcal{A})$: $z(a_0\otimes ...\otimes a_n) = (za_0\otimes ...\otimes a_n)$, each $H_n(\mathcal{A},\mathcal{A})$ becomes a $Z(\mathcal{A})$-module. In fact, for any $z\in Z(\mathcal{A})$ we obtain $zb(a_0\otimes ...\otimes a_n) = b(za_0\otimes ...\otimes a_n)$. If $\mathcal{A}$ is commutative then $H_0(\mathcal{A},\mathcal{A}) = \mathcal{A}$, and if $\mathcal{A}$ is unital then $H_1(\mathcal{A},\mathcal{A})$ is isomorphic to the $\mathcal{A}$-module of *Kähler differentials* which is an $\mathcal{A}$-module generated by 1-forms ada' (see the next subsection 2.2.).

2.2. Kähler differentials. We assume $\mathcal{A}$ is unital and commutative. Set an $\mathcal{A}$-module $O_{\mathcal{A}|k}$ generated by da for any $a\in\mathcal{A}$, where d is merely a symbol. Define two relations (or axioms) on the module $O_{\mathcal{A}|k}$:

$$d(\lambda a + \lambda' a') - \lambda da - \lambda' da' = 0, \tag{8}$$
$$d(aa') - ada' - a'da = 0, \tag{9}$$

where $\lambda, \lambda' \in k$. The quotient module $O_{\mathcal{A}|k}/\sim$ is the module of Kähler differentials, and it is denoted by $\Omega^1_{\mathcal{A}|k}$. It is known that $H_1(\mathcal{A},\mathcal{A}) \cong \Omega^1_{\mathcal{A}|k}$ (see 1.1.10 Proposition in [6]). The isomorphism between $H_1(\mathcal{A},\mathcal{A})$ and $\Omega^1_{\mathcal{A}|k}$ is given by $a_0 \otimes a_1 \cong a_0 da_1$, on the level of cycles. In fact, by the relation (8), $O_{\mathcal{A}|k}$ becomes the tensor product $\mathcal{A}\otimes\mathcal{A}$, and the second relation is the same as the defining relation of the Hochschild homology $H_1(\mathcal{A},\mathcal{A})$.

2.3. Hochschild cohomology. Next, we consider the Hochschild cohomology groups for general algebras. The n-complex $C^n(\mathcal{A},\mathcal{A})$ is $\mathrm{Hom}_k(\mathcal{A}^{\otimes n},\mathcal{A})$ and when $n=0$, $C^0(\mathcal{A},\mathcal{A}) = \mathcal{A}$. The coboundary map β is defined by the following formula. For any $f \in C^n(\mathcal{A},\mathcal{A})$:

$$\begin{aligned}\beta(f)(a_1\otimes\ldots\otimes a_{n+1}) &= a_1 f(a_2\otimes\ldots\otimes a_{n+1})\\ &\quad+\sum_{i=1}^{n}(-1)^i f(a_1\otimes\ldots\otimes a_i a_{i+1}\otimes\ldots\otimes a_{n+1})\\ &\quad+(-1)^{n+1} f(a_1\otimes\ldots\otimes a_n)a_{n+1}\end{aligned}$$

and $\beta : C^0(\mathcal{A},\mathcal{A}) \to C^1(\mathcal{A},\mathcal{A})$ is $a \mapsto [a,\cdot]$ for any $a\in\mathcal{A}$. It is easily checked that $H^0(\mathcal{A},\mathcal{A}) = Z(\mathcal{A})$ and the cocycles of $C^1(\mathcal{A},\mathcal{A})$ is the set of derivations on $\mathcal{A}$. Denote the derivations on $\mathcal{A}$ by $\mathrm{Der}(\mathcal{A})$. We have

$$H^1(\mathcal{A},\mathcal{A}) = \mathrm{Der}(\mathcal{A})/[\mathcal{A},\cdot]\,,$$

where $[\mathcal{A},\cdot]$ is the submodule of $C^1(\mathcal{A},\mathcal{A})$ generated by inner derivations $[a,\cdot] : a' \mapsto [a,a']$. Especially if $\mathcal{A}$ is commutative then $H^1(\mathcal{A},\mathcal{A}) = \mathrm{Der}(\mathcal{A})$. Note that each $H^n(\mathcal{A},\mathcal{A})$ is also a $Z(\mathcal{A})$-module.

2.4. Algebraic derivatives. Secondly, we recall a Lie bracket on $H^1(\mathcal{A},\mathcal{A})$, a Lie derivative $\mathcal{L}_X$, an interior product i_X and Connes' boundary map B on homology.

REMARK 2.1. In [6], the Lie derivative and the interior product are denoted by L_D and e_D respectively. Here we use geometrical notations $\mathcal{L}_X$ and i_X.

For any $\mathcal{A}$, $\mathrm{Der}(\mathcal{A})$ has a canonical Lie bracket by taking the commutator. One can easily check that the module generated by inner-derivations $[a,\cdot]$ is an ideal of $\mathrm{Der}(\mathcal{A})$. Thus a Lie bracket on $H^1(\mathcal{A},\mathcal{A})$ is induced.

A Lie derivative $\mathcal{L}_X : H_n(\mathcal{A},\mathcal{A}) \to H_n(\mathcal{A},\mathcal{A})$ associated with an element $X\in\mathrm{Der}(\mathcal{A})$ is defined by the formula below, on the level of cycles (Section 4.1 of [6]).

$$\mathcal{L}_X(a_0\otimes\ldots\otimes a_n) := \sum_{i=0}^{n} a_0\otimes\ldots\otimes X(a_i)\otimes\ldots\otimes a_n\,.$$

Further, we can show that $\mathcal{L}_{[a,\cdot]}$ is the zero map on the level of homology groups for any $a\in\mathcal{A}$ (see 4.1.5 Proposition in [6]). Thus $\mathcal{L}_X$ is well-defined for any $X\in H^1(\mathcal{A},\mathcal{A})$.

It is known that an interior product is also well-defined on Hochschild homology. For any $X \in \mathrm{Der}(\mathcal{A})$, set an operator $i_X : H_n(\mathcal{A},\mathcal{A}) \to H_{n-1}(\mathcal{A},\mathcal{A})$, on the level of cycles, by the formula:

$$i_X(a_0 \otimes ... \otimes a_n) := (-1)^{n+1}(X(a_n)a_0 \otimes ... \otimes a_{n-1}) .$$

LEMMA 2.2. *For any $a \in \mathcal{A}$, $i_{[a,\cdot]}$ is the zero map on the level of homology groups.*

PROOF. For any $a' \in \mathcal{A}$, set the map $h_{a'} : C_n(\mathcal{A},\mathcal{A}) \to C_n(\mathcal{A},\mathcal{A})$, $a_0 \otimes ... \otimes a_n \mapsto a'a_0 \otimes ... \otimes a_n$. We have

$$h_{a'} \circ b(a_0 \otimes ... \otimes a_n) = a'a_n a_0 \otimes ... \otimes a_{n-1} + \sum_{i=0}^{n-1} P_i(a'a_0 \otimes ... \otimes a_n) ,$$

$$b \circ h_{a'}(a_0 \otimes ... \otimes a_n) = a_n a' a_0 \otimes ... \otimes a_{n-1} + \sum_{i=0}^{n-1} P_i(a'a_0 \otimes ... \otimes a_n) .$$

Thus $h_{a'} \circ b - b \circ h_{a'} = (-1)^{n+1} i_{[a',\cdot]}$ which implies that $i_{[a',\cdot]}$ is homotopic to the zero map. □

Thus the interior product i_X is well-defined for any $X \in H^1(\mathcal{A},\mathcal{A})$. Like smooth manifold cases, the following lemma holds.

LEMMA 2.3. *For any $X, Y \in H^1(\mathcal{A},\mathcal{A})$:*

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X , \quad i_{[X,Y]} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X .$$

PROOF. We only show the second formula. For the first formula, we refer 4.1.6 Corollary in [6]. For any $\mathbf{a} := a_0 \otimes ... \otimes a_n$:

$$\begin{aligned}(-1)^{n+1}\mathcal{L}_X \circ i_Y(\mathbf{a}) &= \mathcal{L}_X Y(a_n)a_0 \otimes ... \otimes a_{n-1} \\ &= XY(a_n)a_0 \otimes ... \otimes a_{n-1} + Y(a_n)X(a_0) \otimes ... \otimes a_{n-1} \\ &\quad + \sum_{i=1}^{n-1} Y(a_n)a_0 \otimes ... \otimes X(a_i) \otimes ... \otimes a_{n-1} , \qquad (10)\end{aligned}$$

and on the other hand,

$$\begin{aligned}(-1)^{n+1} i_Y \circ \mathcal{L}_X(\mathbf{a}) &= (-1)^{n+1} i_Y \sum_{i=0}^{n} a_0 \otimes ... \otimes X(a_i) \otimes ... \otimes a_n \\ &= Y(a_n)X(a_0) \otimes ... \otimes a_{n-1} + \sum_{i=1}^{n-1} Y(a_n)a_0 \otimes ... \otimes X(a_i) \otimes ... \otimes a_{n-1} \\ &\quad + YX(a_n)a_0 \otimes ... \otimes a_{n-1} . \qquad (11)\end{aligned}$$

The difference of (10), (11) is $(-1)^{n+1} i_{[X,Y]}(\mathbf{a})$. □

2.5. Connes' boundary map. In the following, we assume that $\mathcal{A}$ is unital. The Connes' boundary map $B : H_n(\mathcal{A}, \mathcal{A}) \to H_{n+1}(\mathcal{A}, \mathcal{A})$ is defined by using cyclic operators on $C_n(\mathcal{A}, \mathcal{A})$ (Section 2.1.7 of [6]). We use an explicit definition:

$$B(a_0 \otimes ... \otimes a_n) := \sum_{i=0}^{n} (-1)^{ni} (1 \otimes a_i \otimes ... \otimes a_n \otimes a_0 \otimes ... \otimes a_{i-1}) + (-1)^{ni} (a_i \otimes 1 \otimes a_{i+1} \otimes ... \otimes a_n \otimes a_0 \otimes ... \otimes a_{i-1}) ,$$

where 1 is the unit of $\mathcal{A}$. For example,

$$B(a_0 \otimes a_1) = 1 \otimes a_0 \otimes a_1 - 1 \otimes a_1 \otimes a_0 + a_0 \otimes 1 \otimes a_1 - a_1 \otimes 1 \otimes a_0 .$$

REMARK 2.4. It is known that the condition of boundary operator $B^2 = 0$ is satisfied. However, in our explicit definition, it is difficult to show the condition.

It is known that $\mathcal{L}_X = B \circ i_X + i_X \circ B$ for each $H_n(\mathcal{A}, \mathcal{A})$ (4.1.9 Corollary of [6] and see Remark 2.6 below). We directly show the condition: $\mathcal{L}_X = B \circ i_X + i_X \circ B$ for $H_1(\mathcal{A}, \mathcal{A})$. For any cycles α and α' we denote $\alpha \equiv \alpha'$, if $\alpha = \alpha'$ on the level of homology.

LEMMA 2.5. *For any $X \in \mathrm{Der}(\mathcal{A})$, and any cycle $\alpha \in C_1(\mathcal{A}, \mathcal{A})$:*

$$\mathcal{L}_X(\alpha) \equiv B \circ i_X(\alpha) + i_X \circ B(\alpha) .$$

Thus $\mathcal{L}_X = B \circ i_X + i_X \circ B$ on $H_1(\mathcal{A}, \mathcal{A})$ for any $X \in H^1(\mathcal{A}, \mathcal{A})$.

PROOF. We can put $\alpha = a_0 \otimes a_1$ without loss of generality. By $b(1 \otimes 1 \otimes a) = a \otimes 1$, $a \otimes 1 \equiv 0$. Thus we obtain

$$\begin{aligned} i_X \circ B(a_0 \otimes a_1) &= i_X(1 \otimes a_0 \otimes a_1 - 1 \otimes a_1 \otimes a_0 + a_0 \otimes 1 \otimes a_1 - a_1 \otimes 1 \otimes a_0) \\ &\equiv -X(a_1) \otimes a_0 + X(a_0) \otimes a_1 , \end{aligned}$$

and $B \circ i_X(a_0 \otimes a_1) \equiv 1 \otimes X(a_1)a_0$. In addition, we have

$$b(1 \otimes X(a_1) \otimes a_0) = X_1(a_1) \otimes a_0 - 1 \otimes X(a_1)a_0 + a_0 \otimes X(a_1) .$$

Thus $1 \otimes X(a_1)a_0 \equiv X_1(a_1) \otimes a_0 + a_0 \otimes X(a_1)$. This gives a proof of the lemma. □

Here we remark that when $\mathcal{A}$ is commutative, the derivative da into the space of Kähler differentials is the same as the the boundary $B(a)$:

$$B(a) \equiv 1 \otimes a \cong da, \quad B = d : \mathcal{A} \to \Omega^1_{\mathcal{A}|k} .$$

REMARK 2.6. We can take a normalized-Hochschild homology group $\overline{H}_n(\mathcal{A}, \mathcal{A})$ which is defined by the certain quotient $C_n(\mathcal{A}, \mathcal{A})/\sim$ of Hochschild complex. It is known that the normalized-Hochschild homology group is isomorphic with an ordinary one (see 1.1.14 of [6]). The condition $\mathcal{L}_X = B \circ i_X + i_X \circ B$ is shown in the normalized framework for any $H_n(\mathcal{A}, \mathcal{A})$.

2.6. Canonical pairings. We now set the pairing between $H^1(\mathcal{A},\mathcal{A})$ and $H_1(\mathcal{A},\mathcal{A})$ using the interior product by the form:

$$\langle\cdot,\cdot\rangle : H^1(\mathcal{A},\mathcal{A})\times H_1(\mathcal{A},\mathcal{A})\to H_0(\mathcal{A},\mathcal{A}),\quad \langle X,\alpha\rangle := i_X\alpha\,, \tag{12}$$

where $X\in H^1(\mathcal{A},\mathcal{A})$ and $\alpha\in H_1(\mathcal{A},\mathcal{A})$. Note that the pairing is $Z(\mathcal{A})$-bilinear.

We remark here that the pairing (12) is equivalent to the *Kronecker product*. The Kronecker product $\langle\cdot,\cdot\rangle : H^n(\mathcal{A},\mathcal{A})\otimes H_n(\mathcal{A},\mathcal{A})\to \mathcal{A}\otimes_{\mathcal{A}^e}\mathcal{A}$, is a canonical pairing between cohomology groups and homology groups defined by, on the level of (co)chains,

$$\langle f, a_0\otimes a_1\otimes ...\otimes a_n\rangle = f(a_1\otimes ...\otimes a_n)\otimes_{A^e} a_0\,,$$

where $f\in C^n(\mathcal{A},\mathcal{A})$, $\mathcal{A}^e := \mathcal{A}\otimes\mathcal{A}^{op}$ and $\mathcal{A}^{op}$ is the opposite algebra of $\mathcal{A}$ (see 1.5.9 Duality of [6]). One can easily show that $\mathcal{A}\otimes_{\mathcal{A}^e}\mathcal{A}\cong H_0(\mathcal{A},\mathcal{A})$. The isomorphism is $a\otimes_{A^e}a'\cong\overline{aa'}$, where $\overline{aa'}$ is the equivalence class of aa'. In fact, by the definition, we have $a\otimes_{A^e}a' = 1(1\otimes a)\otimes_{A^e}a' = 1\otimes a'a$. On the other hand, $a\otimes_{A^e}a' = 1(a\otimes 1)\otimes_{A^e}a' = 1\otimes aa'$. Thus $a\otimes_{A^e}a' = a'\otimes_{A^e}a$. This commutativity is expressed as the *abelianzation* $\mathcal{A}/[\mathcal{A},\mathcal{A}] = H_0(\mathcal{A},\mathcal{A})$.

Recall the bilinear forms (3) and (6). By means of the bilinear form, the notion of Dirac structure is defined as a maximally isotropic subspace. We use carefully the term "maximally isotropic" in the algebraic framework.

Let k be a unital commutative ring, and let E and M be (left) k-modules, and let $(\cdot,\cdot)$ be a M-valued nondegenerate symmetric k-bilinear form on E. Here $(\cdot,\cdot)$ is nondegenerate, namely $(e,\cdot) : E\to M$ is injective for any nontrivial $e\in E$.

DEFINITION 2.7. Under the notations above, let L be a submodule of E. We say that L is "isotropic" for the bilinear form, if the bilinear form is zero on L. When L is isotropic, we say that L is "maximally isotropic", if $(e,\cdot)$ vanishes on L then e is in L for any $e\in E$.

3. Courant bracket of $H^1(\mathcal{A},\mathcal{A})\oplus H_1(\mathcal{A},\mathcal{A})$

In this section we define a Courant bracket on an associative algebra using the operations of Section 2.

DEFINITION 3.1. Let $\mathcal{A}$ be a unital and associative k-algebra. We call a bracket on $H^1(\mathcal{A},\mathcal{A})\oplus H_1(\mathcal{A},\mathcal{A})$ below a Courant bracket on $\mathcal{A}$.

$$[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!] = ([X_1,X_2],\mathcal{L}_{X_1}\alpha_2-\mathcal{L}_{X_2}\alpha_1+B\langle X_2,\alpha_1\rangle)\,,$$

where $(X_1,\alpha_1),(X_2,\alpha_2)\in H^1(\mathcal{A},\mathcal{A})\oplus H_1(\mathcal{A},\mathcal{A})$. We denote $H^1(\mathcal{A},\mathcal{A})\oplus H_1(\mathcal{A},\mathcal{A})$ by $E(\mathcal{A})$.

When k contains $1/2$, we have Courant's original formula (1) as the skew-symmetrization of the Courant bracket on $\mathcal{A}$.

We set a symmetric $Z(A)$-bilinear form $(\cdot,\cdot)$ on $H^1(\mathcal{A},\mathcal{A})\oplus H_1(\mathcal{A},\mathcal{A})$ using the formula (3) without the factor $1/2$, i.e., for any $e_1 := (X_1,\alpha_1)$, $e_2 := (X_2,\alpha_2) \in E(\mathcal{A})$:

$$(e_1,e_2) := \langle X_2,\alpha_1\rangle + \langle X_1,\alpha_2\rangle\,. \tag{13}$$

Note that this bilinear form is $H_0(\mathcal{A},\mathcal{A})$-valued in general. In addition, we set a map $\rho : E(\mathcal{A}) \to H^1(\mathcal{A},\mathcal{A})$ as the canonical projection:

$$\rho(X,\alpha) := X\,. \tag{14}$$

By the definition, ρ has a $Z(A)$-linearity. We notice a derivative action:

$$H^1(\mathcal{A},\mathcal{A}) \times Z(\mathcal{A}) \to Z(\mathcal{A})\,,\quad (X,z)\mapsto X(z)\,, \tag{15}$$

This action is well-defined on the level of homology, since $[a,z]=0$ for any $a\in\mathcal{A}$.

PROPOSITION 3.2. *Let $\mathcal{A}$ be a unital and associative k-algebra. Then the Courant bracket satisfies the following properties. For any $e_1,e_2,e_3\in E(\mathcal{A})$ and $z\in Z(\mathcal{A})$:*

$$[\![e_1,[\![e_2,e_3]\!]]\!] = [\![[\![e_1,e_2]\!],e_3]\!] + [\![e_2,[\![e_1,e_3]\!]]\!] \tag{16}$$

$$\rho[\![e_1,e_2]\!] = [\rho(e_1),\rho(e_2)]\,, \tag{17}$$

$$[\![e_1,ze_2]\!] = z[\![e_1,e_2]\!] + \rho(e_1)(z)e_2\,, \tag{18}$$

$$2[\![e_1,e_1]\!] = D(e_1,e_1)\,, \tag{19}$$

$$\mathcal{L}_{\rho(e_1)}(e_2,e_3) = ([\![e_1,e_2]\!],e_3) + (e_2,[\![e_1,e_3]\!])\,, \tag{20}$$

where D is a k-homomorphism:

$$D : H_0(\mathcal{A},\mathcal{A}) \to H_1(\mathcal{A},\mathcal{A}),\quad \alpha\mapsto (0,B(\alpha))\,,$$

and $\rho(e_1)(z)$ of (18) *is the action* (15).

PROOF. The formulas (17) and (19) are clearly. For (16), (18) and (20), by Lemma 2.3, 2.5 we can take the same proof as the case of $TM\oplus T^*M$ on a smooth manifold. □

The conditions (16)–(20) above are the set of axioms of *Courant algebroids* in [7] (see also [4], [5]). However $E(\mathcal{A})$ is not a Courant algebroid, because the bilinear form is degenerate in general. In the next section we will study the bilinear form on $E(\mathcal{A})$.

For given algebras $\mathcal{A}$ and $\mathcal{A}'$, we write $E(\mathcal{A}) \cong E(\mathcal{A}')$, if there exists an isomorphism $\phi : H_0(\mathcal{A},\mathcal{A}) \cong H_0(\mathcal{A}',\mathcal{A}')$ and if there exists a Courant bracket isomorphism preserving the bilinear form up to ϕ. We study isomorphisms between Courant brackets.

It is well-known that a unital algebra $\mathcal{A}$ and the matrix algebra $\mathcal{M}_r(\mathcal{A})$ are Morita equivalent, and thus the Hochschild (co)homology groups of $\mathcal{A}$ and $\mathcal{M}_r(\mathcal{A})$ are isomorphic (see 1.2.4 and 1.5.6 in [6]).

PROPOSITION 3.3. *For any $\mathcal{A}$, $E(\mathcal{A}) \cong E(\mathcal{M}_r(\mathcal{A}))$.*

PROOF. We take isomorphisms $cotr : H^1(\mathcal{A},\mathcal{A}) \to H^1(\mathcal{M}_r(\mathcal{A}),\mathcal{M}_r(\mathcal{A}))$ and $inc : H_1(\mathcal{A},\mathcal{A}) \to H_1(\mathcal{M}_r(\mathcal{A}),\mathcal{M}_r(\mathcal{A}))$ in [6]. Here these maps are defined by

$$cotr(X)(m_{ij}) := (X(m_{ij})),\quad inc(a_0\otimes\ldots\otimes a_n) = E_{11}(a_0)\otimes\ldots\otimes E_{11}(a_n)\,,$$

on the level of chains, where $X \in H^1(\mathcal{A}, \mathcal{A})$, $m_{ij} \in \mathcal{M}_r(\mathcal{A})$ and $E_{11}(a)$ is a matrix such that the $(1,1)$-position is a and other positions are all zero. We denote $cotr$ and inc by T and I respectively.

It is obvious that T is a Lie algebra isomorphism. First we show that $T \oplus I$ preserves the bilinear form. It is sufficient to show that $i_{T(X)} \circ I(a_0 \otimes a_1) = I \circ i_X(a_0 \otimes a_1)$.

$$\begin{aligned} i_{T(X)} \circ I(a_0 \otimes a_1) &= i_{T(X)}(E_{11}(a_0) \otimes E_{11}(a_1)) \\ &= T(X)(E_{11}(a_1)) \cdot E_{11}(a_0) \\ &= E_{11}(X(a_1)) \cdot E_{11}(a_0) \\ &= E_{11}(X(a_1)a_0) = I \circ i_X(a_0 \otimes a_1)\,. \end{aligned}$$

Thus the bilinear form is preserved by the isomorphism. Secondly we show that the Courant bracket is preserved. For any $a \in \mathcal{A}$, we have $B \circ I(a) \equiv 1_{\mathcal{M}_r(\mathcal{A})} \otimes E_{11}(a)$ and $I \circ B(a) \equiv E_{11}(1) \otimes E_{11}(a)$, where $1_{\mathcal{M}_r(\mathcal{A})}$ is the unit element of $\mathcal{M}_r(\mathcal{A})$. On the other hand,

$$\begin{aligned} (B \circ I - I \circ B)(a) &\equiv (1_{\mathcal{M}_r(\mathcal{A})} - E_{11}(1)) \otimes E_{11}(a) \\ &= -b\{(1_{\mathcal{M}_r(\mathcal{A})} - E_{11}(1)) \otimes E_{11}(a) \otimes E_{11}(1)\} \\ &\equiv 0\,. \end{aligned}$$

Thus $B \circ I(a) = I \circ B(a)$ on the level of homology. We now obtain below, on the level of homology: $I \circ B \circ i_X(a_0 \otimes a_1) = B \circ i_{T(X)} \circ I(a_0 \otimes a_1)$. One can directly show: $i_{T(X)} \circ B \circ I(a_0 \otimes a_1) = I \circ i_X \circ B(a_0 \otimes a_1)$, on the level of homology. Thus we obtain $\mathcal{L}_{T(X)} \circ I(\alpha) = I \circ \mathcal{L}_X(\alpha)$ and $I \circ B\langle X, \alpha\rangle = B\langle T(X), I(\alpha)\rangle$ for any $X \in H^1(\mathcal{A}, \mathcal{A})$ and $\alpha \in H_1(\mathcal{A}, \mathcal{A})$. Thus $T \oplus I$ preserves the Courant bracket. □

This proposition will be used to give a proof of Theorem 1.3 in the next section. As an example of other isomorphisms we can easily check that $E(\mathcal{A}) \cong E(\mathcal{A}^{op})$, where $\mathcal{A}^{op}$ is the opposite algebra of $\mathcal{A}$. (We refer E.2.1.4 of [6].)

EXAMPLE 3.4. $E(\mathcal{A}) \cong E(\mathcal{A}^{op})$.

4. Dirac algebras and Poisson brackets

4.1. Dirac structures. Let M be a smooth manifold. Dirac structures L on M are defined as maximally isotropic subbundles of $TM \oplus T^*M$ for the bilinear form (3) such that the Courant bracket (2) is closed on the set of sections ΓL. The maximality condition is well-defined because the bilinear form (3) is nondegenerate. In Courant's original example, the pair (M, L) is called a Dirac manifold. In this subsection, we introduce a notion of *Dirac algebra*. First we study the bilinear form of $E(\mathcal{A})$.

Let $\mathcal{A}$ be a unital k-algebra. The bilinear form (13) of $E(\mathcal{A})$ is degenerate in general. But we can show that the kernel of the bilinear form is an ideal of $E(\mathcal{A})$ with respect to the Courant bracket. Denote the kernel by J, i.e.,

$$J := \{e \in E(\mathcal{A}) \mid (e, e') = 0 \text{ for any } e' \in E(\mathcal{A})\}\,. \tag{21}$$

LEMMA 4.1. *The kernel J is an ideal of $E(\mathcal{A})$.*

PROOF. For any $e \in J$, $e_1, e_2 \in E(\mathcal{A})$, by (20) in Proposition 3.2 we have

$$\mathcal{L}_{\rho(e_1)}(e, e_2) = (\llbracket e_1, e \rrbracket, e_2) + (e, \llbracket e_1, e_2 \rrbracket) .$$

Since e is in the kernel, we have $(\llbracket e_1, e \rrbracket, e_2) = 0$. By the definition of Courant bracket, the skew-symmetrization is

$$\llbracket e_1, e \rrbracket - \llbracket e, e_1 \rrbracket = 2\llbracket e_1, e \rrbracket - D(e_1, e) , \tag{22}$$

where D was defined in Proposition 3.2. This implies that J is a two-side ideal. □

From this lemma, when $J \neq E(\mathcal{A})$, we obtain a nontrivial Leibniz algebra $E(\mathcal{A})/J$ with $H_0(\mathcal{A}, \mathcal{A})$-valued nondegenerate symmetric bilinear form $(\cdot, \cdot)$. Here the bilinear form on $E(\mathcal{A})/J$ is $Z(\mathcal{A})$-bilinear. We denote $E(\mathcal{A})/J$ by $\varepsilon(\mathcal{A})$. When $\varepsilon(\mathcal{A}) \neq 0$, we obtain a Leibniz algebra $\varepsilon(\mathcal{A})$ with a nondegenerate bilinear form and a (induced) Courant bracket. So we define Dirac structures on noncommutaitve algebras.

DEFINITION 4.2. Let A be a unital and associative k-algebra. We assume that $\varepsilon(\mathcal{A}) \neq 0$. We call a submodule L of $\varepsilon(\mathcal{A})$ a **Dirac structure** on $\mathcal{A}$, if L is maximally isotropic for the induced bilinear form on $\varepsilon(\mathcal{A})$ and the induced Courant bracket on $\varepsilon(\mathcal{A})$ is closed on L. We call the pair $(\mathcal{A}, L)$ a (noncommutative) **Dirac algebra**.

In general we have no hope of defining the map $\rho : \varepsilon(\mathcal{A}) \to H^1(\mathcal{A}, \mathcal{A})$. When $\mathcal{A}$ is commutative, the kernel J becomes a submodule of $H_1(\mathcal{A}, \mathcal{A})$, thus the map ρ is well-defined (see Lemma 4.8 below).

An algebraic meaning of Dirac structure is that it is a Lie algebra. By the isotropy of Dirac structure we have a corollary below.

COROLLARY 4.3. *A Dirac structure L on k-algebra $\mathcal{A}$ is a k-Lie algebra and the inverse image $p^{-1}(L)$ of the canonical projection $p : E(\mathcal{A}) \to \varepsilon(\mathcal{A})$ satisfies the defining conditions of Lie algebroids. For any $l_1, l_2 \in L$ and $z \in Z(\mathcal{A})$:*

$$\sigma\llbracket l_1, l_2 \rrbracket = [\sigma(l_1), \sigma(l_2)], \quad \llbracket l_1, zl_2 \rrbracket = z\llbracket l_1, l_2 \rrbracket + \sigma(l_1)(z)l_2 ,$$

where σ is an anchor map defined by the composition $p^{-1}(L) \xrightarrow{\rho} H^1(\mathcal{A}, \mathcal{A}) \to \mathrm{Der}(Z(\mathcal{A}))$ and $[\cdot, \cdot]$ is a commutator on $\mathrm{Der}(Z(\mathcal{A}))$.

PROOF. It is obvious that the Courant bracket on $\mathcal{A}$ is closed on $p^{-1}(L)$. The anchor map is well-defined by the action (15). Two conditions above follow from (17), (18) in Proposition 3.2. □

Note that the above σ differs from ρ in Proposition 3.2. Especially when $k = \mathbf{R}$ and $Z(\mathcal{A})$ is the algebra of smooth functions on a manifold M, $p^{-1}(L)$ is just the space of sections of a Lie algebroid on M.

In the next subsection we will show that a Poisson algebra is a Dirac algebra with the corresponding Dirac structure. It is well-known that closed 2-forms on a manifold define Dirac structures (see [1]). Similar to manifold cases, we obtain a proposition below.

PROPOSITION 4.4. *We assume $\varepsilon(\mathcal{A}) \neq 0$. Let $\omega \in H_2(\mathcal{A}, \mathcal{A})$ be a closed 2-form in the sense of $B(\omega) = 0$ and $i_X i_Y \omega = -i_Y i_X \omega$. Then $p(L_\omega)$ is a Dirac structure, where L_ω is the set of elements $(X, i_X\omega)$ and p is the canonical projection $p : E(\mathcal{A}) \to \varepsilon(\mathcal{A})$.*

PROOF. It is obvious that L_ω is isotropic on $E(\mathcal{A})$ and $\varepsilon(\mathcal{A})$. By the same way as geometrical cases in [1], one can easily check that the Courant bracket is closed on L_ω. We show that $p(L_\omega)$ is maximally isotropic. Recall Definition 2.7. For any $(X, i_X\omega) \in L_\omega$, we assume $((X, i_X\omega), (Y, \alpha)) = 0$ on $E(\mathcal{A})$. Then we have $i_X i_Y \omega = i_X \alpha$ for any X, thus $(0, i_Y\omega - \alpha)$ is in the kernel J. Thus in $\varepsilon(\mathcal{A})$ we have $\alpha = i_Y\omega$, i.e., $p(L_\omega)$ is maximally isotropic. □

From the proposition above, when ω is trivial, the projection of $H^1(\mathcal{A}, \mathcal{A})$ is a Dirac structure.

We now give a proof of Theorem 1.3 in Introduction.

PROOF. Using the isomorphism $T \oplus I : E(\mathcal{A}) \cong E(\mathcal{M}_r(\mathcal{A}))$ in Proposition 3.3, we obtain a Courant bracket isomorphism

$$p \circ (T \oplus I) \circ p^{-1} : \varepsilon(\mathcal{A}) \cong \varepsilon(\mathcal{M}_r(\mathcal{A}))$$

which preserves the bilinear form on $\varepsilon(\mathcal{A})$ up to the isomorphism $H_0(\mathcal{A}, \mathcal{A}) \cong H_0(\mathcal{M}_r(\mathcal{A}), \mathcal{M}_r(\mathcal{A}))$. Thus Dirac structures correspond bijectively between $\mathcal{A}$ and $M_r(\mathcal{A})$. This gives the proof of Theorem 1.3. □

From Example 3.4 and Theorem 1.3, we obtain $\varepsilon(\mathcal{A}) \cong \varepsilon(\mathcal{M}_r(\mathcal{A})^{op})$. Using the theorem we give an example of Dirac algebra on a smooth manifold.

EXAMPLE 4.5. Set $\mathcal{A} := C^\infty(M)$ which is the set of smooth functions on a smooth manifold M. Then $\mathcal{M}_r(\mathcal{A})$ is identified with $\Gamma\mathrm{End}(\mathbf{R}^r \times M)$ which is the space of sections of the endmorphism bundle of the trivial bundle. Using the identification $\varepsilon(C^\infty(M)) \cong \Gamma(TM \oplus T^*M)$, we obtain Dirac structures on the algebra $\Gamma\mathrm{End}(\mathbf{R}^r \times M)$ from geometrical (i.e. ordinary) Dirac structures on the manifold. For instance, for a Poisson structure π on M, the graph L_π is a Dirac structure on $C^\infty(M)$. We can denote the derivation $T(X) \in \mathrm{Der}(\mathcal{M}_r(\mathcal{A}))$ for $X \in \Gamma TM$ in the matrix form $\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$, where we put $r = 2$. On the other hand, $I(fdg)$ is $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}$. Thus the Dirac structure $p \circ (T \oplus I) \circ p^{-1}(\Gamma L_\pi)$ has the form, on the level of chains,

$$\left\{ \left(\begin{pmatrix} fX_g & 0 \\ 0 & fX_g \end{pmatrix}, \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \right) \middle| \; X \in \Gamma TM, \; f, g \in C^\infty(M) \right\},$$

where X_f is the Hamilton vector field of f.

REMARK 4.6. Lemma 4.1 is important for Courant algebroids in Poisson geometry. Given a "week"-Courant algebroid with degenerate symmetric bilinear form, we can take

the quotient bundle with the induced Leibniz bracket and the nondegenerate bilinear form. Conversely, we expect that every Courant algebroid is given in this way.

4.2. Poisson algebras. The purpose of this subsection is to show that every Poisson bracket is characterized as the corresponding Dirac structure.

Standing Assumptions. We assume $\mathcal{A}$ is a unital and commutative k-algebra. Thus $H^1(\mathcal{A},\mathcal{A}) = \mathrm{Der}(\mathcal{A})$, $H_1(\mathcal{A},\mathcal{A}) = \Omega^1_{\mathcal{A}|k}$, $H^0(\mathcal{A},\mathcal{A}) = Z(\mathcal{A}) = \mathcal{A} = H_0(\mathcal{A},\mathcal{A})$ and the boundary map $B = d : \mathcal{A} \to \Omega^1_{\mathcal{A}|k}$. In addition we assume that $\varepsilon(\mathcal{A}) \neq 0$. The condition is satisfied if there exists a nontrivial derivation on $\mathcal{A}$. Thus this assumption is always satsified in Poisson geometry.

It is known that the derivative $d : \mathcal{A} \to \Omega^1_{\mathcal{A}|k}$ has the universality below (see 1.3.7–1.3.9 of [6]). For any derivative $\delta : \mathcal{A} \to M$ to an $\mathcal{A}$-module, there exists a unique map $\phi : \Omega^1_{\mathcal{A}|k} \to M$ such that $\delta = \phi \circ d$, here ϕ is $\mathcal{A}$-linear.

REMARK 4.7. Usually, the universal derivation of an algebra is defined as the derivation $d : \mathcal{A} \to I/I^2$, where I is a (non-symmetric) $\mathcal{A}$-bimodule generated by $1 \otimes a - a \otimes 1$ for any $a \in \mathcal{A}$ and I/I^2 is the symmetrization of I. One can check that $I/I^2 \cong \Omega^1_{\mathcal{A}|k}$.

LEMMA 4.8. *If (X,α) is an element of the kernel J (21) of the bilinear form then $X = 0$.*

PROOF. By the assumption, for any $a \in \mathcal{A}$ we have $((0,da),(X,\alpha)) = 0$. When $da \neq 0$, this gives $X(a) = 0$. Even if $da = 0$, by the universality above, we have $X(a) = 0$. □

By this lemma, when $\mathcal{A}$ is commutative, the map $\rho : \varepsilon(\mathcal{A}) \to \mathrm{Der}(\mathcal{A})$ is induced from ρ on $E(\mathcal{A})$. In this case, all conditions (16)–(20) of Proposition 3.2 are satisfied on $\varepsilon(\mathcal{A})$. Thus for a commutative algebra $\mathcal{A}$, $\varepsilon(\mathcal{A})$ can be viewed as an example of *Courant algebra*. In fact if k includes $1/2$ and $\mathcal{A}$ is commutative then $\varepsilon(\mathcal{A})$ becomes an example of $(k,\mathcal{A})$ C-algebra. In [8] an algebraic edition of Courant algebroids is defined on a non-unital commutative algebra, this is called a *C-algebra*. It was shown that omni-Lie algebra $gl(V) \oplus V$ is a C-algebra on the algebra V with trivial multiplication. In the next section, we will show that the brackets of omni-Lie algebras are given by the purely algebraic Courant brackets.

Now we consider Poisson algebras (on commutative algebras). In Poisson manifold cases, it is well-known that a Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ is equivalent with the Poisson structure $\pi \in \Gamma \bigwedge^2 TM$ using the definition $\{f,g\} = \pi(df,dg)$ for any $f,g \in C^\infty(M)$. Recall that the Poisson condition $[\pi,\pi] = 0$ is equivalent to the Jacobi law of the bracket $\{\cdot,\cdot\}$. The Poisson structure π is identified with the bundle map $\tilde{\pi} : T^*M \to TM$ by the canonical pairing $\pi(df,dg) = \langle \tilde{\pi}(df), dg\rangle$, and thus the Poisson bracket is identified with the Dirac structure L_π given by the graph of $\tilde{\pi}$. For an arbitrary Poisson algebra $\mathcal{A}$ these identifications are not always defined. But we can get the Dirac structure of a Poisson algebra.

Let $\{\cdot,\cdot\}$ be a k-bilinear biderivation on a k-algebra $\mathcal{A}$, not necessarily Poisson bracket. From the universality above, a Hamiltonianzation $\mathcal{A} \to \mathrm{Der}(\mathcal{A})$, $a \mapsto \{a,\cdot\}$ is given by the formula: $\{a,\cdot\} = \tilde{\pi}(da)$ using the unique map $\tilde{\pi} : \Omega^1_{\mathcal{A}|k} \to \mathrm{Der}(\mathcal{A})$. So we obtain the graph of the map $\tilde{\pi}$, which we denote by L_π:

$$L_\pi := \{(\tilde{\pi}(\alpha), \alpha) \mid \alpha \in \Omega^1_{\mathcal{A}|k}\}.$$

Note that L_π is a $\mathcal{A}$-submodule of $E(\mathcal{A})$.

PROPOSITION 4.9. *Let $\{\cdot,\cdot\}$ be a k-bilinear biderivation on $\mathcal{A}$, and we put the corresponding map $\tilde{\pi}$. The bracket is a Poisson bracket if and only if the pairing $(\cdot,\cdot)$ on $E(\mathcal{A})$ is zero on L_π and the Courant bracket is closed on L_π.*

PROOF. We assume that L_π is isotropic and the Courant bracket is closed on L_π. For any elements $(\tilde{\pi}(da_1), da_1), (\tilde{\pi}(da_2), da_2) \in L_\pi$, by the isotropy condition, we have $\langle\tilde{\pi}(da_2), da_1\rangle = -\langle\tilde{\pi}(da_1), da_2\rangle$. Here $\langle\tilde{\pi}(da_2), da_1\rangle = i_{\tilde{\pi}(da_2)}(da_1) = \{a_2, a_1\}$. This gives the skewsymmetry of the bracket. The Courant bracket of $(\tilde{\pi}(da_1), da_1)$ and $(\tilde{\pi}(da_2), da_2)$ has the form: $([\tilde{\pi}(da_1), \tilde{\pi}(da_2)], \overline{1 \otimes \{a_1, a_2\}})$, here $\overline{1 \otimes \{a_1, a_2\}}$ is the equivalence class of $1 \otimes \{a_1, a_2\}$, i.e., $\overline{1 \otimes \{a_1, a_2\}} = d\{a_1, a_2\}$ on $\Omega^1_{\mathcal{A}|k}$. Then we have

$$\tilde{\pi}(d\{a_1, a_2\}) = \{\{a_1, a_2\}, \cdot\} = [\tilde{\pi}(da_1), \tilde{\pi}(da_2)], \tag{23}$$

this implies that $\{\cdot,\cdot\}$ is a Poisson bracket.

Conversely, we assume that $\{\cdot,\cdot\}$ is a Poisson bracket. Then we have (23) by the Jacobi identity, i.e., generators of L_π is closed under the Courant bracket. Since $\Omega^1_{\mathcal{A}|k}$ is generated by $\{da | a \in \mathcal{A}\}$ as $\mathcal{A}$-module, by (18) in Proposition 3.2 the Courant bracket is closed on L_π. The isotropy condition of L_π is equivalent to the skewsymmetry of $\{\cdot,\cdot\}$. □

LEMMA 4.10. *The submodule L_π of Proposition 4.9 is maximally isotropic on $E(\mathcal{A})$, hence $p(L_\pi)$ is maximally isotropic on $\varepsilon(\mathcal{A})$, where $p : E(\mathcal{A}) \to \varepsilon(\mathcal{A})$ is the canonical projection.*

PROOF. For some element $(X, b'db)$ in $E(\mathcal{A})$, we assume that $((X, b'db), \cdot) = 0$ on L_π. Then for any $a \in \mathcal{A}$, $((X, b'db), (\tilde{\pi}da, da)) = X(a) + b'\{a, b\} = 0$. When $da \neq 0$, this implies that $X(a) = \tilde{\pi}(b'db)(a)$. Even if $da = 0$, by the universality we obtain $X(a) = \tilde{\pi}(b'db)(a) = 0$. Thus $X = \tilde{\pi}(b'db)$ which gives that L_π is maximally isotropic on $E(\mathcal{A})$. This implies that $p(L_\pi)$ is maximally isotropic in $\varepsilon(\mathcal{A})$. □

Here we obtain the main result of this subsection.

PROPOSITION 4.11. *Let $\{\cdot,\cdot\}$ be a binary and biderivation on $\mathcal{A}$. The bracket is a Poisson bracket if and only if $p(L_\pi)$ is a Dirac structure, where L_π is the same as L_π in Proposition* 4.9.

PROOF. By $p^{-1}(p(L_\pi)) = L_\pi$. □

Note that since $\mathcal{A}$ is commutative, $\varepsilon(\mathcal{A})$ is identified with $\mathrm{Der}(\mathcal{A}) \oplus (\Omega^1_{\mathcal{A}|k}/J)$. Thus $p(L_\pi)$ is still the graph of the induced map.

By Proposition 4.11, every Poisson bracket on a unital commutative algebra $\mathcal{A}$ is characterized by the Dirac structure of $\varepsilon(\mathcal{A})$.

REMARK 4.12. There exists the case $\varepsilon(\mathcal{A}) = 0$, for example $\mathcal{A} = k$. For this case we may always take the trivial Poisson bracket on $\mathcal{A}$. But this zero Poisson bracket is not characterized by Dirac structures. This is the difficulty of the algebraic formulation.

In Section 5 an example of $\varepsilon(3\mathcal{A})$ will be given and studied.

Noncommutaive Poisson algebras. Finally at this subsection, we consider Poisson structures associated with Poisson brackets. Let $\{\cdot,\cdot\}$ be a Poisson bracket on $\mathcal{A}$. Then we have k-homomorphism $\pi : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ by $\pi(a\otimes a') = \{a, a'\}$. Since $\{\cdot,\cdot\}$ is a biderivation, one can easily check that π is a Hochschild 2-cocycle, thus there exists the equivalence class $\pi \in H^2(\mathcal{A},\mathcal{A})$. We do not know whether the class satisfies the Poisson condition $[\pi,\pi] = 0$ on $H^3(\mathcal{A},\mathcal{A})$ under the Gerstenhaber bracket. P. Xu [9] showed the converse in noncommutative algebra cases. If $\Pi \in H^2(\mathcal{A},\mathcal{A})$ satisfies the Poisson condition then the center $Z(\mathcal{A})$ becomes a Poisson algebra by the bracket $\{z, z'\} := [z,[\Pi, z']]$. We do not know whether a noncommutative Poisson structure Π defines the Dirac structure or not, in general. Here we consider a particular case. If the matrix algebra $\mathcal{M}_r(\mathcal{A})$ of a commutative algebra $\mathcal{A}$ has a Poisson structure Π then $Z(\mathcal{M}_r(\mathcal{A})) \cong \mathcal{A}$ is a Poisson algebra, and thus we have a Dirac structure L_π on $\mathcal{A}$. By Theorem 1.3 we obtain the corresponding Dirac structure on $\mathcal{M}_r(\mathcal{A})$.

5. Omni-Lie algebras v.s. $\varepsilon(\mathcal{A})$

In this subsection we will show Theorem 1.4 in Introduction.

Let V be a vector space over the field $\mathbf{R}$. Set the vector bundle $V^* \to \{o\}$ over a point, where V^* is the dual space of V. The fiber-linearized functions on the bundle is a vector space $V[1] := V \oplus \mathbf{R}\cdot 1$ with almost trivial multiplication:

$$v_1 \cdot v_2 = 0 \text{ and } v\cdot 1 = 1\cdot v = v \text{ for any } v_1, v_2, v \in V\,.$$

It is clear that $V[1]$ is commutative.

We now move on to the proof of Theorem 1.4. To show the theorem, we determine the module of Kähler differentials $\Omega^1_{V[1]|\mathbf{R}}$. Recall the definition (8) and (9) in Section 2. Let $\{v_i\}$ be a basis of V. First we consider the module $O_{V[1]|\mathbf{R}}$. It is generated by all elements $\{d1, vd1, dv, vdv' \mid v, v' \in V\}$ as infinite $\mathbf{R}$-module. Since the multiplication on V is trivial, $v''vdv' = 0$. First, we assume the linearity (8) of d. Then the dimension of $O_{V[1]|\mathbf{R}}$ is reduced to in $1 + 2\dim V + (\dim V)^2$ and the induced module is generated by $\{d1, v_i d1, dv_i, v_i dv_j\}$, because every element of $V[1]$ is generated by $\{1, v_i\}$. Remark that the dimension is the same as the one of tensor product $V[1]\otimes V[1]$. Secondly, we assume the derivation property (9) of d. All defining relations are generated by $d1 = 0$, $v_i d1 = 0$ and $v_i dv_j = -v_j dv_i$. Here we

used $d(v_i v_j) = 0$. Thus the dimension of $\Omega_{V[1]|\mathbf{R}}$ is $\dim V +_{\dim V} C_2$ and it is generated by $\{dv_i, v_i dv_{j\,(i<j)}\}$ as $\mathbf{R}$-module, where $.C.$ is the combination. Thus we have $\mathbf{R}$-isomorphism $\Omega^1_{V[1]|\mathbf{R}} \cong V \oplus \bigwedge^2 V$ by $dv \cong v$, $vdv' = v \wedge v'$.

It is easy to determine the space of derivations $\mathrm{Der}(V[1])$. For any $X \in \mathrm{Der}(V[1])$, if $X(v) = v' + r \cdot 1$ for any $v, v' \in V$ then $0 = X(v^2) = 2rv$. Thus we have $r = 0$, i.e., $\mathrm{Der}(V[1]) \subset gl(V)$. On the other hand $gl(V)$ becomes the space of derivations of $V[1]$ by the rule $\xi(1) := 0$ for any $\xi \in gl(V)$. Thus we obtain $\mathrm{Der}(V[1]) \cong gl(V)$ and

PROPOSITION 5.1. $E(V[1]) \cong gl(V) \oplus V \oplus \bigwedge^2 V$.

Now we compute the Courant bracket on $V[1]$. For any $(\xi_1, dv_1), (\xi_2, dv_2)$, the bracket has Weinstein's formula (5):

$$\begin{aligned}[\![(\xi_1, dv_1), (\xi_2, dv_2)]\!] &= ([\xi_1, \xi_2], \mathcal{L}_{\xi_1} dv_2 - \mathcal{L}_{\xi_2} dv_1 + d\langle \xi_2, dv_1 \rangle) \\ &= ([\xi_1, \xi_2], d(\xi_1(v_2)))\,,\end{aligned}$$

where $(\xi_1, dv_1), (\xi_2, vdv_2) \in \varepsilon(V[1])$ and the Connes boundary map B is d. For $(\xi_1, dv_1), (\xi_2, vdv_2)$, the bracket is $([\xi_1, \xi_2], 0)$ from the triviality of the multiplication.

LEMMA 5.2. *The kernel J of the bilinear form of $E(V[1])$ is generated by $\{(0, v_i dv_j)\}$, i.e., $J \cong \bigwedge^2 V$*

PROOF. By $i_\xi(v_i dv_j) = \xi(v_j)v_i = 0$ and Lemma 4.8. □

From the above lemma, we obtain an isomorphism between Leibniz algebras:

$$\varepsilon(V[1]) \cong gl(V) \oplus V\,, \quad (\xi, dv) \cong (\xi, v)\,.$$

One can easily check that by the isomorphism the bilinear forms are isomorphic. Thus Theorem 1.4 is proved. We easily obtain the corollary of the theorem.

COROLLARY 5.3. *A Poisson bracket on $V[1]$ corresponds bijectively with the Lie bracket on V. Thus a Lie-Poisson bracket on V^* corresponds bijectively to the Poisson bracket on $V[1]$.*

PROOF. By the isomorphism, the Dirac structure of a Poisson bracket on $V[1]$ corresponds to the graph of a Lie algebra structure on V. □

References

[1] COURANT, THEODORE JAMES: Dirac manifolds, Trans. Amer. Math. Soc. **319** (1990), no. 2, 631–661.

[2] GERSTENHABER, MURRAY: The cohomology structure of an associative ring, Ann. of Math. (2) **78** (1963), 267–288.

[3] KINYON, MICHAEL K and WEINSTEIN, ALAN: Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces, Amer. J. Math. **123** (2001), no. 3, 525–550.

[4] KOSMANN-SCHWARZBACH, YVETTE: Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory, The breadth of symplectic and Poisson geometry, 363–389, Progr. Math. **232**, Birkhauser Boston, Boston, MA, 2005.

[5] LIU, ZHANG-JU, WEINSTEIN, ALAN and XU, PING: Manin triples for Lie bialgebroids, J. Differential Geom. **45** (1997), no. 3, 547–574.

[6] LODAY, JEAN-LOUIS: *Cyclic homology*, Grundlehren der Math. Wissenschaften, 301, Springer-Verlag, Berlin, 1998.

[7] ROYTENBERG, DMITRY: Courant algebroids, derived brackets and even symplectic supermanifolds, math.DG/9910078.

[8] WEINSTEIN, ALAN: Omni-Lie algebras, Microlocal analysis of the Schrodinger equation and related topics (Japanese) (Kyoto, 1999), Surikaisekikenkyusho Kokyuroku No. 1176 (2000), 95–102.

[9] XU, PING: Noncommutative Poisson algebras, Amer. J. Math **116** (1994), no. 1, 101–125.

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