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Pseudo-null Iwasawa Modules for $\mathbf{Z}_2^2$-extensions

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Abstract. We shall give some criteria for the validity of Greenberg's generalized conjecture (GGC) for imaginary quadratic fields and the prime 2. Using this, we shall give examples of imaginary quadratic fields satisfying GGC for the prime 2.

1. Introduction

Let k be an algebraic number field and l a prime number. We denote by $\tilde{k}$ the compositum of all $\mathbf{Z}_l$-extensions of k. Then $\tilde{k}/k$ forms a $\mathbf{Z}_l^d$-extension with a positive integer d, that is, $G := \mathrm{Gal}(\tilde{k}/k)$ is topologically isomorphic to the d-copies of the additive group of $\mathbf{Z}_l$. Let $L(\tilde{k})$ be the maximal unramified abelian pro-l extension field of $\tilde{k}$. Then the Galois group $X(\tilde{k}) := \mathrm{Gal}(L(\tilde{k})/\tilde{k})$ can be considered as a module over the completed group ring $\Lambda(G) := \mathbf{Z}_l[[G]]$. Greenberg showed that $X(\tilde{k})$ is a finitely generated torsion $\Lambda(G)$-module ([12]). Moreover, the following is conjectured ([15]):

GREENBERG'S GENERALIZED CONJECTURE (GGC). *$X(\tilde{k})$ is a pseudo-null $\Lambda(G)$-module. That is, the annihilator ideal $\mathrm{Ann}_{\Lambda(G)}(M)$ has height at least* 2.

This is a generalization of (usual) Greenberg's conjecture for totally real number fields (see [13], [15]). As same as Greenberg's conjecture, no counterexample for GGC is known yet. However, not so many cases are known to be true when k is not totally real. Hence it is important to find new examples for which GGC is valid.

In the following, we shall restrict our attention to the case that k is an imaginary quadratic field. We will state some of known results. Minardi showed that if the class number of k is not divisible by l, then GGC holds for k and l ([23], [24]). Recently, Fukuda and Komatsu gave many examples of imaginary quadratic fields satisfying GGC for the prime 3 by using computer calculations ([8]).

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In the present paper, we shall consider the case that $l = 2$. We will state some criteria for the validity of GGC in Section 2 (Theorems 1, 2, 3). In Section 3, we shall give some families of imaginary quadratic fields satisfying GGC for the prime 2.

2. Criteria

Let F be an algebraic number field. We denote by $A(F)$ the Sylow 2-subgroup of the ideal class group of F. We write $\lambda_2(F)$ for the Iwasawa λ-invariant of the **cyclotomic** $\mathbf{Z}_2$-extension of F. For a $\mathbf{Z}_2^d$-extension F'/F with a positive integer d, we denote by $L(F')$ the maximal unramified abelian pro-2 extension field over F', and by $X(F')$ the Galois group $\mathrm{Gal}(L(F')/F')$. We call $X(F')$ the Iwasawa module. For a finite set S, we denote by $|S|$ the number of elements in S. For a finitely generated $\mathbf{Z}_2$-module N, we denote by $\mathrm{rank}_{\mathbf{Z}_2}(N)$ the dimension of the $\mathbf{Q}_2$-vector space $N \otimes_{\mathbf{Z}_2} \mathbf{Q}_2$ (where $\mathbf{Q}_2$ is the field of 2-adic numbers). For a group $\mathcal{G}$ which is topologically isomorphic to the direct sum of finite copies of the additive group of $\mathbf{Z}_2$, we put $\Lambda(\mathcal{G}) = \mathbf{Z}_2[[\mathcal{G}]]$. We denote by $\left(\frac{\cdot}{\cdot}\right)$ the quadratic residue symbol.

Let $\tilde{k}_1$ be the maximal elementary abelian 2-extension field of k which is contained in $\tilde{k}$. Our main theorems of the present paper are the following:

THEOREM 1. *We put $k = \mathbf{Q}(\sqrt{-d})$ with a positive square-free integer d which satisfies $d \equiv 3 \pmod 8$. Assume that k satisfies one of the following two conditions*:

(i) $k(\sqrt{-1}, \sqrt{2}) = \tilde{k}_1$, *and* $\lambda_2(\mathbf{Q}(\sqrt{d})) = 0$,

(ii) $k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1$ *with a rational prime divisor p of d satisfying $p \equiv 5 \pmod 8$, and* $\lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0$.

Then GGC holds for k and the prime 2.

THEOREM 2. *We put $k = \mathbf{Q}(\sqrt{-d})$ with a positive square-free integer d which satisfies $d \equiv 7 \pmod 8$. Assume that $k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1$ with a rational prime divisor p of d satisfying $p \equiv 3 \pmod 8$, and $\lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0$, then GGC holds for k and the prime 2.*

THEOREM 3. *We put $k = \mathbf{Q}(\sqrt{-d})$ with a positive square-free integer d satisfying $d > 4$. If k satisfies one of the following* (a)–(f), *then GGC holds for k and the prime 2.* (*In the following list, p is an **odd prime divisor** of d.*)

(a) $d \equiv 5 \pmod 8$, *and*

$$\begin{cases} k(\sqrt{-1}, \sqrt{2}) = \tilde{k}_1, & \lambda_2(\mathbf{Q}(\sqrt{d})) = 0, \textit{ or} \\ k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1 \textit{with } p \equiv 3 \pmod 8, & \lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0. \end{cases}$$

(b) $d \equiv 1 \pmod 8$, $k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1$ *with* $p \equiv \pm 5 \pmod 8$, *and* $\lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0$.

(c) $d = 2d'$ *with* $d' \equiv 3 \pmod 8$, *and*

$$\begin{cases} k(\sqrt{-1}, \sqrt{2}) = \tilde{k}_1, & \lambda_2(\mathbf{Q}(\sqrt{d})) = 0, \textit{ or} \\ k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1 \textit{ with } p \equiv 5 \pmod 8, & \lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0 \end{cases}$$

(d) $d = 2d'$ *with* $d' \equiv 5 \pmod 8$, *and*

$$\begin{cases} k(\sqrt{-1}, \sqrt{2}) = \tilde{k}_1\,, & \lambda_2(\mathbf{Q}(\sqrt{d})) = 0, \textit{ or} \\ k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1 \textit{ with } p \equiv 3 \pmod 8\,, & \lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0\,. \end{cases}$$

(e) $d = 2d'$ *with* $d' \equiv 1 \pmod 8$, $k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1$ *with* $p \equiv \pm 5 \pmod 8$, *and* $\lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0$.

(f) $d = 2d'$ *with* $d' \equiv 7 \pmod 8$, $k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1$ *with* $p \equiv 5 \pmod 8$, *and* $\lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0$.

Note that our method works only when all quadratic subextension of $\tilde{k}/k$ are abelian over $\mathbf{Q}$. In this case, there is a quadratic subextension of $\tilde{k}/k$ such that every prime ideal lying above an odd prime number which does not split in k splits completely (see Theorem (11) of [4]). Moreover, if the prime 2 splits in k, then k must has an unramified quadratic extension contained in $\tilde{k}$ (see Remark 3).

From now on, we shall prove the above theorems. Since $\tilde{k}$ contains the cyclotomic $\mathbf{Z}_2$-extension field k_∞ of k, we put $H = \mathrm{Gal}(\tilde{k}/k_\infty)$. We note that $\tilde{k}/k$ is a $\mathbf{Z}_2^2$-extension (see, e.g., Theorem 13.4 of [31]). Then H is isomorphic to the additive group of $\mathbf{Z}_2$. We can consider $X(\tilde{k})$ as a $\Lambda(H)$-module. It is well known that $X(\tilde{k})$ is a finitely generated $\Lambda(H)$-module because $X(k_\infty)$ is finitely generated over $\mathbf{Z}_2$ (see Theorem 2 of Bloom [2], and Iwasawa [19]).

In this situation, we know the following result:

PROPOSITION A. *If $X(\tilde{k})$ is a finitely generated* ***torsion*** *$\Lambda(H)$-module, then $X(\tilde{k})$ is pseudo-null as a $\Lambda(G)$-module.*

PROOF. This proposition is a special case of Lemma 4.10 of Venjakob [30]. □

REMARK 1. Venjakob's result includes the case of non-commutative l-adic Lie extensions. See also Hachimori-Sharifi [16]. We also note that a similar assertion is mentioned in pp. 352–353 of Greenberg [15].

We shall fix a topological generator γ of H. We put $\omega_n = \gamma^{2^n} - 1$ and $\nu_{m,n} = \omega_m/\omega_n$ for non-negative integers $m > n$ as usual.

LEMMA 1. *Assume that there is a $\Lambda(H)$-submodule Y of $X(\tilde{k})$ such that $X(\tilde{k})/\nu_{1,0}Y$ is finitely generated as a $\mathbf{Z}_2$-module. If*

$$\mathrm{rank}_{\mathbf{Z}_2}(X(\tilde{k})/Y) = \mathrm{rank}_{\mathbf{Z}_2}(X(\tilde{k})/\nu_{1,0}Y)\,,$$

then $X(\tilde{k})$ is $\Lambda(H)$-torsion.

PROOF. We shall consider the following exact sequence:

$$0 \to Y/\nu_{1,0}Y \to X(\tilde{k})/\nu_{1,0}Y \to X(\tilde{k})/Y \to 0\,.$$

From the assumption, the order of $Y/\nu_{1,0}Y$ is finite. Hence we can see that Y is a finitely generated torsion $\Lambda(H)$-module (see, e.g., [1]). Then our assertion follows because $X(\tilde{k})/Y$ is also $\Lambda(H)$-torsion. □

Next, we will state a preliminary result which plays a central role in the proof of our main theorems.

PROPOSITION 1. *Let $k = \mathbf{Q}(\sqrt{-d})$ with a positive square-free integer d which satisfies $d > 2$.*

(i) *If $\lambda_2(\mathbf{Q}(\sqrt{d})) = 0$, then $\lambda_2(k(\sqrt{-1})) = \lambda_2(k)$.*

(ii) *If d has a prime divisor p satisfying $p \equiv \pm 5 \pmod 8$, and $\lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0$, then $\lambda_2(k(\sqrt{-p})) = \lambda_2(k)$.*

REMARK 2. For every quadratic extension of an imaginary quadratic field, its Iwasawa μ-invariant of the cyclotomic $\mathbf{Z}_2$-extension is always zero (Iwasawa [19]).

PROOF OF PROPOSITION 1. Let m be a positive divisor of d which satisfies $k(\sqrt{-m}) \neq k(\sqrt{2})$. Then the following formula is known:

$$\lambda_2(k(\sqrt{-m})) = \lambda_2(k) + \lambda_2(\mathbf{Q}(\sqrt{-m})) + \lambda_2(\mathbf{Q}(\sqrt{d/m})).$$

(This follows from Lemma 3 of [21].)

First, we put $m = 1$. If $k(\sqrt{-1}) = k(\sqrt{2})$, then the assertion trivially follows. Otherwise, we see that $\lambda_2(k(\sqrt{-1})) = \lambda_2(k) + \lambda_2(\mathbf{Q}(\sqrt{d}))$. The part (i) follows.

We note that if p is an odd prime number satisfying $p \equiv \pm 5 \pmod 8$, then $\lambda_2(\mathbf{Q}(\sqrt{-p})) = 0$ (See Ferrero [5] and Kida [20]). Hence, the part (ii) can be shown similarly. □

We will finish to prove our main theorems. Let k_∞^1 be the unique quadratic extension of k_∞ which is contained in $\tilde{k}$. Note that $X(k_\infty^1)$ is also finitely generated as a $\mathbf{Z}_2$-module (see Remark 2).

2.1. Proof of Theorem 1. We only give a proof of the part (i). (The part (ii) can be shown quite similarly.)

By the assumption, there is only one prime of $\tilde{k}$ lying above 2, and it is totally ramified in $\tilde{k}/k$. In this case, we can obtain the following:

$$X(\tilde{k})/\omega_0 X(\tilde{k}) \cong X(k_\infty), \quad X(\tilde{k})/\omega_1 X(\tilde{k}) \cong X(k_\infty^1) \tag{1}$$

(cf. p. 248, Corollary of Bloom [2]).

Note that k_∞^1 is the cyclotomic $\mathbf{Z}_2$-extension field of $k(\sqrt{-1})$. Hence by Proposition 1, we see that the Iwasawa λ-invariant of $k_\infty^1/k(\sqrt{-1})$ coincides with the Iwasawa λ-invariant of k_∞/k. This implies that $\mathrm{rank}_{\mathbf{Z}_2}(X(k_\infty)) = \mathrm{rank}_{\mathbf{Z}_2}(X(k_\infty^1))$. Then by using (1), Lemma 1 (taking $Y = \omega_0 X(\tilde{k})$), and Proposition A, we can show the theorem in this case. □

2.2. Proof of Theorem 2. In this case, the prime 2 splits completely in k. Hence, we know that $\tilde{k}/k_\infty$ is an unramified extension (see, e.g., Ozaki [27]) and we can see that

$$X(\tilde{k})/\omega_0 X(\tilde{k}) \oplus \mathbf{Z}_2 \cong X(k_\infty)\,, \quad X(\tilde{k})/\omega_1 X(\tilde{k}) \oplus \mathbf{Z}_2 \cong X(k_\infty^1) \tag{2}$$

as $\mathbf{Z}_2$-modules (cf. Theorem 4 of Bloom [2]).

By Proposition 1, we have the equality $\mathrm{rank}_{\mathbf{Z}_2}(X(k_\infty)) = \mathrm{rank}_{\mathbf{Z}_2}(X(k_\infty^1))$. Then by repeating the same argument given in the proof of Theorem 1, we can prove Theorem 2. □

REMARK 3. Under the assumption of Theorem 2 ($d \equiv 7 \pmod 8$), we can show that GGC holds if k satisfies one of the following conditions:

- $k(\sqrt{-1}) \subset \tilde{k}$ and $\lambda_2(\mathbf{Q}(\sqrt{d})) = 0$,
- $k(\sqrt{-p}) \subset \tilde{k}$ with a rational prime divisor p of d satisfying $p \equiv 5 \pmod 8$, and $\lambda_2(\mathbf{Q}(\sqrt{d/p})) = 0$.

However, these do not occur. Since the prime 2 splits in k, there is a $\mathbf{Z}_2$-extension such that only one prime divisor of 2 ramifies. If k satisfies one of the above conditions, both primes lying above 2 are ramified in every quadratic subextension of $\tilde{k}/k$. It is a contradiction.

2.3. Proof of Theorem 3. In all cases expect for (f), we can see that there is only one prime of $\tilde{k}$ lying above 2. We note that k_∞^1/k_∞ is an unramified extension in some cases. However, we can show that

$$\mathrm{rank}_{\mathbf{Z}_2}(X(\tilde{k})/\omega_0 X(\tilde{k})) = \mathrm{rank}_{\mathbf{Z}_2}(X(k_\infty))\,,$$
$$\mathrm{rank}_{\mathbf{Z}_2}(X(\tilde{k})/\omega_1 X(\tilde{k})) = \mathrm{rank}_{\mathbf{Z}_2}(X(k_\infty^1))$$

for (a)–(e) (see the proof of Lemma 1 of [27]).

The case (f) is slightly complicated. Let $\mathfrak{l}$ be the unique prime of k which is lying above 2. Note that $k(\sqrt{2})/k$ is an unramified extension, and $\mathfrak{l}$ splits exactly two distinct primes in k_∞. We claim that the inertia field of $\tilde{k}/k$ for the prime $\mathfrak{l}$ is $k(\sqrt{2})$. (It is sufficient to show that $L(k) \cap \tilde{k} = k(\sqrt{2})$. Assume that it does not hold, then there is an unramified cyclic quartic extension L'/k contained $\tilde{k}$ because $k(\sqrt{-p})/k$ is not an unramified extension. Let $\mathfrak{p}$ be the unique prime of k which is lying above p. By class field theory, the order of the Frobenius automorphism $\left(\frac{L(k)/k}{\mathfrak{p}}\right)$ is exactly 2. Note that L' contains $k(\sqrt{2})$, and $\mathfrak{p}$ does not decomposed in $k(\sqrt{2})$ because $p \equiv 5 \pmod 8$. It is a contradiction. The claim follows.) This implies that all primes of k_∞ lying above 2 are totally ramified in $\tilde{k}$. Hence by Theorem 3 of [2], there is a $\Lambda(H)$-submodule Y of $X(\tilde{k})$ which satisfies

$$X(\tilde{k})/Y \cong X(k_\infty)\,, \quad X(\tilde{k})/\nu_{1,0} Y \cong X(k_\infty^1).$$

The rest of the proof is similar to that of Theorem 1. □

REMARK 4. Let $k = \mathbf{Q}(\sqrt{-d})$ with a positive square-free integer d. By using a different method, Fujii recently showed in [6] that if $d \not\equiv 1, 2, 7, 9, 15 \pmod{16}$, $k(\sqrt{-1}) \subset \tilde{k}$, and $\lambda_2(\mathbf{Q}(\sqrt{d}))=0$, then GGC holds for k and the prime 2. Fujii's result contains the following result that if $d=2d'$ with $d'\equiv 7 \pmod 8$, $k(\sqrt{-1},\sqrt{2})=\tilde{k}_1$, and $\lambda_2(\mathbf{Q}(\sqrt{d}))=0$, then GGC holds. Note that this is not contained in our main theorems. The author did not know it when he submitted the present paper. However, we can show this result by using our method. The proof is similar to that of Theorem 3 (f). It is sufficient to show that $L(k)\cap\tilde{k}=k(\sqrt{2})$. Assume that it does not hold, then there is an unramified cyclic quartic extension L'/k contained in $\tilde{k}$ because $k(\sqrt{-1})/k$ is not an unramified extension. Since L' contains $k(\sqrt{2})$ and any quadratic subextension of $\tilde{k}/k(\sqrt{2})$ is abelian over $\mathbf{Q}$, we see that L' is an abelian extension of $\mathbf{Q}$. It is a contradiction.

3. Examples

When we use our criteria in practice, some problems arise, namely, the determination of the first layer of non-cyclotomic $\mathbf{Z}_2$-extensions, and Greenberg's conjecture for real quadratic fields. However, these problems were well studied by many authors. Thanks to their results, we can obtain examples of imaginary quadratic fields which satisfies GGC for the prime 2.

Let the notations be as in the previous sections. Recall that $\tilde{k}_1$ is the maximal elementary abelian 2-extension field of k which is contained in $\tilde{k}$. The determination problem of $\tilde{k}_1$ is considered in Carroll [3], Carroll-Kisilevsky [4], Gras [11], ... In particular, according to Gras [11], one can determine $\tilde{k}_1$ exactly for any given imaginary quadratic field k. However, we shall mainly use the following result due to Carroll-Kisilevsky [4] because it is convenient to construct a family of examples.

PROPOSITION B (cf. [4], Corollary 8, Example 2). *Let $k=\mathbf{Q}(\sqrt{-d})$ with a positive square-free integer d. Assume that $d\not\equiv 1 \pmod 8$, d has a prime divisor p which satisfies $p\equiv\pm 3 \pmod 8$, and the exponent of $A(k)$ is 2. Then there is a unique integer $m\neq 1,2,-d,-2d$ dividing $2d$ (m is allowed to be negative) such that every odd prime divisor of d splits completely in $k(\sqrt{m})/k$. Moreover, $k(\sqrt{m},\sqrt{2})=\tilde{k}_1$.*

On the other hand, various families of real quadratic fields which satisfies Greenberg's conjecture for the prime 2 are given by many authors.

First, we shall consider the case that $A(k)$ is a cyclic group. We shall mention the cases where GGC follows without using our main theorems. (Almost all cases seem well known.)

PROPOSITION C. *We put $k=\mathbf{Q}(\sqrt{-d})$ with a positive square-free integer d. If d satisfies one of the following (1)–(6), then GGC holds for k and the prime 2 (in the following table, p and q are distinct prime numbers).*

	d	*conditions*
(1)	1 *or* 2	
(2)	p *or* $2p$	$p \equiv \pm 3 \pmod 8$
(3)	p	$p \equiv 7 \pmod 8$
(4)	pq	$p \equiv 5, q \equiv 3 \pmod 8$
(5)	pq	$p \equiv 5, q \equiv 7 \pmod 8$, $\left(\frac{p}{q}\right) = -1$
(6)	pq	$p \equiv 1, q \equiv 7 \pmod 8$, $\left(\frac{p}{q}\right) = -1$, $2^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{8}} \pmod p$

PROOF. By Minardi's result ([23], [24], see also Theorem B of [27]), it is sufficient to find a $\mathbf{Z}_2$-extension K/k such that there is only one prime in K which is ramified in K/k, and the Iwasawa module $X(K)$ is finite.

Let $L(k)$ be the maximal unramified abelian 2-extension field over k. For the cases (1) and (2), we can see that $A(k)$ is trivial or cyclic, $L(k)$ is contained in $\tilde{k}$, and there is only one prime lying above 2 in $L(k)$ (cf. Carroll-Kisilevsky [4], Bloom [2], Gras [10]). Then for any $\mathbf{Z}_2$-extension K/k which contains $L(k)$, the Iwasawa module $X(K)$ is finite (in fact, it is trivial).

For the case (5), it follows that $k(\sqrt{-p})$ is contained in $\tilde{k}$ (by Proposition B, or Example (15) of [3]). Note that $k(\sqrt{-p})/k$ is totally ramified at the unique prime lying above 2. We can see $|A(k)| = 2$ (by genus theory and the criterion of Rédei-Reichardt [29]) and $|A(k(\sqrt{-p}))| = 2$ (by the analytic class number formula, e.g., Washington [31]). Then for any $\mathbf{Z}_2$-extension K/k which contains $k(\sqrt{-p})$, the Iwasawa module $X(K)$ is finite (see Fukuda [7]).

For the cases (3), (4), and (6), we note that the prime 2 splits into two distinct primes in k. Fix a prime $\mathfrak{l}$ of k dividing 2. Then there is a unique $\mathbf{Z}_2$-extension N/k which is unramified outside $\mathfrak{l}$. For all cases, $\mathfrak{l}$ does not split in N/k. Moreover, it is known that $X(N)$ is finite for these cases (see Theorem 1 and Theorem 2 of [17]).

Hence in all cases, we can see that GGC holds for k and the prime 2. □

We are also able to prove the above result by using our main theorems, except for the cases (1), (3), and (6) (in these cases, $\tilde{k}_1$ is not abelian over $\mathbf{Q}$). We shall give an example which is not contained in Proposition C.

EXAMPLE 1. Let $k = \mathbf{Q}(\sqrt{-pq})$ with prime numbers p, q satisfying

$$p \equiv 1 \pmod 8, \quad q \equiv 3 \pmod 8, \quad \left(\frac{p}{q}\right) = -1 .$$

Then by genus theory and the criterion of Rédei-Reichardt, we see $A(k) \cong \mathbf{Z}/2\mathbf{Z}$. By Proposition B or Example (15) of [3], we can see that $k(\sqrt{-1})$ is contained in $\tilde{k}$. Moreover, it is known that if $2^{\frac{p-1}{4}} \not\equiv 1 \pmod p$, then $\lambda_2(\mathbf{Q}(\sqrt{pq})) = 0$ (Fukuda-Komatsu [9]). Then under (all of) the above conditions, GGC for k and the prime 2 holds by Theorem 1.

Next, we shall consider the case that $A(k)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$. Such fields can be completely determined (see Kisilevsky [22]).

EXAMPLE 2. The following imaginary quadratic fields k satisfy $A(k) \cong (\mathbf{Z}/2\mathbf{Z})^2$. We can show the validity of GGC for these fields and the prime 2 by using our main theorems, Proposition B, and known results on Greenberg's conjecture for real quadratic fields. In the following, p, q, and r denote distinct prime numbers.

(1) $k = \mathbf{Q}(\sqrt{-pqr})$, $p \equiv q \equiv 5, r \equiv 3 \pmod 8$, $\left(\frac{p}{q}\right) = \left(\frac{r}{p}\right) = -1$, $\left(\frac{q}{r}\right) = 1$.
$$\begin{cases} k(\sqrt{-q}, \sqrt{2}) = \tilde{k}_1 . \\ \lambda_2(\mathbf{Q}(\sqrt{pr})) = 0 \text{ by Ozaki-Taya [28]} . \end{cases}$$
We can apply Theorem 1 (ii).

(2) $k = \mathbf{Q}(\sqrt{-pqr})$, $p \equiv 1, q \equiv 5, r \equiv 7 \pmod 8$, $\left(\frac{p}{q}\right) = 1$, $\left(\frac{r}{q}\right) = \left(\frac{r}{p}\right) = -1$.
$$\begin{cases} k(\sqrt{-q}, \sqrt{2}) = \tilde{k}_1 . \\ \text{“}p \equiv 9 \pmod{16}\text{” or “}p \equiv 1, r \equiv 7 \pmod{16}, 2^{\frac{p-1}{4}} \equiv -1 \pmod p\text{”} \\ \Rightarrow \lambda_2(\mathbf{Q}(\sqrt{pr})) = 0 \text{ by Nishino [26]} . \end{cases}$$
We can apply Theorem 1 (ii).

(3) $k = \mathbf{Q}(\sqrt{-pqr})$, $p \equiv q \equiv 3, r \equiv 7 \pmod 8$, $\left(\frac{p}{q}\right) = \left(\frac{q}{r}\right) = \left(\frac{r}{p}\right) = 1$.
$$\begin{cases} k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1 . \\ \lambda_2(\mathbf{Q}(\sqrt{qr})) = 0 \text{ by Iwasawa's theorem ([18], see also [28])} . \end{cases}$$
We can apply Theorem 2.

(4) $k = \mathbf{Q}(\sqrt{-pq})$, $p \equiv 1, q \equiv 5 \pmod 8$, $\left(\frac{p}{q}\right) = -1$.
$$\begin{cases} k(\sqrt{-1}, \sqrt{2}) = \tilde{k}_1 . \\ 2^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{8}} \pmod p \Rightarrow \lambda_2(\mathbf{Q}(\sqrt{pq})) = 0 \text{ by Yamamoto [32]} . \end{cases}$$
We can apply Theorem 3 (a).

(5) $k = \mathbf{Q}(\sqrt{-pq})$, $p \equiv 3, q \equiv 7 \pmod 8$, $\left(\frac{p}{q}\right) = -1$.
$$\begin{cases} k(\sqrt{-p}, \sqrt{2}) = \tilde{k}_1 . \\ \lambda_2(\mathbf{Q}(\sqrt{q})) = 0 \text{ by Iwasawa's theorem} . \end{cases}$$
We can apply Theorem 3 (a).

(6) $k = \mathbf{Q}(\sqrt{-2pq})$, $p \equiv 1, q \equiv 3 \pmod 8$, $\left(\frac{p}{q}\right) = -1$.
$$\begin{cases} k(\sqrt{-1}, \sqrt{2}) = \tilde{k}_1 . \\ 2^{\frac{p-1}{4}} \not\equiv -1 \pmod p \Rightarrow \lambda_2(\mathbf{Q}(\sqrt{2pq})) = 0 \text{ by Fukuda-Komatsu [9]} . \end{cases}$$

We can apply Theorem 3 (c).

(7) $k=\mathbf{Q}(\sqrt{-2pq})$, $p\equiv 5, q\equiv 7 \pmod 8$, $\left(\frac{p}{q}\right)=-1$.

$$\begin{cases} k(\sqrt{-p},\sqrt{2})=\tilde{k}_1\,. \\ \lambda_2(\mathbf{Q}(\sqrt{2q}))=0 \text{ by Iwasawa's theorem}\,. \end{cases}$$

We can apply Theorem 3 (c).

(8) $k=\mathbf{Q}(\sqrt{-2pq})$, $p\equiv 5, q\equiv 1 \pmod 8$, $\left(\frac{p}{q}\right)=-1$.

$$\begin{cases} k(\sqrt{-1},\sqrt{2})=\tilde{k}_1\,. \\ 2^{\frac{p-1}{4}}\not\equiv(-1)^{\frac{p-1}{8}} \pmod p \Rightarrow \lambda_2(\mathbf{Q}(\sqrt{2pq}))=0 \text{ by Yamamoto [32]}\,. \end{cases}$$

We can apply Theorem 3 (d).

(9) $k=\mathbf{Q}(\sqrt{-2pq})$, $p\equiv 3, q\equiv 7 \pmod 8$, $\left(\frac{p}{q}\right)=-1$.

$$\begin{cases} k(\sqrt{-p},\sqrt{2})=\tilde{k}_1\,. \\ \lambda_2(\mathbf{Q}(\sqrt{2q}))=0 \text{ by Iwasawa's theorem}\,. \end{cases}$$

We can apply Theorem 3 (d).

(10) $k=\mathbf{Q}(\sqrt{-2pq})$, $p\equiv 5, q\equiv 3 \pmod 8$, $\left(\frac{p}{q}\right)=1$.

$$\begin{cases} k(\sqrt{-p},\sqrt{2})=\tilde{k}_1\,. \\ \lambda_2(\mathbf{Q}(\sqrt{2q}))=0 \text{ by Iwasawa's theorem}\,. \end{cases}$$

We can apply Theorem 3 (f).

It seems that one can show the validity of GGC for some of cases in Example 2 by using the same argument given in the proof of the case (5) of Proposition C.

Moreover, assume that $k=\mathbf{Q}(\sqrt{-pqr})$ with distinct primes p,q, and r satisfying one of the following conditions (A) – (C):

(A) $p\equiv q\equiv r\equiv 3 \pmod 8$, $\left(\frac{q}{p}\right)=\left(\frac{r}{q}\right)=\left(\frac{p}{r}\right)$,

(B) $p\equiv q\equiv 1$, $r\equiv 3 \pmod 8$, $\left(\frac{q}{p}\right)=1, \left(\frac{r}{q}\right)=\left(\frac{r}{p}\right)=-1$,

(C) $p\equiv q\equiv 1$, $r\equiv 3 \pmod 8$, $\left(\frac{q}{p}\right)=-1$, $\left(\frac{r}{p}\right)=-1$.

Then, we can see that $k(\sqrt{-1})$ is contained in $\tilde{k}$. Hence if $\lambda_2(\mathbf{Q}(\sqrt{pqr}))=0$ is shown, GGC holds for k and the prime 2.

It seems that many other examples can be found by using our main theorems. However, we will end the present paper by giving only one more example.

EXAMPLE 3. We put $k=\mathbf{Q}(\sqrt{-pqrs})$ with distinct prime numbers p,q,r, and s satisfying:

$$p\equiv q\equiv 3\,,\quad r\equiv 7\,,\quad s\equiv 5 \pmod 8\,,$$

and

$$\left(\frac{s}{p}\right) = \left(\frac{s}{q}\right) = 1\,, \quad \left(\frac{s}{r}\right) = -1\,, \quad \left(\frac{q}{p}\right) = \left(\frac{r}{q}\right) = \left(\frac{p}{r}\right).$$

In this case, we can see that $A(k)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^3$. Since $k(\sqrt{-s})$ is contained in $\tilde{k}$ (by Proposition B) and $\lambda_2(\mathbf{Q}(\sqrt{pqr})) = 0$ (Yamamoto [32], see also Mizusawa [25]), GGC for k and the prime 2 holds by Theorem 1.

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References

[1] P. N. BALISTER and S. HOWSON, Note on Nakayama's lemma for compact Λ-modules, Asian J. Math. **1** (1997), 224–229.

[2] J. R. BLOOM, On the invariants of some $\mathbf{Z}_l$-extensions, J. Number Theory **11** (1979), 239–256.

[3] J. E. CARROLL, On determining the quadratic subfields of $\mathbf{Z}_2$-extensions of complex quadratic fields, Compositio Math. **30** (1975), 259–271.

[4] J. E. CARROLL and H. KISILEVSKY, Initial layers of $\mathbf{Z}_l$-extensions of complex quadratic fields, Compositio Math. **32** (1976), 157–168.

[5] B. FERRERO, The cyclotomic $\mathbf{Z}_2$-extension of imaginary quadratic fields, Amer. J. Math. **102** (1980), 447–459.

[6] S. FUJII, The torsion of p-ramified Iwasawa modules, preprint.

[7] T. FUKUDA, Remarks on $\mathbf{Z}_p$-extensions of number fields, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), 264–266.

[8] T. FUKUDA and K. KOMATSU, Noncyclotomic $\mathbf{Z}_p$-extensions of imaginary quadratic fields, Experiment. Math. **11** (2002), 469–475.

[9] T. FUKUDA and K. KOMATSU, On the Iwasawa λ-invariant of the cyclotomic $\mathbf{Z}_2$-extension of a real quadratic field, Tokyo J. Math. **28** (2005), 259–264.

[10] G. GRAS, Sur les $\mathbf{Z}_2$-extensions d'un corps quadratique imaginaire, Ann. Inst. Fourier (Grenoble) **33** (1983), 1–18.

[11] G. GRAS, Plongements kummeriens dans les $\mathbf{Z}_p$-extensions, Compositio Math. **55** (1985), 383–396.

[12] R. GREENBERG, The Iwasawa invariants of Γ-extensions of a fixed number field, Amer. J. Math. **95** (1973), 204–214.

[13] R. GREENBERG, On the Iwasawa invariants of totally real number fields, Amer. J. Math. **98** (1976), 263–284.

[14] R. GREENBERG, On the structure of certain Galois groups, Invent. Math. **47** (1978), 85–99.

[15] R. GREENBERG, Iwasawa theory—past and present, Class field theory—its centenary and prospect, Adv. Stud. Pure Math. **30**, Math. Soc. Japan, Tokyo, 2001, 335–385.

[16] Y. HACHIMORI and R. SHARIFI, On the failure of pseudo-nullity of Iwasawa modules, J. Algebraic Geom. **14** (2005), 567–591.

[17] T. ITOH, S. KIYOFUJI, and Y. MIZUSAWA, On the Iwasawa invariants of certain non-cyclotomic $\mathbf{Z}_2$-extensions, Manuscripta Math. **118** (2005), 227–233.

[18] K. IWASAWA, A note on class numbers of algebraic number fields, Abh. Math. Sem. Univ. Hamburg **20** (1956), 257–258.

[19] K. IWASAWA, *On the μ-invariants of $\mathbf{Z}_l$-extensions*, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Yasuo Akizuki, Kinokuniya, Tokyo, 1973, 1–11.

[20] Y. KIDA, On cyclotomic $\mathbf{Z}_2$-extensions of imaginary quadratic fields, Tôhoku Math. J. **31** (1979), 91–96.

[21] Y. KIDA, Cyclotomic $\mathbf{Z}_2$-extensions of J-fields, J. Number Theory **14** (1982), 340–352.

[22] H. KISILEVSKY, Number fields with class number congruent to 4 mod 8 and Hilbert's theorem 94, J. Number Theory **8** (1976), 271–279.

[23] J. MINARDI, Iwasawa modules for $\mathbf{Z}_p^d$-extensions of algebraic number fields, Thesis (1986), University of Washington.

[24] J. MINARDI, Iwasawa modules for $\mathbf{Z}_p^d$-extensions of number fields, Number theory, CMS Conf. Proc. **7** Amer. Math. Soc., Providence, RI, 1987, 237–242.

[25] Y. MIZUSAWA, On the Iwasawa invariants of $\mathbf{Z}_2$-extensions of certain real quadratic fields, Tokyo J. Math. **27** (2004), 255–261.

[26] Y. NISHINO, On the Iwasawa invariants of the cyclotomic $\mathbf{Z}_2$-extensions of certain real quadratic fields, Tokyo J. Math. **29** (2006), 239–245.

[27] M. OZAKI, Iwasawa invariants of $\mathbf{Z}_p$-extensions over an imaginary quadratic field, Class field theory—its centenary and prospect, Adv. Stud. Pure Math. **30**, Math. Soc. Japan, Tokyo, 2001, 387–399.

[28] M. OZAKI and H. TAYA, On the Iwasawa λ_2-invariants of certain families of real quadratic fields, Manuscripta Math. **94** (1997), 437–444.

[29] L. RÉDEI and H. REICHARDT, Die Anzahl der durch 4 teilbaren Invarianten der Klassengruppe eines beliebigen quadratischen Zahlkörpers, J. Reine Angew. Math. **170** (1933), 69–74.

[30] O. VENJAKOB, On the structure theory of the Iwasawa algebra of a p-adic Lie group, J. Eur. Math. Soc. **4** (2002), 271–311.

[31] L. C. WASHINGTON, *Introduction to Cyclotomic Fields*, 2nd ed., Graduate Texts in Math. **83**, Springer-Verlag, New York, Berlin, Heidelberg, 1997.

[32] G. YAMAMOTO, Iwasawa invariants of abelian p-extension fields, Thesis (2001), Waseda University.

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