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L^p Estimates for Some Schrödinger Type Operators and a Calderón-Zygmund Operator of Schrödinger Type

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Abstract. We consider the Schrödinger and Schrödinger type operators $H_1 = -\Delta + V$ and $H_2 = (-\Delta)^2 + V^2$ with non-negative potentials V on $\mathbf{R}^n$. We assume that the potential V belongs to the reverse Hölder class which includes non-negative polynomials. We establish estimates of the fundamental solution for H_2 and show some L^p estimates for Schrödinger type operators. Moreover, we show that the operator $\nabla^4 H_2^{-1}$ is a Calderón-Zygmund operator.

1. Introduction and Theorems

Let $V(x)$ be a non-negative potential and consider the Schrödinger and Schrödinger type operators $H_1 = -\Delta + V$ and $H_2 = (-\Delta)^2 + V^2$ on $\mathbf{R}^n$. When V is a non-negative polynomial, Zhong ([Zh]) proved the estimates of the fundamental solution for H_1 and H_2 and showed some estimates for H_1 and H_2. He showed the L^p boundedness of the operators $V^{2-j/2}\nabla^j H_2^{-1}$, where $j = 0, 1, 2, 3, 4$, and $V^k H_1^{-k}$, $V^{k-1/2}\nabla H_1^{-k}$, where $k \in \mathbf{N}$. He also proved that the operators $\nabla^2 H_1^{-1}$ and $\nabla^4 H_2^{-1}$ are Calderón-Zygmund operators.

For the potential V which belongs to the reverse Hölder class, which includes non-negative polynomials, Shen ([Sh1]) generalized Zhong's results on H_1. Actually, he established estimates of the fundamental solution for H_1 and showed the L^p estimates of the operators VH_1^{-1}, $V^{1/2}\nabla H_1^{-1}$, $\nabla^2 H_1^{-1}$, etc. On the operator H_1 these Shen's results were generalized to other directions. See [KS1], [Su]. Moreover, in [KS2] the authors studied the magnetic Schrödinger operator with potentials V which belong to a certain reverse Hölder class and showed some estimates. In particular they showed that the operator $\nabla^2 H_1^{-1}$ is a Calderón-Zygmund operator.

In this paper we study H_2 with reverse Hölder class potentials. We establish estimates of the fundamental solution for H_2 and show the L^p boundedness of the operators

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$V^{2-j/2}\nabla^j H_2^{-1}$, where $j = 0, 1, 2, 3, 4$. Moreover, we show that the operator $\nabla^4 H_2^{-1}$ is a Calderón-Zygmund operator.

To be precise, we recall the definitions of the reverse Hölder class (e.g. [Sh1]). Throughout this paper we denote by $B_r(x)$ the ball centered at x with radius r, and the letter C stands for a constant not necessarily the same at each occurrence.

DEFINITION 1 (Reverse Hölder class). Let $V \geq 0$.

(1) For $1 < p < \infty$ we say $V \in (RH)_p$, if $V \in L^p_{loc}(\mathbf{R}^n)$ and there exists a constant C such that

$$\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} V(y)^p dy\right)^{1/p} \leq \frac{C}{|B_r(x)|}\int_{B_r(x)} V(y)dy \tag{1}$$

holds for every $x \in \mathbf{R}^n$ and $0 < r < \infty$.

(2) We say $V \in (RH)_\infty$, if $V \in L^\infty_{loc}(\mathbf{R}^n)$ and there exists a constant C such that

$$\|V\|_{L^\infty(B_r(x))} \leq \frac{C}{|B_r(x)|}\int_{B_r(x)} V(y)dy \tag{2}$$

holds for every $x \in \mathbf{R}^n$ and $0 < r < \infty$.

REMARK 1. If $P(x)$ is a polynomial and $\alpha > 0$, then $V(x) = |P(x)|^\alpha$ belongs to $(RH)_\infty$ ([Fe]). For $1 < p < \infty$, it is easy to see $(RH)_\infty \subset (RH)_p$.

In [Sh1], Shen defined the auxiliary function $m(x, V)$ and established the estimates of the fundamental solution of H_1. For the operator H_2, we show the estimates of the fundamental solution with Shen's auxiliary function $m(x, V)$. We recall the definition of the function $m(x, V)$.

DEFINITION 2 ([Sh1, Definition 1.3]). Let $V \in (RH)_{n/2}$ and $V \not\equiv 0$. Then it is well-known that there exists $\varepsilon > 0$ such that $V \in (RH)_{n/2+\varepsilon}$ ([Ge]). Then the function $m(x, V)$ is well-defined by

$$\frac{1}{m(x, V)} = \sup\left\{r > 0 : \frac{r^2}{|B_r(x)|}\int_{B_r(x)} V(y)dy \leq 1\right\}$$

and satisfies $0 < m(x, V) < \infty$ for every $x \in \mathbf{R}^n$.

REMARK 2. If $V \in (RH)_\infty$ then there exists a constant C such that $V(x) \leq Cm(x, V)^2$ ([Sh1, Remark 2.9]). We also remark that, if $V \in (RH)_p$, $p \geq n/2$, then there exists a constant C such that

$$\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} V(y)^p dy\right)^{1/p} \leq Cm(x, V)^2$$

(cf. [Sh1, Lemma 1.8] and [KS1, Lemma 2.2(a)]).

Now we state our theorems. In this paper we study H_1 and H_2 only for $n \geq 3$ and $n \geq 5$ respectively. We denote by $\Gamma_{H_j}(x, y)$ the fundamental solution for H_j, $j = 1, 2$. The operator H_j^{-1} is the integral operator with $\Gamma_{H_j}(x, y)$ as its kernel.

THEOREM 1. (1) *Let $j = 0, 1, 2, 3$. Suppose $V \in (RH)_{n/2}$ and there exists a constant C such that $V(x) \leq Cm(x, V)^2$. Then there exist constants C_j such that*

$$\|V^{2-j/2}\nabla^j H_2^{-1} f\|_{L^p(\mathbf{R}^n)} \leq C_j \|f\|_{L^p(\mathbf{R}^n)}\,, \tag{3}$$

where $1 < p \leq \infty$ and $\nabla^j = \nabla_x^j = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}$, $j = |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.
(2) *Suppose $V \in (RH)_{n/2}$ and there exists a constant C such that $V(x) \leq Cm(x, V)^2$. Then there exists a constant C' such that*

$$\|\nabla^4 H_2^{-1} f\|_{L^p(\mathbf{R}^n)} \leq C' \|f\|_{L^p(\mathbf{R}^n)}\,, \tag{4}$$

where $1 < p < \infty$.

For the operator $\nabla^4 H^{-1}$, we prove that the operator $\nabla^4 H^{-1}$ is a Calderón-Zygmund operator under a little stronger assumption (see Theorem 4).

To prove Theorem 1 estimates of the fundamental solution are needed. The following Theorems 2 and 3 generalize the results in [Zh, Theorem 5.1 and Proposition 5.7] to the operator H_2 with potentials V which belong to the reverse Hölder class.

THEOREM 2. *Suppose $V \in (RH)_{n/2}$. Then for any positive integer N there exists a constant C_N such that*

$$(0 \leq) \Gamma_{H_2}(x, y) \leq \frac{C_N}{\{1 + m(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-4}}\,. \tag{5}$$

THEOREM 3. *Let $j = 1, 2, 3$. Suppose $V \in (RH)_{n/2}$ and there exists a constant C such that $V(x) \leq Cm(x, V)^2$. Then for any positive integer N there exists a constant C_N such that*

$$|\nabla_x^j \Gamma_{H_2}(x, y)| \leq \frac{C_N}{\{1 + m(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-4+j}}\,. \tag{6}$$

REMARK 3. Estimate (6) can be proved under the assumption $V \in (RH)_{2n/(4-j)}$, $j = 1, 2, 3$ (see Theorem 6). When we assume $V \in (RH)_q$ for some $q \geq n/2$ and use Theorem 6 (also Theorem 2) and the same method as in [Sh1, Theorem 4.13] (also [Sh1, Theorem 3.1]), we can prove the operators $V^{2-j/2}\nabla^j H_2^{-1}$, $j = 0, 1, 2, 3$, are bounded on $L^p(\mathbf{R}^n)$, $1 \leq p \leq q$. We note that, if we take the limit $q \to +\infty$, then the class $(RH)_q$ becomes $(RH)_\infty$ and $V \in (RH)_\infty$ implies "$V \in (RH)_{n/2}$ and $V(x) \leq Cm(x, V)^2$".

REMARK 4. For $\Gamma_{H_1}(x, y)$, some exponential decay estimates are known ([Ku], [Sh3]). For $\Gamma_{H_2}(x, y)$, we only prove polynomial decay estimates, since it suffices to show them to obtain our L^p estimates.

We prove Theorems 2 and 3 in Sections 3 and 4 respectively. In Section 2, we show an estimate for H_1 (Corollary 1) needed to prove Theorem 2. In Section 5, we prove Theorem 1 by using Theorems 2 and 3.

We now recall the definition of the Calderón-Zygmund operator. Let $\mathcal{D}'$ denote the space of distributions dual to $C_0^\infty(\mathbf{R}^n)$. An operator T taking $C_0^\infty(\mathbf{R}^n)$ into $\mathcal{D}'$ is called a Calderón-Zygmund operator if

(i) T extends to a bounded linear opeator on $L^2(\mathbf{R}^n)$,

(ii) there exists a kernel K such that for every $f \in C_0^\infty(\mathbf{R}^n)$,

$$Tf(x) = \int_{\mathbf{R}^n} K(x, y) f(y) dy \quad \text{a.e.} \quad \text{on} \quad \{\text{supp } f\}^c ,$$

(iii) there exist positive constants δ and C such that for all distinct $x, y \in \mathbf{R}^n$ and all z such that $|x - z| < |x - y|/2$,

$$|K(x, y)| \leq \frac{C}{|x - y|^n} , \tag{7}$$

$$|K(x, y) - K(z, y)| \leq \frac{C|x - z|^\delta}{|x - y|^{n+\delta}} , \tag{8}$$

$$|K(y, x) - K(y, z)| \leq \frac{C|x - z|^\delta}{|x - y|^{n+\delta}} . \tag{9}$$

See e.g. [Ch, page 12].

THEOREM 4. *Suppose $V \in C^5(\mathbf{R}^n)$. Assume also that $V \in (RH)_{n/2}$ and there exists a constant C such that*

$$|\nabla^j V(x)| \leq Cm(x, V)^{2+j} , \qquad j = 1, 2, 3, 4, 5 . \tag{10}$$

Then $\nabla^4 H_2^{-1}$ is a Calderón-Zygmund operator.

Once we obtain Theorem 4, we can obtain the result that the operator $\nabla^4 H^{-1}$ is of weak-type (1,1) under the same assumption as in Theorem 4.

REMARK 5. It is known that $|\nabla V(x)| \leq Cm(x, V)^3$ implies $V(x) \leq Cm(x, V)^2$ ([Sh2, Remark 1.8]). We note that the condition (10) holds if V is a non-negative polynomial and there exist potentials V which satisfy our assumptions but are not non-negative polynomials (see [KS2, Remark 5]). We also note that, in [KS2, Theorem 2], the authors showed that $\nabla^2 H_1^{-1}$ is a Calderón-Zygmund operator under the assmuption $V \in (RH)_{n/2}$ and $|\nabla^j V(x)| \leq Cm(x, V)^{2+j}$, $j = 1, 2, 3$.

We note that the estimates (8) and (9) are implied by a condition

$$|\nabla K(x, y)| \leq \frac{C}{|x - y|^{n+1}}$$

([Ch, page 12]). Hence, to prove Theorem 3, it suffices to show that the estimates

$$|\nabla^4 \Gamma_{H_2}(x, y)| \le \frac{C}{|x-y|^n}, \qquad |\nabla^5 \Gamma_{H_2}(x, y)| \le \frac{C}{|x-y|^{n+1}}$$

hold. In fact, stronger and higher order derivative estimates hold as the following theorem states.

THEOREM 5. *Let j be a positive integer and $j \ge 4$. Suppose $V \in C^j(\mathbf{R}^n)$. Assume also that $V \in (RH)_{n/2}$ and there exists a constant C such that $|\nabla^i V(x)| \le Cm(x, V)^{2+i}$, $i = 1, 2, 3, \cdots, j$. Then for any positive integer N there exists a constant C_N such that*

$$|\nabla_x^j \Gamma_{H_2}(x, y)| \le \frac{C_N}{\{1 + m(x, V)|x-y|\}^N} \cdot \frac{1}{|x-y|^{n-4+j}} . \tag{11}$$

We prove Theorem 5 in Section 6. Section 7, which is an appendix, is devoted to L^p boundedness of the operator $V^{2k} H_2^{-k}$, $k \in \mathbf{N}$.

2. An estimate for H_1

In this section we show an estimate for the operator H_1 (Lemma 2). Before we state it, we recall the estimates related to the function $m(x, V)$ sometimes needed later.

LEMMA 1 ([Sh1, Lemma 1.4 (b), (c)]). *Suppose $V \in (RH)_{n/2}$. Then there exist constants C_1, C_2, and k_0 such that*

$$m(y, V) \le C_1\{1 + |x-y| m(x, V)\}^{k_0} m(x, V) , \tag{12}$$

$$m(y, V) \ge \frac{C_2 m(x, V)}{\{1 + |x-y| m(x, V)\}^{k_0/(k_0+1)}} . \tag{13}$$

LEMMA 2 (cf. [Sh1, Theorem 4.13]). *Suppose $V \in (RH)_{q_0}$ for some $n/2 \le q_0 < n$. Then for $1 \le p \le p_0$ there exists a constant C such that*

$$\|m(\cdot, V)\nabla H_1^{-1} f\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)} , \tag{14}$$

where $1/p_0 = 1/q_0 - 1/n$.

REMARK 6. Using the same way as in the proof of [Sh1, Corollary 2.8], we can obtain L^p boundedness of the operator $m(\cdot, V)\nabla H_1^{-1}$ with potentials V which belong to $(RH)_{q_0}$ for some $q_0 \ge n$.

The following Corollary 1 is needed to prove Theorem 2.

COROLLARY 1. *Suppose $V \in (RH)_{n/2}$. Then there exists a constant C such that*

$$\|m(\cdot, V)\nabla H_1^{-1} f\|_{L^2(\mathbf{R}^n)} \le C\|f\|_{L^2(\mathbf{R}^n)} . \tag{15}$$

PROOF OF LEMMA 2. We show Lemma 2 by a method similar to the one used in the proof of [Sh1, Theorem 4.13]. Suppose $V \in (RH)_{q_0}$ for some $q_0 \geq n/2$. Then $V \in (RH)_{q_1}$ for some q_1, satisfying $n > q_1 > q_0$. We denote by $\Gamma_{H_1}(x, y)$ the fundamental solution and let

$$Tf(x) = m(x, V) \int_{\mathbf{R}^n} \nabla_x \Gamma_{H_1}(x, y) f(y) dy .$$

The adjoint of T is given by

$$T^* f(x) = \int_{\mathbf{R}^n} \nabla_y \Gamma_{H_1}(y, x) m(y, V) f(y) dy .$$

By duality, it suffices to show that

$$\|T^* f\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for} \quad p_0' \leq p \leq \infty , \tag{16}$$

where $1/p_0 + 1/p_0' = 1$. Let $r = 1/m(x, V)$. We choose t and p_1 such that $1/t = 1/q_1 - 1/n$, $1/p_1 = 1 - 1/q_1 + 1/n$. Thus $1/t + 1/p_1 = 1$. Hence, by Hölder's inequality,

$$\begin{aligned} |T^* f(x)| &\leq \sum_{j=-\infty}^{+\infty} \int_{2^{j-1}r < |y-x| \leq 2^j r} |\nabla_y \Gamma_{H_1}(y, x)| m(y, V) |f(y)| dy \\ &\leq \sum_{j=-\infty}^{+\infty} \left(\int_{2^{j-1}r < |y-x| \leq 2^j r} \{|\nabla_y \Gamma_{H_1}(y, x)| m(y, V)\}^t dy \right)^{1/t} \\ &\qquad \cdot \left(\int_{|y-x| \leq 2^j r} |f(y)|^{p_1} dy \right)^{1/p_1} . \end{aligned}$$

It follows from (12) and [Sh1, Lemma 4.6 and Theorem 2.7] that

$$\begin{aligned} &\left(\int_{2^{j-1}r < |y-x| \leq 2^j r} \{|\nabla_y \Gamma_{H_1}(y, x)| m(y, V)\}^t dy \right)^{1/t} \\ &\quad \leq \sum_{k=1}^{K} \left(\int_{\substack{|y_k - x| = 3 \cdot 2^{j-2} r \\ |z - y_k| \leq 2^{j-1} r}} \{|\nabla_z \Gamma_{H_1}(z, x)| m(z, V)\}^t dz \right)^{1/t} \\ &\quad \leq C (2^{j-1} r)^{n/q_1 - 2} \{1 + 2^j r m(x, V)\}^{2k_0} m(x, V) \sup_{z \in B_{5 \cdot 2^{j-3} r}(y_k)} |\Gamma_{H_1}(z, x)| \\ &\quad \leq C (2^{j-1} r)^{n/q_1 - 2} (1 + 2^j)^{2k_0} \frac{1}{r} \cdot \frac{C_N}{\{1 + m(x, V)|z - x|\}^N} \cdot \frac{1}{|z - x|^{n-2}} \\ &\quad \leq C_N (2^{j-1} r)^{n/q_1 - 2} (1 + 2^j)^{2k_0} \frac{1}{r} \cdot \frac{1}{(1 + 2^{j-3})^N} \cdot \frac{1}{(2^{j-3} r)^{n-2}} \\ &\quad \leq C_N \frac{(2^j r)^{n/q_1 - n}}{(1 + 2^{j-3})^N r} (1 + 2^j)^{2k_0} , \end{aligned}$$

where K is a finite integer not depending on j and r. Thus

$$|T^* f(x)| \le C_N \sum_{j=-\infty}^{+\infty} \frac{2^{(1+2k_0)j}}{(1+2^{j-3})^N} \left\{ \frac{1}{(2^j r)^n} \int_{B_{2^j r}(x)} |f(y)|^{p_1} dy \right\}^{1/p_1}$$
$$\le C\{M(|f|^{p_1})(x)\}^{1/p_1},$$

where we choose $N \ge 2 + 2k_0$ and M is the Hardy-Littlewood maximal operator. It follows that

$$\|T^* f\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)} \quad \text{for} \quad p_1 < p \le \infty. \tag{17}$$

Then (16) follows since $p_0' > p_1$. □

3. Proof of Theorem 2

In this section we prove Theorem 2. It follows easily from the following Lemma 3.

LEMMA 3. *Suppose $V \in (RH)_{n/2}$ and $(-\Delta)^2 u + V^2 u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then for any positive integer N there exists a constant C_N such that*

$$\sup_{y \in B_{R/2}(x_0)} |u(y)| \le \frac{C_N}{\{1 + Rm(x_0, V)\}^N} \sup_{y \in B_R(x_0)} |u(y)|. \tag{18}$$

Assuming Lemma 3 for the moment, we give

PROOF OF THEOREM 2. Fix $x_0, y_0 \in \mathbf{R}^n$ and put $R = |x_0 - y_0|$. Then $u(x) = \Gamma_{H_2}(x, y_0)$ is a solution of $(-\Delta)^2 u + V^2 u = 0$ on $B_{R/4}(x_0)$. Using the estimate $0 \le \Gamma_{H_2}(x, y) \le C/|x-y|^{n-4}$ and (18), we arrive at the desired estimate. □

To prove Lemma 3 we need some lemmas.

LEMMA 4. *Let $V \in (RH)_{n/2}$. Then there exists a constant C such that*

$$\int_{\mathbf{R}^n} m(x, V)^4 |u(x)|^2 dx + \int_{\mathbf{R}^n} m(x, V)^2 |\nabla u(x)|^2 dx$$
$$\le C \int_{\mathbf{R}^n} |\Delta u(x)|^2 dx + C \int_{\mathbf{R}^n} V(x)^2 |u(x)|^2 dx,$$

where $u \in C_0^\infty(\mathbf{R}^n)$.

PROOF. By Corollary 1 and [Sh1, Corollary 2.8] we have

$$\int_{\mathbf{R}^n} m(x, V)^4 |u(x)|^2 dx + \int_{\mathbf{R}^n} m(x, V)^2 |\nabla u(x)|^2 dx$$
$$\le C \int_{\mathbf{R}^n} |(-\Delta + V)u(x)|^2 dx$$

$$\leq C \int_{\mathbf{R}^n} |\Delta u(x)|^2 dx + C \int_{\mathbf{R}^n} V(x)^2 |u(x)|^2 dx . \qquad \square$$

LEMMA 5 ([Zh, Lemma 5.5])(Caccioppoli type inequality). *Assume* $(-\Delta)^2 u + V^2 u = 0$ *in* $B_R(x_0)$. *Then there exists a constant C such that*

$$\begin{aligned} &\int_{B_{R/2}(x_0)} |\Delta u(x)|^2 dx + \int_{B_{R/2}(x_0)} V(x)^2 |u(x)|^2 dx \\ &\quad \leq \frac{C}{R^4} \int_{B_R(x_0)} |u(x)|^2 dx + \frac{C}{R^2} \int_{B_R(x_0)} |\nabla u(x)|^2 dx . \end{aligned} \tag{19}$$

LEMMA 6 ([Zh, Corollary 5.6]). *Assume* $(-\Delta)^2 u + V^2 u = 0$, $u \geq 0$, *in* $B_R(x_0)$. *Then*

$$\begin{aligned} |u(x_0)| \leq C\left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2} \\ + CR\left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla u(x)|^2 dx \right)^{1/2} . \end{aligned} \tag{20}$$

REMARK 7. From (20) we have for all $y \in B_{R/2}(x_0)$,

$$\begin{aligned} |u(y)| \leq C\left(\frac{1}{|B_{R/4}(y)|} \int_{B_{R/4}(y)} |u(x)|^2 dx \right)^{1/2} \\ + CR\left(\frac{1}{|B_{R/4}(y)|} \int_{B_{R/4}(y)} |\nabla u(x)|^2 dx \right)^{1/2} . \end{aligned} \tag{21}$$

Then we have

$$\begin{aligned} \sup_{y \in B_{R/2}(x_0)} |u(y)| \leq C\left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2} \\ + CR\left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |\nabla u(x)|^2 dx \right)^{1/2} . \end{aligned} \tag{22}$$

LEMMA 7. *Let* $j = 1, 2, 3$. *Suppose* $V \in (RH)_{q_0}$ *for some* $n/2 \leq q_0 < 2n/(4-j)$. *Assume also that* $(-\Delta)^2 u + V^2 u = 0$ *in* $B_R(x_0)$ *for some* $x_0 \in \mathbf{R}^n$. *Then there exists a constant C such that*

$$\left(\int_{B_{R/2}(x_0)} |\nabla^j u(x)|^t dx \right)^{1/t} \leq CR^{(2n/q_0)-4} \{1 + Rm(x_0, V)\}^4 \sup_{y \in B_R(x_0)} |u(y)| , \tag{23}$$

where $1/t = 2/q_0 - (4-j)/n$.

PROOF. We show Lemma 7 by a method similar to the one used in the proof of [Sh1, Lemma 4.6]. Let $\eta \in C_0^\infty(B_R(x_0))$ such that $\eta \equiv 1$ on $B_{3R/4}(x_0)$ and $|\nabla\eta| \le C/R$, $|\nabla^2\eta| \le C/R^2$, $|\nabla(\Delta\eta)| \le C/R^3$, and $|\Delta^2\eta| \le C/R^4$. We denote by $\Gamma_{H_2,0}(x,y)$ the fundamental solution for $(-\Delta)^2$. Note that

$$\begin{aligned} u(x)\eta(x) &= \int_{\mathbf{R}^n} \Gamma_{H_2,0}(x,y)(-\Delta)^2(u\eta)(y)dy \\ &= \int_{\mathbf{R}^n} \Gamma_{H_2,0}(x,y)\{-V(y)^2u(y)\eta(y) + 4\Delta(\nabla u(y)\cdot\nabla\eta(y)) \\ &\quad +2\Delta(u(y)\Delta\eta(y)) - 4\nabla^2u(y)\cdot\nabla^2\eta(y) - 4\nabla u(y)\cdot\nabla(\Delta\eta(y)) \\ &\quad -u(y)(\Delta^2\eta(y))\}dy\,, \end{aligned} \tag{24}$$

where $\nabla^2u(y)\cdot\nabla^2\eta(y) = \sum_{j,k=1}^n \partial^2u(y)/\partial y_j\partial y_k \cdot \partial^2\eta(y)/\partial y_j\partial y_k$. Then by integration by parts, for $x \in B_{R/2}(x_0)$ we have

$$\begin{aligned} |\nabla^j u(x)| &\le C\int_{B_R(x_0)} \frac{V(y)^2|u(y)||\eta(y)|}{|x-y|^{n-4+j}}dy + \frac{C}{R^{n+j}}\int_{B_R(x_0)} |u(y)|dy \\ &\le C\sup_{y\in B_R(x_0)}|u(y)|\cdot\int_{B_R(x_0)}\frac{V(y)^2|\eta(y)|}{|x-y|^{n-4+j}}dy + \frac{C}{R^{n+j}}\int_{B_R(x_0)}|u(y)|dy\,. \end{aligned}$$

It then follows from the well known theorem on fractional integrals that

$$\begin{aligned} &\left(\int_{B_{R/2}(x_0)} |\nabla^j u(x)|^t dx\right)^{1/t} \\ &\le C\sup_{y\in B_R(x_0)}|u(y)|\left(\int_{B_R(x_0)} V(x)^{q_0}dx\right)^{2/q_0} + CR^{(2n/q_0)-4}\sup_{y\in B_R(x_0)}|u(y)| \\ &\le CR^{(2n/q_0)-4}\{1+Rm(x_0,V)\}^4 \sup_{y\in B_R(x_0)}|u(y)|\,, \end{aligned}$$

where $1/t = 2/q_0 - (4-j)/n$ and we have used Remark 2. □

Since $n \ge 5$, we have

COROLLARY 2. *Let $j = 1, 2$. Suppose $V \in (RH)_{n/2}$ and $(-\Delta)^2u + V^2u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exists a constant C such that*

$$\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)} |\nabla^j u(x)|^2 dx\right)^{1/2} \le \frac{C\{1+Rm(x_0,V)\}^4}{R^j}\sup_{y\in B_R(x_0)}|u(y)|\,. \tag{25}$$

Now we are ready to give

PROOF OF LEMMA 3. Let $\eta \in C_0^\infty(B_{R/2}(x_0))$ such that $\eta \equiv 1$ on $B_{R/4}(x_0)$, $|\nabla\eta| \le C/R$, and $|\nabla^2\eta| \le C/R^2$. Applying Lemma 4 to $u\eta$ and using Lemma 5 we have

$$\int_{B_{R/4}(x_0)} m(x,V)^4|u(x)|^2dx + \int_{B_{R/4}(x_0)} m(x,V)^2|\nabla u(x)|^2dx$$
$$\le \frac{C}{R^4}\int_{B_R(x_0)} |u(x)|^2dx + \frac{C}{R^2}\int_{B_R(x_0)} |\nabla u(x)|^2dx\,.$$

By (13) it follows that

$$\int_{B_{R/4}(x_0)} |u(x)|^2dx$$
$$\le \frac{C\{1+Rm(x_0,V)\}^{4k_0/(k_0+1)}}{R^4m(x_0,V)^4}\left(\int_{B_R(x_0)} |u(x)|^2dx + R^2\int_{B_R(x_0)} |\nabla u(x)|^2dx\right)$$
$$\le \frac{C}{\{1+Rm(x_0,V)\}^{4/(k_0+1)}}\left(\int_{B_R(x_0)} |u(x)|^2dx + R^2\int_{B_R(x_0)} |\nabla u(x)|^2dx\right).$$

Then we have

$$\left(\frac{1}{|B_{R/4}(x_0)|}\int_{B_{R/4}(x_0)} |u(x)|^2dx\right)^{1/2} \le \frac{C}{\{1+Rm(x_0,V)\}^{2/(k_0+1)}}$$
$$\cdot\left\{\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |u(x)|^2dx\right)^{1/2} + R\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |\nabla u(x)|^2dx\right)^{1/2}\right\}.$$

Similarly

$$R\left(\frac{1}{|B_{R/4}(x_0)|}\int_{B_{R/4}(x_0)} |\nabla u(x)|^2dx\right)^{1/2} \le \frac{C}{\{1+Rm(x_0,V)\}^{1/(k_0+1)}}$$
$$\cdot\left\{\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |u(x)|^2dx\right)^{1/2} + R\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |\nabla u(x)|^2dx\right)^{1/2}\right\}.$$

By repeating above argument, for any $N > 0$ we have

$$\left(\frac{1}{|B_{R/4^N}(x_0)|}\int_{B_{R/4^N}(x_0)} |u(x)|^2dx\right)^{1/2} + R\left(\frac{1}{|B_{R/4^N}(x_0)|}\int_{B_{R/4^N}(x_0)} |\nabla u(x)|^2dx\right)^{1/2}$$
$$\le \frac{C_N}{\{1+Rm(x_0,V)\}^{N/(k_0+1)}}\left\{\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |u(x)|^2dx\right)^{1/2}\right.$$
$$\left.+R\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} |\nabla u(x)|^2dx\right)^{1/2}\right\}. \tag{26}$$

Then using Estimates (22), (25), and (26) we arrive at the desired estimate. □

4. Proof of Theorem 3

In this section we prove Theorem 3 which states the first, second, and third order derivative estimates of the fundamental solution for H_2. We arrive at Theorem 3 combining the following Lemma 8 with Lemma 3.

LEMMA 8. *Let $j = 1, 2, 3$. Suppose $V \in (RH)_{n/2}$ and there exists a constant C such that $V(x) \le Cm(x, V)^2$. Assume also that $(-\Delta)^2 u + V^2 u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Then there exist constants C_j and C'_j such that*

$$\sup_{y \in B_{R/2}(x_0)} |\nabla^j u(y)| \le \frac{C_j \{1 + Rm(x_0, V)\}^{C'_j}}{R^j} \sup_{y \in B_R(x_0)} |u(y)| . \tag{27}$$

PROOF. Let $\eta \in C_0^\infty(B_{R/2}(x_0))$ such that $\eta \equiv 1$ on $B_{R/4}(x_0)$ and $|\nabla \eta| \le C/R$, $|\nabla^2 \eta| \le C/R^2$, $|\nabla(\Delta \eta)| \le C/R^3$, and $|\Delta^2 \eta| \le C/R^4$. From (24) and (12) we have

$$\begin{aligned}
|\nabla^j u(x_0)| &\le C \int_{B_R(x_0)} \frac{V(y)^2 |u(y)|}{|x_0 - y|^{n-4+j}} dy + \frac{C}{R^{n+j}} \int_{B_R(x_0)} |u(y)| dy \\
&\le C\{1 + Rm(x_0, V)\}^{4k_0} m(x_0, V)^4 R^{4-j} \sup_{y \in B_R(x_0)} |u(y)| \\
&\quad + \frac{C}{R^j} \left(\frac{1}{R^n} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2} \\
&\le \frac{C\{1 + Rm(x_0, V)\}^{4(k_0+1)}}{R^j} \sup_{y \in B_R(x_0)} |u(y)| ,
\end{aligned} \tag{28}$$

From (28) we have for all $y \in B_{R/2}(x_0)$,

$$|\nabla^j u(y)| \le \frac{C\{1 + Rm(y, V)\}^{4(k_0+1)}}{R^j} \sup_{x \in B_{R/4}(y)} |u(x)| .$$

Using (12) we have

$$\sup_{y \in B_{R/2}(x_0)} |\nabla^j u(y)| \le \frac{C\{1 + Rm(x_0, V)\}^{4(k_0+1)^2}}{R^j} \sup_{y \in B_R(x_0)} |u(y)| .$$

Then the proof is complete. □

As we mentioned in Section 1, we can prove derivative estimates of the fundamental solution under another assumption as the following theorem states.

THEOREM 6. *Let $j = 1, 2, 3$, and suppose $V \in (RH)_{2n/(4-j)}$. Then for any positive integer N there exists a constant C_N such that*

$$|\nabla_x^j \Gamma_{H_2}(x, y)| \le \frac{C_N}{\{1 + m(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-4+j}} . \tag{29}$$

We arrive at Theorem 6 combining the following Lemma 9 with Lemma 3.

LEMMA 9. *Let* $j = 1, 2, 3,$ *and suppose* $V \in (RH)_{2n/(4-j)}$. *Assume also that* $(-\Delta)^2 u + V^2 u = 0$ *in* $B_R(x_0)$ *for some* $x_0 \in \mathbf{R}^n$. *Then there exist constants* C_j *and* C'_j *such that*

$$\sup_{y \in B_{R/2}(x_0)} |\nabla^j u(y)| \le \frac{C_j \{1 + Rm(x_0, V)\}^{C'_j}}{R^j} \sup_{y \in B_R(x_0)} |u(y)| . \tag{30}$$

PROOF. As in the proof of Lemma 7, we have

$$|\nabla^j u(x_0)| \le C \int_{B_R(x_0)} \frac{V(y)^2 |u(y)|}{|x_0 - y|^{n-4+j}} dy + \frac{C}{R^{n+j}} \int_{B_R(x_0)} |u(y)| dy .$$

Since $V \in (RH)_{2n/(4-j)}$, it follows that $V \in (RH)_q$ for some $q > 2n/(4-j)$. We choose r such that $2/q + 1/r = 1$ and $r > 1$. By Hölder's inequality, it follows that

$$\begin{aligned} |\nabla^j u(x_0)| &\le C R^n \left(\frac{1}{R^n} \int_{B_R(x_0)} V(y)^q dy \right)^{2/q} \left(\frac{1}{R^n} \int_{B_R(x_0)} \frac{dy}{|x_0 - y|^{(n-4+j)r}} \right)^{1/r} \\ &\quad \cdot \sup_{y \in B_R(x_0)} |u(y)| + \frac{C}{R^{n+j}} \int_{B_R(x_0)} |u(y)| dy \\ &\le \frac{C\{1 + Rm(x_0, V)\}^4}{R^j} \sup_{y \in B_R(x_0)} |u(y)| , \end{aligned} \tag{31}$$

where we have used Remark 2. Then as in the proof of Lemma 8, we arrive at the desired estimate. □

5. Proof of Theorem 1

Theorem 1(1) immediately follows from the following Lemma 10.

LEMMA 10. (1) *Suppose* $V \in (RH)_{n/2}$. *Then there exists a constant* C *such that*

$$|m(x, V)^4 H_2^{-1} f(x)| \le C M(|f|)(x) \quad \text{for} \quad f \in C_0^\infty(\mathbf{R}^n) , \tag{32}$$

where M *is the Hardy-Littlewood maximal operator.*

(2) *Let* $j = 1, 2, 3$. *Suppose* $V \in (RH)_{n/2}$ *and there exists a constant* C *such that* $V(x) \le Cm(x, V)^2$. *Then there exists a constant* C' *such that*

$$|m(x, V)^{4-j} \nabla^j H_2^{-1} f(x)| \le C' M(|f|)(x) \quad \text{for} \quad f \in C_0^\infty(\mathbf{R}^n) , \tag{33}$$

where M *is the Hardy-Littlewood maximal operator.*

PROOF OF LEMMA 10. Estimate (32) can be proved as follows. Let $r = 1/m(x, V)$. Then it follows from Theorem 2 that

$$
\begin{aligned}
|m(x,V)^4 H_2^{-1} f(x)| &\leq C_N \int_{\mathbf{R}^n} \frac{m(x,V)^4 |f(y)|}{\{1 + m(x,V)|x-y|\}^N |x-y|^{n-4}} dy \\
&\leq C_N \sum_{j=-\infty}^{+\infty} \int_{2^{j-1}r < |x-y| \leq 2^j r} \frac{|f(y)|}{r^4 (1 + r^{-1}|x-y|)^N |x-y|^{n-4}} dy \\
&\leq C_N \sum_{j=-\infty}^{+\infty} \frac{2^{4(j-1)+n}}{(1 + 2^{j-1})^N} \cdot \frac{1}{(2^j r)^n} \int_{|x-y| \leq 2^j r} |f(y)| dy \\
&\leq C C_N \sum_{j=-\infty}^{+\infty} \frac{2^{4j}}{(1+2^j)^N} M(|f|)(x) \,.
\end{aligned}
$$

Therefore we obtain the desired estimate, if we take $N = 5$ for example.

The proof of (33) can be done in the same way as above by using Theorem 3. □

PROOF OF THEOREM 1(1). Since $V(x) \leq Cm(x, V)^2$, Estimate (3) immediately follows from (32), (33), and the fact that the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbf{R}^n)$, $1 < p \leq \infty$. □

PROOF OF THEOREM 1(2). Since $\nabla^4 (\Delta^2)^{-1}$ is bounded on L^p, $1 < p < \infty$, we obtain

$$
\begin{aligned}
\|\nabla^4 H_2^{-1} f\|_{L^p(\mathbf{R}^n)} &\leq C \|(\Delta^2 - V^2 + V^2) H_2^{-1} f\|_{L^p(\mathbf{R}^n)} \\
&\leq C \|f\|_{L^p(\mathbf{R}^n)} \,.
\end{aligned}
$$
□

6. Proof of Theorem 5

In this section we prove Theorem 5. First we show some lemmas needed to prove it.

LEMMA 11 (Caccioppoli type inequality). *Assume* $(-\Delta)^2 u + V^2 u = f$ *in* $B_R(x_0)$. *Then there exists a constant* C *such that*

$$
\begin{aligned}
&\int_{B_{R/2}(x_0)} |\nabla(\Delta u(x))|^2 dx + \int_{B_{R/2}(x_0)} V(x)^2 |u(x)||\Delta u(x)| dx \\
&\leq \int_{B_R(x_0)} |f(x)||\Delta u(x)| dx + \frac{C}{R^2} \int_{B_R(x_0)} |\Delta u(x)|^2 dx \,.
\end{aligned}
$$

LEMMA 12 (cf. [Sh2, Lemma 1.3]). *Assume* $(-\Delta)^2 u + V^2 u = f$ *in* $B_R(x_0)$. *Then there exists a constant* C *such that*

$$
\left(\frac{1}{|B_{R/16}(x_0)|} \int_{B_{R/16}(x_0)} |u(x)|^q dx \right)^{1/q}
$$

$$\leq C\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2dx\right)^{1/2}+CR\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|\nabla u(x)|^2dx\right)^{1/2}$$
$$+CR^2\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|\nabla^2 u(x)|^2dx\right)^{1/2}+CR^4\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|f(x)|^pdx\right)^{1/p}, \tag{34}$$

where $2\leq p\leq q\leq\infty$ *and* $1/q>1/p-4/n$.

PROOF. Let $\eta\in C_0^\infty(B_{R/8}(x_0))$ such that $\eta\equiv 1$ on $B_{R/12}(x_0)$ and $|\nabla\eta|\leq C/R$, $|\nabla^2\eta|\leq C/R^2$, $|\nabla(\Delta\eta)|\leq C/R^3$, and $|\Delta^2\eta|\leq C/R^4$. Note that

$$\begin{aligned}\{(-\Delta)^2+V^2\}(u\eta)&=\eta\{(-\Delta)^2+V^2\}u+4\nabla(\Delta u)\cdot\nabla\eta+2(\Delta u)(\Delta\eta)\\&\quad+4\nabla^2u\cdot\nabla^2\eta+4\nabla u\cdot\nabla(\Delta\eta)+u(\Delta^2\eta)\,.\end{aligned}$$

It follows that

$$\begin{aligned}&|u(x)\eta(x)|\\&\leq C\int_{\mathbf{R}^n}\frac{1}{|x-y|^{n-4}}\{|f(y)\eta(y)|+|\nabla(\Delta u(y))||\nabla\eta(y)|+|\Delta u(y)||\Delta\eta(y)|\\&+|\nabla^2u(y)||\nabla^2\eta(y)|+|\nabla u(y)||\nabla(\Delta\eta(y))|+|u(y)||\Delta^2\eta(y)|\}dy\,.\end{aligned}$$

Thus, for $x\in B_{R/16}(x_0)$,

$$\begin{aligned}|u(x)|&\leq C\int_{B_{R/4}(x_0)}\frac{|f(y)|}{|x-y|^{n-4}}dy+\frac{C}{R^{n-3}}\int_{B_{R/4}(x_0)}|\nabla(\Delta u(y))|dy\\&+\frac{C}{R^{n-2}}\int_{B_{R/4}(x_0)}|\Delta u(y)|dy+\frac{C}{R^{n-2}}\int_{B_{R/4}(x_0)}|\nabla^2u(y)|dy\\&+\frac{C}{R^{n-1}}\int_{B_{R/4}(x_0)}|\nabla u(y)|dy+\frac{C}{R^{n}}\int_{B_{R/4}(x_0)}|u(y)|dy\\&\leq C\int_{B_{R/4}(x_0)}\frac{|f(y)|}{|x-y|^{n-4}}dy+R^4\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|f(y)|^2dy\right)^{1/2}\\&+C\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(y)|^2dy\right)^{1/2}+CR\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|\nabla u(y)|^2dy\right)^{1/2}\\&+CR^2\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|\nabla^2u(y)|^2dy\right)^{1/2},\end{aligned}$$

where we have used Lemmas 5 and 11 in the second inequality. Now by Young's inequality, if $2\leq p\leq q\leq\infty$, $1/q=1/r+1/p-1$, and $(n-4)r<n$,

$$\begin{aligned}&\|u\|_{L^q(B_{R/16}(x_0))}\\&\leq CR^{-n(1-(1/r))+4}\|f\|_{L^p(B_R(x_0))}+CR^{n/q+4}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|f(y)|^2dy\right)^{1/2}\end{aligned}$$

$$+ CR^{n/q}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(y)|^2dy\right)^{1/2}$$
$$+ CR^{n/q+1}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|\nabla u(y)|^2dy\right)^{1/2}$$
$$+ CR^{n/q+2}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|\nabla^2 u(y)|^2dy\right)^{1/2}.$$

The lemma then follows since $p \geq 2$. □

LEMMA 13. *Let j be a positive integer and $j \geq 2$. Suppose $V \in C^{j-2}(\mathbf{R}^n)$ and $(-\Delta)^2u + V^2u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Assume also that $V \in (RH)_{n/2}$ and there exists a constant C such that $|\nabla^i V(x)| \leq Cm(x, V)^{2+i}$, where $i = 1, 2, 3, \ldots, j-2$.Then there exist constants C_j and C'_j such that*

$$\left(\frac{1}{|B_{R/2^j}(x_0)|}\int_{B_{R/2^j}(x_0)}|\nabla^j u(x)|^2dx\right)^{1/2} \leq \frac{C_j\{1+Rm(x_0,V)\}^{C'_j}}{R^j}\sup_{y\in B_R(x_0)}|u(y)|. \quad (35)$$

PROOF. We prove (35) by induction on j. If $j = 2, 3$, Estimate (35) holds under weaker assumption than that of Lemma 13 (see Corollary 2 and Lemma 8). For $j \geq 4$ we assume it is true for $2, 3, 4, \ldots, j-1$, and show the case j. Note that

$$\{(-\Delta)^2 + V^2\}\nabla^{j-2}u = \nabla^{j-2}\{(-\Delta)^2u\} + V^2\nabla^{j-2}u$$
$$= \nabla^{j-2}(-V^2u) + V^2\nabla^{j-2}u = \sum_{k=1}^{j-2}\sum_{l=0}^{k}c(l,k)(\nabla^l V)(\nabla^{k-l}V)\nabla^{j-2-k}u\,, \quad (36)$$

where $c(l, k)$ is a constant depending on l and k. Let $\eta \in C_0^\infty(B_{R/2^{j-1}}(x_0))$ such that $\eta \equiv 1$ on $B_{R/2^j}(x_0)$ and $|\nabla\eta| \leq C/R$, $|\nabla^2\eta| \leq C/R^2$. Multiplying the equation (36) by $\eta^4\nabla^{j-2}u$ and integrating over $\mathbf{R}^n$ by integration by parts, we have

$$\int_{\mathbf{R}^n}\sum_{s,t=1}^{n}(\partial_t\partial_s^2\nabla^{j-2}u(x))\partial_t(\eta^4\nabla^{j-2}u(x))dx$$
$$\leq C\int_{\mathbf{R}^n}\sum_{k=1}^{j-2}m(x,V)^{4+k}\nabla^{j-2-k}u(x)\eta(x)^4\nabla^{j-2}u(x)dx\,, \quad (37)$$

where $\partial_t = \partial/\partial x_t$, $\partial_s^2 = \partial^2/\partial x_s^2$, $1 \leq t \leq n$, $1 \leq s \leq n$. The left hand side of (37) is equal to

$$\int_{\mathbf{R}^n}\sum_{s,t=1}^{n}(\partial_t\partial_s\nabla^{j-2}u(x))^2\eta(x)^4 + (\partial_t\partial_s\nabla^{j-2}u(x))\{4\eta(x)^3\partial_t\eta(x)(\partial_s\nabla^{j-2}u(x))$$
$$+4\eta(x)^3(\partial_s\eta(x))(\partial_t\nabla^{j-2}u(x)) + 12\eta(x)^2(\partial_s\eta(x))(\partial_t\eta(x))\nabla^{j-2}u(x)$$

$$+4\eta(x)^3(\partial_s\partial_t\eta(x))(\nabla^{j-2}u(x))\}dx\,.$$

Then we have

$$\begin{aligned}
&\int_{\mathbf{R}^n}|\nabla^j u(x)|^2\eta(x)^4dx\\
&\le C\int_{\mathbf{R}^n}\{|\nabla\eta(x)|^2|\nabla^{j-1}u(x)|^2+\eta(x)^2(|\nabla\eta(x)|^2+|\nabla^2\eta(x)|)^2|\nabla^{j-2}u(x)|^2\\
&\quad+\sum_{k=1}^{j-2}m(x,V)^{4+k}|\nabla^{j-2-k}u(x)|\eta(x)^4|\nabla^{j-2}u(x)|\}dx\,.
\end{aligned}$$

By (12) we obtain

$$\begin{aligned}
&\int_{B_{R/2^j}(x_0)}|\nabla^j u(x)|^2dx\\
&\le \frac{C}{R^2}\int_{B_{R/2^{j-1}}(x_0)}|\nabla^{j-1}u(x)|^2dx+\frac{C}{R^4}\int_{B_{R/2^{j-1}}(x_0)}|\nabla^{j-2}u(x)|^2dx\\
&\quad+C\sum_{k=1}^{j-2}\{1+Rm(x_0,V)\}^{(4+k)k_0}m(x_0,V)^{4+k}\int_{B_{R/2^{j-1}}(x_0)}|\nabla^{j-2-k}u(x)||\nabla^{j-2}u(x)|dx\\
&\le \frac{C}{R^2}\cdot\frac{C_{j-1}\{1+Rm(x_0,V)\}^{C'_{j-1}}}{R^{2(j-1)}}\cdot R^n\left(\sup_{y\in B_R(x_0)}|u(y)|\right)^2\\
&\quad+\frac{C}{R^4}\cdot\frac{C_{j-2}\{1+Rm(x_0,V)\}^{C'_{j-2}}}{R^{2(j-2)}}\cdot R^n\left(\sup_{y\in B_R(x_0)}|u(y)|\right)^2\\
&\quad+C\sum_{k=1}^{j-2}\{1+Rm(x_0,V)\}^{(4+k)k_0}m(x_0,V)^{4+k}R^k\\
&\quad\cdot\left(\int_{B_{R/2^{j-1}}(x_0)}|\nabla^{j-2}u(x)|\cdot\frac{1}{R^k}|\nabla^{j-2-k}u(x)|dx\right)\\
&\le \frac{C\{1+Rm(x_0,V)\}^{C'_{j-1}}}{R^{2j}}\cdot R^n\left(\sup_{y\in B_R(x_0)}|u(y)|\right)^2+C\sum_{k=1}^{j-2}\{1+Rm(x_0,V)\}^{(4+k)k_0}\\
&\quad\cdot\{Rm(x_0,V)\}^{4+k}\frac{1}{R^4}\left(\int_{B_{R/2^{j-1}}(x_0)}|\nabla^{j-2}u(x)|^2dx\right.\\
&\quad\left.+\frac{1}{R^{2k}}\int_{B_{R/2^{j-1}}(x_0)}|\nabla^{j-2-k}u(x)|^2dx\right)\\
&\le \frac{C_j\{1+Rm(x_0,V)\}^{C'_j}}{R^{2j}}\cdot R^n\left(\sup_{y\in B_R(x_0)}|u(y)|\right)^2,
\end{aligned}$$

where $C'_j = C'_{j-2} + (j+2)(k_0+1)$. □

Theorem 5 immediately follows from the following Lemma 14 and Lemma 3.

LEMMA 14. *Let j be a positive integer. Suppose $V \in C^j(\mathbf{R}^n)$ and $(-\Delta)^2 u + V^2 u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$. Assume also that $V \in (RH)_{n/2}$ and there exists a constant C such that $|\nabla^i V(x)| \le Cm(x,V)^{2+i}$, where $i = 1, 2, 3, \ldots, j$. Then there exist constants C_j and C''_j such that*

$$\sup_{y \in B_{R/2}(x_0)} |\nabla^j u(y)| \le \frac{C_j \{1 + Rm(x_0, V)\}^{C''_j}}{R^j} \sup_{y \in B_R(x_0)} |u(y)| . \tag{38}$$

PROOF. We prove (38) by induction on j. If $j = 1, 2, 3$, Estimate (38) holds under weaker assumption than that of Lemma 14 (see Lemma 8). For $j \ge 4$, we assume it is true for $1, 2, 3, \ldots, j-1$, and show the case j. Note that

$$\{(-\Delta)^2 + V^2\}\nabla^j u = \sum_{k=1}^{j} \sum_{l=0}^{k} c(l,k)(\nabla^l V)(\nabla^{k-l} V)\nabla^{j-k} u . \tag{39}$$

Let $p \ge 2$ and $p > n/4$. Then it follows from (39) and Lemma 12 that

$$\begin{aligned}
&|\nabla^j u(x_0)| \\
&\le C\left(\frac{1}{|B_{R/2^j}(x_0)|}\int_{B_{R/2^j}(x_0)} |\nabla^j u(x)|^2 dx\right)^{1/2} \\
&\quad + CR\left(\frac{1}{|B_{R/2^j}(x_0)|}\int_{B_{R/2^j}(x_0)} |\nabla^{j+1} u(x)|^2 dx\right)^{1/2} \\
&\quad + CR^2\left(\frac{1}{|B_{R/2^j}(x_0)|}\int_{B_{R/2^j}(x_0)} |\nabla^{j+2} u(x)|^2 dx\right)^{1/2} \\
&\quad + CR^4\left\{\frac{1}{|B_{R/2^j}(x_0)|}\int_{B_{R/2^j}(x_0)} \left(\sum_{k=1}^{j}\sum_{l=1}^{k} |\nabla^l V(x)||\nabla^{k-l} V(x)||\nabla^{j-k} u(x)|\right)^p dx\right\}^{1/p} \\
&\le \frac{CC_{j+2}\{1 + Rm(x_0,V)\}^{C'_{j+2}}}{R^j} \sup_{y \in B_R(x_0)} |u(y)| + CR^4 \sum_{k=1}^{j} \{1 + Rm(x_0,V)\}^{(4+k)k_0} \\
&\quad \cdot m(x_0,V)^{4+k}\left(\frac{1}{|B_{R/2^j}(x_0)|}\int_{B_{R/2^j}(x_0)} |\nabla^{j-k} u(x)|^p dx\right)^{1/p} \\
&\le \frac{C\{1 + Rm(x_0,V)\}^{C'_{j+2}}}{R^j} \sup_{y \in B_R(x_0)} |u(y)| + C \sum_{k=1}^{j} \{1 + Rm(x_0,V)\}^{(4+k)k_0}
\end{aligned}$$

$$\cdot R^{4+k} m(x_0, V)^{4+k} \cdot \frac{1}{R^k} \cdot \frac{C_{j-k}\{1 + Rm(x_0, V)\}^{C'_{j-k}}}{R^{j-k}} \sup_{y \in B_R(x_0)} |u(y)|$$

$$\leq \frac{C_j\{1 + Rm(x_0, V)\}^{C''_j}}{R^j} \sup_{y \in B_R(x_0)} |u(y)| ,$$

where we have used (12) and Lemma 13 in the second inequality and the assumption of induction in the third. □

7. Appendix

In this section we show the L^p boundedness of the operator $V^{2k} H_2^{-k}$, $k \in \mathbf{N}$. Let $f \in C_0^\infty(\mathbf{R}^n)$ and assume that $V \in (RH)_{n/2}$ and there exists a constant C such that $V(x) \leq Cm(x, V)^2$. Then for any integer $k \geq 2$, we define H_2^{-k} as follows.

$$H_2^{-k} f(x) = \int_{\mathbf{R}^n} \Gamma_{H_2}(x, y) H_2^{-(k-1)} f(y) dy .$$

THEOREM 7. *Suppose $V \in (RH)_{n/2}$ and there exists a constant C such that $V(x) \leq Cm(x, V)^2$. Then there exists a constant C' such that*

$$\|V^{2k} H_2^{-k} f\|_{L^p(\mathbf{R}^n)} \leq C' \|f\|_{L^p(\mathbf{R}^n)} , \tag{40}$$

where $1 < p \leq \infty$ and $k \in \mathbf{N}$.

Theorem 7 is easily proved by the following pointwise estimate.

LEMMA 15. *Let k be a positeve integer. The opeator M^k stands for the k times composition of the Hardy-Littlewood maximal operator M. Suppose $V \in (RH)_{n/2}$. Then there exists a constant C such that*

$$|m(x, V)^{4k} H_2^{-k} f(x)| \leq CM^k(|f|)(x) \quad \textit{for} \quad f \in C_0^\infty(\mathbf{R}^n) . \tag{41}$$

PROOF OF LEMMA 15. Let $f \in C_0^\infty(\mathbf{R}^n)$. We prove Estimate (41) by induction on k. For $k \geq 2$, we assume it is true for $k-1$ and show the case k. It follows from Theorem 2 and (13) that

$$|m(x, V)^{4k} H_2^{-k} f(x)|$$
$$\leq \left| Cm(x, V)^4 \int_{\mathbf{R}^n} \Gamma_{H_2}(x, y) m(x, V)^{4(k-1)} H_2^{-(k-1)} f(y) dy \right|$$
$$\leq CC_N m(x, V)^4 \int_{\mathbf{R}^n} \frac{\{1 + m(x, V)|x-y|\}^{4(k-1)k_0/(k_0+1)} |m(y, V)^{4(k-1)} H_2^{-(k-1)} f(y)|}{\{1 + m(x, V)|x-y|\}^N |x-y|^{n-4}} dy .$$

Therefore we obtain the desired estimate in the same way as the case $k = 1$. □

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