

## Some Remarks on the Generalized St. Petersburg Games and Formal Laurent Series Expansions

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**Abstract.** We discuss the asymptotic behavior of the largest amount of winnings in the generalized St. Petersburg games and show that the results on the random variables associated with the amount of winnings can be applied to the random variables associated with the degree of the polynomial digits of the continued fraction and the Oppenheim expansions of formal Laurent series.

### 1. Introduction

The Generalized St. Petersburg game is described as follows: consider tossing a coin, which needs not be fair; suppose that ‘heads’ occur with probability  $p$ ,  $0 < p < 1$  so that the probability that ‘tails’ occur is  $q_1 = 1 - p$ , see [3]. For a single game, a coin is repeatedly tossed until a head appears. If the head occurs for the first time on the  $k$ -th toss, the player wins  $q_1^{-k}$  dollars. Let  $X_n$  be the amount of winning in the  $n$ -th game. Then the winnings  $\{X_n\}_{n \geq 1}$  become a sequence of independent and identically distributed random variables with

$$\mathbf{P}\{X_n = q_1^{-k}\} = pq_1^{k-1}, \quad k \geq 1.$$

In this paper, we are interested in the asymptotic behavior of the sequence of random variables  $\{X_n\}$ . Suppose that the player pays the entrance fee  $m_n$  for the opportunity to play in the  $n$ -th game. For the first  $n$  games, the sum  $S_n = \sum_{i=1}^n X_i$  represents the total amount of winning and  $M_n = \sum_{i=1}^n m_i$  represents the total or accumulated entrance fees,  $n \geq 1$ . In order that the game will be fair, it is necessary that  $S_n \sim M_n$ . For several results concerning  $S_n$  and  $M_n$ , we refer to [2], [3], [6] and [12]. The main result of this paper concerns with the largest value of  $X_i$ —which corresponds to the largest win of the player.

We note that the classical St. Petersburg game is obtained when  $p = 1/2$ . In other words, if  $X$  represents the amount of winning in a single classical St. Petersburg game,  $\mathbf{P}(X = 2^k) = 2^{-k}$ , for each integer  $k \geq 1$ . The expectation of  $X$  is infinite and since there is no ‘fair’ entry fee exists, this was considered a paradox—a self contradictory statement. However, if  $n$  independent games are considered, Feller showed that the weak law of large numbers holds:  $S_n/(n \log_2 n)$  converges to 1 in probability. In other words,  $n \log_2 n$  seems

to be a fair entry fee for the first  $n$  games. In [19], Vardi emphasized that the partial sums of the continued fraction digits of a random real number also satisfy the same law, thus one might consider the  $n$ -th term of continued fraction digit of a real number as the entry fees for  $n$  games in the classical St. Petersburg game. Furthermore, Vardi showed that known results for continued fraction can be obtained for the classical St. Petersburg game by using the same proofs. His results focused on how the player is favored even with a fair entry fee. More precisely, he showed that for the first  $n$  games,  $n \log_2 n$  should be the accumulated fair entry fee even if the largest amount of winnings is neglected.

Our main result (Theorem 1) is the following: Let  $\{X_n\}$  represent the amount of winnings in the generalized St. Petersburg game. Then

$$\liminf_{N \rightarrow \infty} \frac{\max_{1 \leq i \leq N} X_i}{N / \log \log N} = 1, \quad \text{a.s.}$$

This explains how the player is favored in the generalized St. Petersburg game even with a fair entry fee. We formally state Theorem 1 in Section 2 together with its proof. It is based on the proof of Vardi [19] for the classical St. Petersburg game and the proof of Philipp [18] for the continued fraction expansion of real numbers. The first and second Borel-Cantelli lemmas play a very significant role in the proof. We note that in the first part of the proof, we don't use the result of Barndorff-Nielsen [5] on the generalization of the convergence part of Borel-Cantelli lemma because it is hard to find a sequence of real numbers  $\{\alpha_n\}$  satisfying  $\mathbf{P}(\{X_n \leq \alpha_n\})^n$  in our case. Furthermore, we can not apply the similar theorem of Nakada-Natsui [16] since there exists no constant  $H > 0$  such that  $\mathbf{P}\{X_n \geq j\} = H/j + o(1/j)$  as  $j \rightarrow \infty$ . However, the conditions of the results on the theory of trimmed sums of Aaronson-Nakada [1] are satisfied by  $\{X_n\}$  as a special case, in which the result for the generalized St. Petersburg game is stated in Remark 1 (iii). Other known results for this game are also mentioned in Remark 1 at the end of Section 2. We apply these results to continued fraction and Oppenheim expansions for formal Laurent series over a finite base field in Section 3. It begins with the basic definition and concepts of the formal Laurent series, then the application of the results to the continued fraction and Oppenheim expansions of Laurent series. We refer to [5] and [8] for the general theory of these expansions.

## 2. The Theorem and Its Proof

Take  $q = 1/q_1$ , for  $0 < q_1 < 1$ . We consider an independent and identically distributed sequence of random variables  $\{X_n, n \geq 1\}$  on the probability space  $(\Omega, \mathcal{B}, \mathbf{P})$  with

$$\mathbf{P}\{\omega \in \Omega : X_n(\omega) = q^k\} = \frac{q-1}{q^k}, \quad k \geq 1,$$

for any  $n \geq 1$ .

THEOREM 1. *We have*

$$\liminf_{N \rightarrow \infty} \frac{\max_{1 \leq i \leq N} X_i}{N / \log \log N} = 1, \quad a.s.$$

First note that since

$$|\log(1-x) - (-x)| \leq x^2 \quad \text{if } |x| \leq \frac{1}{2},$$

which follows easily from the Taylor expansion of  $\log x$ , we have

$$(1-z)^k = \exp\{k \log(1-z)\} \geq \exp\{-kz - kz^2\} \quad \text{if } |z| \leq \frac{1}{2}. \quad (1)$$

Clearly, we also have

$$(1-z)^k \leq \exp(-kz) \quad \text{if } z \geq 0.$$

PROOF. For each  $k$ , let  $j_k$  be the positive integer for which

$$q^{j_k} \leq k^{2k} < q^{j_k+1}$$

and let  $d(k)$  be the real number satisfying

$$\frac{d(k)}{\log \log d(k)} = q^{j_k}.$$

Then  $d(k)$  is strictly increasing in  $k$ , and furthermore both

$$\frac{1}{q} k^{2k} < d(k) < k^{4k} \quad (2)$$

and

$$\frac{d(k-1)}{d(k)} < \frac{q}{k^2} \quad (3)$$

hold for all sufficiently large  $k$ .

Let

$$B_k = \left\{ \max_{d(k-1) \leq n < d(k)} X_n \leq \frac{d(k)}{\log \log d(k)} \right\}.$$

Then  $\{B_k\}$  is an independent sequence of events. Since for each  $k$ ,  $B_k \supset C_k$ , where

$$C_k = \left\{ \max_{1 \leq n < [d(k)]+1} X_n \leq \frac{d(k)}{\log \log d(k)} \right\},$$

we have

$$\mathbf{P}(B_k) \geq \mathbf{P}(C_k) \geq \left(1 - \frac{\log \log d(k)}{d(k)}\right)^{d(k)}$$

$$\begin{aligned} &\geq \exp \left[ -d(k) \cdot \left\{ \frac{\log \log d(k)}{d(k)} + \left( \frac{\log \log d(k)}{d(k)} \right)^2 \right\} \right] \text{ (by(1))} \\ &= \alpha_k, \end{aligned}$$

where  $\alpha_k = (\log d(k))^{-1} \cdot \exp \left[ -\frac{\{\log \log d(k)\}^2}{d(k)} \right]$ .

Since  $\lim_{k \rightarrow \infty} \exp \left[ -\frac{\{\log \log d(k)\}^2}{d(k)} \right] = 1$ , convergence and divergence of the infinite series  $\sum_k \alpha_k$  is the same as that of the series  $\sum_k (\log d(k))^{-1}$ , but from (2) it follows that

$$\sum_k (\log d(k))^{-1} \geq \sum_k (4k \log k)^{-1} = \infty.$$

Thus, we obtain that  $\sum_k \mathbf{P}(B_k) = \infty$ , and therefore, we conclude from the Borel-Cantelli Lemma that  $\mathbf{P}(B_k \text{ occurs for infinitely many } k) = 1$ . On the other hand, if we define

$$F_k := \left\{ \max_{1 \leq n \leq d(k-1)} X_n > \frac{d(k)}{\log \log d(k)} \right\}.$$

then

$$\begin{aligned} \sum_k \mathbf{P}(F_k) &\leq \sum_k d(k-1) \mathbf{P} \left( X > \frac{d(k)}{\log \log d(k)} \right) \\ &= \sum_k d(k-1) \frac{\log \log d(k)}{d(k)} \\ &\leq q \sum_k \log \{4k \log k\} \frac{1}{k^2} < \infty \quad \text{by (2) and (3)}. \end{aligned}$$

Therefore, by the convergence part of the Borel-Cantelli Lemma, the event  $F_k$  occurs for at most finitely many  $k$  with probability 1. Putting together with the result we established above, we conclude that with probability 1, the event  $B_k - F_k = \left\{ \max_{1 \leq n \leq d(k)} X_n \leq \frac{d(k)}{\log \log d(k)} \right\}$  occurs for finitely many  $k$ , from which it follows that

$$\mathbf{P} \left( \liminf_{N \rightarrow \infty} \frac{\max_{1 \leq i \leq N} X_i(\omega)}{N / \log \log N} \leq 1 \right) = 1.$$

On the other hand, to show the reverse inequality, choose a real number  $r > 1$ . Put

$$G_k := \left\{ X_n(\omega) \leq \frac{r^{k+1}}{r^2 \log \log r^{k+1}}, \quad 1 \leq n \leq [r^k] \right\}.$$

Then for  $t(k)$  the greatest integer satisfying  $q^{t(k)} \leq \frac{r^{k+1}}{r^2 \log \log r^{k+1}}$ , we have

$$\mathbf{P}(G_k) = \left( 1 - \frac{1}{q^{t(k)}} \right)^{[r^k]} < \left( 1 - \frac{1}{q^{t(k)}} \right)^{r^k - 1}$$

$$< e \cdot \exp\{-r \log \log r^{k+1}\} < \frac{e}{(\log r)^r} \cdot \frac{1}{(k+1)^r}.$$

Since we choose  $r > 1$ ,  $\sum_{k=1}^{\infty} k^{-r} < \infty$ , again by the convergence part of the Borel-Cantelli Lemma, the events  $\{G_k\}$  occur for at most finitely many  $k$ . Hence, for almost every  $\omega \in \Omega$ , there exists integer  $k_0 = k_0(\omega)$  such that for all  $k \geq k_0$

$$\max_{1 \leq i \leq [r^k]} X_i > \frac{r^{k+1}}{r^2 \log \log r^{k+1}}.$$

For  $r^k \leq N < r^{k+1}$  and  $k \geq k_0$ , since

$$\max_{1 \leq i \leq N} X_i \geq \max_{1 \leq i \leq [r^k]} X_i \quad \text{and} \quad \frac{r^{k+1}}{r^2 \log \log r^{k+1}} > \frac{N}{r^2 \log \log N},$$

consequently, we have

$$\max_{1 \leq i \leq N} X_i > \frac{N}{r^2 \log \log N}.$$

By letting  $r \downarrow 1$ , we obtain

$$\liminf_{N \rightarrow \infty} \frac{\max_{1 \leq i \leq N} X_i}{N / \log \log N} \geq 1, \quad \text{a.s.}$$

This completes the proof of the theorem.  $\square$

We state in the following remark some of the results connected with the generalized St. Petersburg game. We apply these results in the next section.

REMARK 1. We have

(i) ([3], Theorem 4) For all  $\varepsilon > 0$ ,

$$\lim_{N \rightarrow \infty} P \left\{ \left| \frac{\sum_{i=1}^N X_i}{N \log_q N} - (q-1) \right| > \varepsilon \right\} = 0.$$

(ii) ([2], Example 4)

$$\liminf_{N \rightarrow \infty} \frac{\sum_{i=1}^N X_i}{N \log_q N} = (q-1), \quad \text{a.s.}$$

and

$$\limsup_{N \rightarrow \infty} \frac{\sum_{i=1}^N X_i}{N \log_q N} = \infty, \quad \text{a.s.}$$

(iii) ([1], Theorem 1.1 (ii))

$$\lim_{N \rightarrow \infty} \frac{\sum_{i=1}^N X_i - \max_{1 \leq i \leq N} X_i}{N \log_q N} = q-1, \quad \text{a.s.}$$

PROOF OF (iii). From [1], it is sufficient to show that

$$\mathbb{E}(\min(X_1, T)) \sim (q-1) \log_q T,$$

as  $T \rightarrow \infty$ . For  $q^k \leq T < q^{k+1}$ , for some integer  $k$ , then it is easy to see that  $\mathbb{E}(\min(X_1, T)) = (q-1)k + \mathcal{O}(1/q^k)$  which implies the desired result.  $\square$

### 3. Application to Formal Laurent Series

Consider the field of formal Laurent series  $\mathbb{F}_q((X^{-1}))$  over a finite base field  $\mathbb{F}_q$  of  $q$  elements,  $q \geq 1$ , that is,

$$\mathbb{F}_q((X^{-1})) = \{f : f = a_n X^n + \cdots + a_0 + a_{-1} X^{-1} + \cdots : a_i \in \mathbb{F}_q\}.$$

Let  $f$  be in  $\mathbb{F}_q((X^{-1}))$  of the form

$$f = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 + a_{-1} X^{-1} + a_{-2} X^{-2} + \cdots.$$

We define the *degree* of  $f$  and the *valuation* of  $f$  by  $\deg(f) = n$  and  $|f| = q^{\deg f} = q^n$  if  $a_n \neq 0$ . If  $f = 0$ , we put  $\deg(0) = -\infty$  and  $|0| = 0$ . Then we can define a metric  $d$  on  $\mathbb{F}_q((X^{-1}))$  by  $d(f, g) = |f - g|$  for  $f, g \in \mathbb{F}_q((X^{-1}))$ .

Denote by  $\mathbb{F}_q[X]$  the ring of  $\mathbb{F}_q$ -coefficients polynomials and let  $\mathbb{L} = \{f \in \mathbb{F}_q((X^{-1})) : \deg(f) < 0\}$ .  $\mathbb{L}$  is a compact abelian group with addition and the metric  $d$ . We denote by  $\mathbf{m}$  the unique normalized Haar measure on  $\mathbb{L}$ .

**3.1. Continued Fraction.** For  $f \in \mathbb{L}$ , one has the following continued fraction expansion:

$$f = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \frac{1}{a_6 + \frac{1}{a_7 + \frac{1}{a_8 + \frac{1}{a_9 + \frac{1}{a_{10} + \frac{1}{\ddots}}}}}}}}}}}}}} := [a_1, a_2, \dots, a_{n-1}, a_n, \dots], \quad n \geq 1.$$

We call the sequence  $\{a_i\} = \{a_i(f)\} \in \mathbb{F}_q[X]$  the polynomial digits of the continued fraction expansion of a Laurent series  $f$ . We put

$$X_i(f) = |a_i(f)| = q^{\deg a_i(f)}.$$

It is well-known that  $(X_i)_{i \geq 1}$  is an independent and identically distributed sequence of random variables on the probability space  $(\mathbb{L}, \mathbf{m})$  with

$$\mathbf{m}(X_i = q^k) = \frac{q-1}{q^k}.$$

We refer to [5] for the general theory of the continued fraction expansion for  $f \in \mathbb{L}$ . We apply the results in Section 2 for continued fraction of Laurent series in the next theorem with Theorem 1 for (i) and Remarks for (ii)–(iv).

**THEOREM 2.** *For the sequence of polynomial digits  $\{a_i(f)\}$  of the continued fraction expansion of Laurent series  $f$ , we have the following:*

(i)

$$\liminf_{N \rightarrow \infty} \frac{\max_{1 \leq i \leq N} q^{\deg a_i(f)}}{N / \log \log N} = 1, \quad \mathbf{m}\text{-a.e.}$$

(ii) *For all  $\varepsilon > 0$ ,*

$$\lim_{N \rightarrow \infty} \mathbf{m} \left\{ \left| \frac{\sum_{i=1}^N q^{\deg a_i(f)}}{N \log_q N} - (q - 1) \right| > \varepsilon \right\} = 0.$$

(iii)

$$\liminf_{N \rightarrow \infty} \frac{\sum_{i=1}^N q^{\deg a_i(f)}}{N \log_q N} = (q - 1), \quad \mathbf{m}\text{-a.e.}$$

and

$$\limsup_{N \rightarrow \infty} \frac{\sum_{i=1}^N q^{\deg a_i(f)}}{N \log_q N} = \infty, \quad \mathbf{m}\text{-a.e.}$$

(iv)

$$\lim_{N \rightarrow \infty} \frac{\sum_{i=1}^N q^{\deg a_i(f)} - \max_{1 \leq i \leq N} q^{\deg a_i(f)}}{N \log_q N} = q - 1, \quad \mathbf{m}\text{-a.e.}$$

**3.2. Oppenheim Expansion.** Let  $\{r_n\}_{n \geq 1}, \{s_n\}_{n \geq 1}$  be sequences of nonzero polynomials over the field  $\mathbb{F}_q$  satisfying

$$\deg s_n - \deg r_n \leq 2, \quad \forall n \geq 1. \tag{H}$$

In [13], Theorem 2.1, it was shown that every  $f \in \mathbb{L}$  has a finite or infinite convergent (relative to  $d$ ) expansion of the form

$$f = \frac{1}{b_1(x)} + \sum_{n=1}^{\infty} \frac{r_1(b_1) \cdots r_n(b_n)}{s_1(b_1) \cdots s_n(b_n)} \frac{1}{b_{n+1}},$$

where  $b_n \in \mathbb{F}_q[X]$ ,  $\deg b_1 \geq 1$  and for any  $n \geq 1$

$$\deg b_{n+1} \geq 2 \deg b_n + 1 - \deg s_n(b_n) + \deg r_n(b_n).$$

The expansion is unique under the preceding conditions on the polynomial digits  $b_n = b_n(f) \in \mathbb{F}_q[X]$  of the Laurent series  $f$ .

Special cases of the Oppenheim Expansion of Laurent Series include:

Lüroth-type expansion:  $s_n(g) = g(g - 1), r_n(g) = 1$ ;

Engel-type expansion:  $s_n(g) = g, r_n(g) = 1$ ;

Sylvester-type expansion:  $s_n(g) = 1, r_n(g) = 1$ ;

Cantor-type Infinite Product:  $s_n(g) = g, r_n(g) = g + 1$ ;

DKB-type expansion:  $s_n(g) = 1, r_n(g) = g$ ;

Consider the following random variables on the probability space for the polynomial digits of the Oppenheim expansion of  $g \in (\mathbb{L}, \mathbf{m})$ ,

$$\Delta_0(g) := \deg b_1(g),$$

$$\Delta_n(g) := \deg b_{n+1}(g) - 2 \deg b_n(g) - \deg r_n(b_n(g)) + \deg s_n(b_n(g)), \quad \text{for } n \geq 1.$$

Fan and Wu [8] showed that  $\{\Delta_n\}_{n \geq 0}$  is an independent and identically distributed sequence of random variables. In particular, for  $n \geq 0$  and  $k \geq 1$ , they computed

$$\mathbf{m}\{g \in \mathbb{L} : \Delta_n(g) = k\} = \frac{q-1}{q^k}.$$

We put

$$X_n = |\Delta_{n-1}(f)| = q^{\Delta_{n-1}(f)}$$

for  $n \geq 1$ . It follows that  $\{X_n\}_{n \geq 1}$  is a sequence of independent and identically distributed random variables with infinite expectations.

**THEOREM 3.** *For a sequence of random variables  $\{q^{\Delta_n(f)}\}_{n \geq 0}$  associated with the polynomial digits  $\{b_i(f)\}$  of the Oppenheim expansion of a Laurent series  $f$ , the following hold:*

(i)

$$\liminf_{N \rightarrow \infty} \frac{\max_{0 \leq i \leq N-1} q^{\Delta_i(f)}}{N / \log \log N} = 1 \quad \mathbf{m}\text{-a.e.}$$

(ii) For all  $\varepsilon > 0$ ,

$$\lim_{N \rightarrow \infty} \mathbf{m} \left\{ \left| \frac{\sum_{i=0}^{N-1} q^{\Delta_i(f)}}{N \log_q N} - (q-1) \right| > \varepsilon \right\} = 0.$$

(iii)

$$\liminf_{N \rightarrow \infty} \frac{\sum_{i=0}^{N-1} q^{\Delta_i(f)}}{N \log_q N} = (q-1), \quad \mathbf{m}\text{-a.e.}$$

and

$$\limsup_{N \rightarrow \infty} \frac{\sum_{i=0}^{N-1} q^{\Delta_i(f)}}{N \log_q N} = \infty, \quad \mathbf{m}\text{-a.e.}$$

(iv)

$$\lim_{N \rightarrow \infty} \frac{\sum_{i=0}^{N-1} q^{\Delta_i(f)} - \max_{0 \leq i \leq N-1} q^{\Delta_i(f)}}{N \log_q N} = q-1, \quad \mathbf{m}\text{-a.e.}$$



We note that in [8] Theorem 2.4, Fan and Wu proved Theorem 3 (ii) following the idea in [14] for Lüroth case. However, this result follows easily from Remark (i) in §2.

REMARK 2. The same results also hold for  $p$ -adic Oppenheim expansions. We refer to [20] for the definition of  $\Delta_n$  and some properties of  $q^{\Delta_n}$  for the  $p$ -adic case.

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