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Asymptotic Estimates for the Spectral Gaps of the Schrödinger Operators with Periodic δ'-Interactions

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Abstract. In this note we investigate the spectral gaps of the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} \big(\beta_1\delta'(x - 2\pi l) + \beta_2\delta'(x - \kappa - 2\pi l)\big) \quad \text{in} \quad L^2(\mathbf{R})\,,$$

where $\beta_1, \beta_2 \in \mathbf{R} \setminus \{0\}$ and $\kappa/\pi \in \mathbf{Q}$. By G_j we denote the j-th gap of the spectrum of H. We provide the asymptotic expansion of the length of G_j as $j \to \infty$.

1. Introduction

In this paper we discuss the spectrum of the Schrödinger operator which is formally expressed as

$$H = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} \big(\beta_1\delta'(x - 2\pi l) + \beta_2\delta'(x - \kappa - 2\pi l)\big) \quad \text{in} \quad L^2(\mathbf{R})\,,$$

where $\kappa \in (0, 2\pi)$ and $\beta_1, \beta_2 \in \mathbf{R} \setminus \{0\}$ are parameters, the symbol $'$ stands for the derivative with respect to x, and $\delta(x)$ is the Dirac δ-function at the origin. The precise definition of this operator is given through boundary conditions as follows. Let

$$Z_1 = 2\pi\mathbf{Z}\,, \quad Z_2 = \{\kappa\} + 2\pi\mathbf{Z}\,, \quad Z = Z_1 \cup Z_2\,,$$

and

$$A_l = \begin{pmatrix} 1 & \beta_l \\ 0 & 1 \end{pmatrix} \quad \text{for} \ \ l = 1, 2\,.$$

We define

$$(Hy)(x) = -y''(x)\,, \quad x \in \mathbf{R} \setminus Z\,,$$

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$$\mathrm{Dom}(H)=\Bigg\{y\in H^2(\mathbf{R}\setminus Z)\ ;\quad \begin{pmatrix} y(x+0)\\ y'(x+0)\end{pmatrix}=A_l\begin{pmatrix} y(x-0)\\ y'(x-0)\end{pmatrix}\quad \text{for}\ \ x\in Z_l,\ \ l=1,2\Bigg\}.$$

In order to formulate our main result, we recall basic spectral properties of H from [10]. The operator H is self-adjoint. Let us consider the equations

$$\begin{cases} -y''(x)=\lambda y(x)\,, \quad x\in\mathbf{R}\setminus Z\,,\\ \begin{pmatrix} y(x+0)\\ y'(x+0)\end{pmatrix}=\begin{pmatrix} 1 & \beta_l\\ 0 & 1\end{pmatrix}\begin{pmatrix} y(x-0)\\ y'(x-0)\end{pmatrix}\quad \text{for}\ \ x\in Z_l,\ \ l=1,2\,,\end{cases}\tag{1}$$

where λ is a complex parameter. By $y_1(x,\lambda)$ and $y_2(x,\lambda)$ we denote the solutions of (1) subject to the initial conditions

$$(y_1(+0,\lambda)\,,\quad y_1'(+0,\lambda))=(1,\ 0)$$

and

$$(y_2(+0,\lambda)\,,\quad y_2'(+0,\lambda))=(0,\ 1)\,,$$

respectively. We introduce the discriminant of the equations (1):

$$D(\lambda)=y_1(2\pi+0,\lambda)+y_2'(2\pi+0,\lambda)\,,$$

which is an entire function. Throughout this paper we use the following convention to simplify expressions. A sentence which contains either $\pm$ or $\mp$ means two sentences; one of which corresponds to the upper sign, the other the lower sign. For example, $a^{\pm}=b^{\mp}$ means two formulas $a^{+}=b^{-}$ and $a^{-}=b^{+}$. All the zeros of $D(\cdot)\mp 2$ are real, and they form an increasing sequence which diverges to $+\infty$. For $j\in\mathbf{N}=\{1,2,3,\dots\}$, we denote by $\lambda_j^{\pm}$ the j-th zero of $D(\cdot)\mp 2$ counted with multiplicity. Then we have

$$\lambda_1^{\mp}<\lambda_1^{\pm}\le\lambda_2^{\pm}<\lambda_2^{\mp}\le\lambda_3^{\mp}<\cdots<\lambda_{2k-1}^{\pm}\le\lambda_{2k}^{\pm}<\lambda_{2k}^{\mp}\le\lambda_{2k+1}^{\mp}<\cdots$$

for $\pm\beta_1\beta_2<0$ (see Proposition 1(d), (e) of [10]). For $\pm\beta_1\beta_2<0$, we define

$$B_j=\begin{cases}[\lambda_j^{\mp},\ \lambda_j^{\pm}] & \text{if}\ \ j\ \text{is odd}\,,\\ [\lambda_j^{\pm},\ \lambda_j^{\mp}] & \text{if}\ \ j\ \text{is even}\,,\end{cases}$$

$$G_j=\begin{cases}(\lambda_j^{\pm},\ \lambda_{j+1}^{\pm}) & \text{if}\ \ j\ \text{is odd}\,,\\ (\lambda_j^{\mp},\ \lambda_{j+1}^{\mp}) & \text{if}\ \ j\ \text{is even}\,.\end{cases}$$

The spectrum of H is then given by

$$\sigma(H)=\bigcup_{j=1}^{\infty}B_j\,.$$

The closed interval B_j is called the j-th band of $\sigma(H)$, the open interval G_j the j-th gap.

The aim of this paper is to analyze the asymptotic behavior of $|G_j|$, the length of the j-th gap, as $j \to \infty$. Hereafter we impose the following assumption on κ.

$$\text{(A.1)} \qquad \frac{\kappa}{2\pi} = \frac{m}{n}, \quad (m, n) \in \mathbf{N}^2 \quad \text{and} \quad \gcd(m, n) = 1\,.$$

We further assume that the prime period of the interactions is 2π, i.e.,

$$\text{(A.2)} \qquad \text{either} \quad (m, n) \neq (1, 2) \quad \text{or} \quad \beta_1 \neq \beta_2 \quad \text{holds}\,.$$

Let

$$a_k = \frac{n}{2m} k \quad \text{for} \quad k = 1, 2, \dots, m-1\,, \tag{2}$$

$$b_l = \frac{n}{2(n-m)} l \quad \text{for} \quad l = 1, 2, \dots, n-m-1\,. \tag{3}$$

Since gcd(m, n)=1, we have $\{a_k\}_{k=1}^{m-1} \cap \{b_l\}_{l=1}^{n-m-1} = \emptyset$. Let

$$c_1 < c_2 < \cdots < c_{n-2} \tag{4}$$

be the rearrangement of the elements of $\{a_k\}_{k=1}^{m-1} \cup \{b_l\}_{l=1}^{n-m-1}$. We set $c_0 = 0$, $c_{n-1} = n/2$, and

$$d_k = c_k - c_{k-1} \quad \text{for} \quad k = 1, 2, \dots, n-1\,.$$

Our main result is now stated as follows.

THEOREM 1. *Adopt the assumptions* (A.1) *and* (A.2).

(i) *For each* $k \in \{1, 2, \dots, n-1\}$, *we have*

$$|G_{nj+1+k}| = nd_k j + O(1) \quad as \quad j \to \infty\,.$$

(ii) *If* $\beta_1\beta_2 < 0$, *then*

$$|G_{nj+1}| = \left| \frac{4(\beta_1 + \beta_2)\pi}{\beta_1\beta_2\kappa(2\pi - \kappa)} \right| + O(j^{-1}) \quad as \quad j \to \infty\,.$$

(iii) *If* $\beta_1\beta_2 > 0$, *then*

$$|G_{nj+1}| = \frac{4\sqrt{(\beta_1 + \beta_2)^2\pi^2 - 4\beta_1\beta_2\kappa(2\pi - \kappa)}}{\beta_1\beta_2\kappa(2\pi - \kappa)} + O(j^{-1}) \quad as \quad j \to \infty\,.$$

The one-dimensional Schrödinger operators with periodic point interactions have been investigated by numerous authors; we refer to [1], [3], [4], [6], [8], [10] and [2] for a thorough review. Such an operator was first inspired by Kronig and Penney; they introduced the operator

$$L_1 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta(x - 2\pi l) \quad \text{in} \quad L^2(\mathbf{R})\,, \quad \beta \in \mathbf{R} \setminus \{0\}$$

and illustrated the graph of its discriminant. This operator is nowadays called the Kronig-Penney Hamiltonian and is referred as the most fundamental model in the textbooks of solid-state physics. This operator was generalized by Gesztesy, Holden and Kirsch [3], [4]; they originated the operator

$$L_2 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta'(x - 2\pi l) \quad \text{in} \quad L^2(\mathbf{R})\,, \quad \beta \in \mathbf{R} \setminus \{0\}$$

and proved that the length of the k-th gap of $\sigma(L_2)$ is equal to

$$\frac{1}{2}k + O(1)$$

as $k \to \infty$. They also showed that the length of the k-th gap of $\sigma(L_1)$ admits the asymptotic expansion

$$\frac{|\beta|}{\pi} + O(k^{-1})$$

as $k \to \infty$. In [7] Kappeler and Möhr obtained asymptotic estimates for the spectral gaps of the Schrödinger operator whose potential is a complex-valued, periodic distribution in the Sobolev space of order $s > -1$. We note that our operator H is not included in such a class. We stress that the asymptotic nature of $\sigma(H)$ is completely different from that of $\sigma(L_2)$; our Theorem 1 says that the length of the $(nj + 1)$-st gap of $\sigma(H)$ converges as $j \to \infty$, while the length of the j-th gap of $\sigma(L_2)$ diverges to $+\infty$ as $j \to \infty$.

The rest of this paper is organized as follows. In section 2 we prove Theorem 1 for even n. In Lemmas 2–6 we locate rough positions of the gaps by using the Rouché theorem and the intermediate value theorem. These ways are fundamental methods to find the positions of gaps and are used in [5] and section 2.4 of [9]. Combining these lemmas with the Taylor series expansion of the discriminant, we complete the proof of Theorem 1. Section 3 is devoted to the demonstration of Theorem 1 for odd n, which is a minor modification of that for even n.

2. Proof of Theorem 1 for even n

By a straightforward calculation, we obtain

$$D(\lambda) = 2\cos 2\pi\sqrt{\lambda} - (\beta_1 + \beta_2)\sqrt{\lambda}\sin 2\pi\sqrt{\lambda} + \beta_1\beta_2\lambda \sin\kappa\sqrt{\lambda}\sin(2\pi - \kappa)\sqrt{\lambda}\,.$$

For the sake of definiteness, we fix the branch cut of the square root as the positive real axis. We define

$$\begin{aligned} \mu &= \sqrt{\lambda}\,, \\ \Phi_+(\mu) &= D(\lambda) - 2 \\ &= -4\sin^2 \pi\mu - (\beta_1 + \beta_2)\mu \sin 2\pi\mu + \beta_1\beta_2\mu^2 \sin\kappa\mu \sin(2\pi - \kappa)\mu\,, \end{aligned} \tag{5}$$

$$\begin{aligned}\Phi_-(\mu) &= D(\lambda) + 2 \\ &= 4\cos^2 \pi\mu - (\beta_1 + \beta_2)\mu \sin 2\pi\mu + \beta_1\beta_2\mu^2 \sin\kappa\mu \sin(2\pi - \kappa)\mu\,. \end{aligned} \tag{6}$$

Let us recall (A.1). In this section we suppose that

$$n \in 2\mathbf{N}\,.$$

In the subsequent lemmas we locate the zeros of $\Phi_\pm(\mu)$. First we prove the following claim.

LEMMA 2. *There exists $j_0 \in \mathbf{N}$ such that the function $\Phi_\pm(\mu)$ has exactly n zeros in the interval $(nj/2 + n/4,\ n(j+1)/2 + n/4)$ for all integers $j \geq j_0$.*

PROOF. We use the Rouché theorem:

Let Ω be a region in $\mathbf{C}$, Γ the boundary of Ω. If f and g are analytic on $\overline{\Omega}$ and $|f| < |g|$ on Γ, then $f + g$ and g have the same number of zeros inside Ω.

Let $j \in \mathbf{N}$. We put

$$f(z) = -4\sin^2 \pi z - (\beta_1 + \beta_2)z \sin 2\pi z\,, \tag{7}$$

$$g(z) = \beta_1\beta_2 z^2 \sin \frac{2m}{n}\pi z \sin 2\left(1 - \frac{m}{n}\right)\pi z\,, \tag{8}$$

$$\Omega_j = \left\{ z \in \mathbf{C}\,; \quad \frac{nj}{2} + \frac{n}{4} < \mathrm{Re}\, z < \frac{n(j+1)}{2} + \frac{n}{4},\ |\mathrm{Im}\, z| < \frac{nj}{2} + \frac{n}{4} \right\},$$

and $\Gamma_j = \partial\Omega_j$. We show the inequality $|f(z)| < |g(z)|$ on Γ_j. Notice that $\Phi_+(z) = f(z) + g(z)$. We have

$$\left|\frac{f(z)}{g(z)}\right| \leq \left|\frac{4\sin^2 \pi z}{\beta_1\beta_2 z^2 \sin \frac{2m}{n}\pi z \sin 2\left(1 - \frac{m}{n}\right)\pi z}\right| + \left|\frac{(\beta_1 + \beta_2)\sin 2\pi z}{\beta_1\beta_2 z \sin \frac{2m}{n}\pi z \sin 2\left(1 - \frac{m}{n}\right)\pi z}\right|. \tag{9}$$

For $\alpha \in \mathbf{R}$ and $z = \sigma + i\tau$ with $\sigma,\ \tau \in \mathbf{R}$, the equality

$$|\sin \alpha z|^2 = \frac{1}{4}(e^{2\alpha\tau} - 2\cos 2\alpha\sigma + e^{-2\alpha\tau}) \tag{10}$$

holds.

First, we consider the vertical sides of Γ_j. Let $\sigma = nj/2 + n/4$. It follows by $n \in 2\mathbf{N}$ and gcd(m, n)=1 that $m \in 2\mathbf{N} - 1$. By (10) we have

$$\left|\sin \frac{2m}{n}\pi z\right|^2 = \frac{1}{4}\left(e^{\frac{2m}{n}\pi\tau} + e^{-\frac{2m}{n}\pi\tau}\right)^2,$$

$$\left|\sin 2\left(1 - \frac{m}{n}\right)\pi z\right|^2 = \frac{1}{4}\left(e^{2(1-\frac{m}{n})\pi\tau} + e^{-2(1-\frac{m}{n})\pi\tau}\right)^2.$$

Using the Schwarz inequality, we get

$$\left|\sin\frac{2m}{n}\pi z\right|\left|\sin 2\left(1-\frac{m}{n}\right)\pi z\right| \geq \frac{1}{4}(e^{\pi\tau}+e^{-\pi\tau})^2 .$$

It follows from (10) that

$$|\sin \pi z|^2 \leq \frac{1}{4}(e^{\pi\tau}+e^{-\pi\tau})^2 , \tag{11}$$

$$|\sin 2\pi z| \leq \frac{1}{2}(e^{\pi\tau}+e^{-\pi\tau})^2 . \tag{12}$$

Hence

$$\left|\frac{f(z)}{g(z)}\right| \leq \left|\frac{4}{\beta_1\beta_2(nj/2+n/4)^2}\right| + \left|\frac{2(\beta_1+\beta_2)}{\beta_1\beta_2(nj/2+n/4)}\right| .$$

So, there exists $j_1 \in \mathbf{N}$ such that

$$\left|\frac{f(z)}{g(z)}\right| < 1 \quad \text{on} \quad \left\{z\in\mathbf{C}\ ;\ \ \mathrm{Re}\, z = \frac{nj}{2}+\frac{n}{4},\ |\mathrm{Im}\, z| \leq \frac{nj}{2}+\frac{n}{4}\right\}$$

for all integers $j \geq j_1$. The same is true on the other vertical side of Γ_j.

Next we discuss the horizontal sides of Γ_j. Let $\tau = \pm(nj/2+n/4)$. By (10) we get

$$\left|\sin\frac{2m}{n}\pi z\right|^2 \geq \frac{1}{4}\left(e^{\frac{2m}{n}\pi|\tau|} - e^{-\frac{2m}{n}\pi|\tau|}\right)^2 ,$$

$$\left|\sin 2\left(1-\frac{m}{n}\right)\pi z\right|^2 \geq \frac{1}{4}\left(e^{2(1-\frac{m}{n})\pi|\tau|} - e^{-2(1-\frac{m}{n})\pi|\tau|}\right)^2 ,$$

and hence

$$\left|\sin\frac{2m}{n}\pi z\right|\left|\sin 2\left(1-\frac{m}{n}\right)\pi z\right| \geq \frac{1}{4}e^{2\pi|\tau|}\left(1-e^{-4\frac{m}{n}\pi|\tau|}\right)\left(1-e^{-4(1-\frac{m}{n})\pi|\tau|}\right) .$$

Since

$$|\sin\pi z|^2 \leq \frac{1}{4}e^{2\pi|\tau|}(1+e^{-2\pi|\tau|})^2 ,$$

$$|\sin 2\pi z| \leq \frac{1}{2}e^{2\pi|\tau|}(1+e^{-2\pi|\tau|})^2 ,$$

we get

$$\left|\frac{f(z)}{g(z)}\right| \leq \frac{(1+e^{-2\pi|\tau|})^2}{|\beta_1\beta_2|\left(1-e^{-4\frac{m}{n}\pi|\tau|}\right)\left(1-e^{-4(1-\frac{m}{n})\pi|\tau|}\right)}\left(\frac{4}{\tau^2}+\frac{2|\beta_1+\beta_2|}{|\tau|}\right) .$$

So, there exists $j_2 \in \mathbf{N}$ such that

$$\left|\frac{f(z)}{g(z)}\right| < 1 \quad \text{on} \quad \left\{z \in \mathbf{C}\,;\ \frac{nj}{2}+\frac{n}{4} \le \operatorname{Re} z \le \frac{n(j+1)}{2}+\frac{n}{4},\ \operatorname{Im} z = \pm\left(\frac{nj}{2}+\frac{n}{4}\right)\right\}$$

for any $j \geq j_2$. Thus

$$\left|\frac{f(z)}{g(z)}\right| < 1 \quad \text{on} \quad \Gamma_j$$

for any $j \geq j_0$:=max$\{j_1, j_2\}$. Using the Rouché theorem, we infer that $g(z)$ and $f(z)+g(z)$ admit the same number of zeros inside Ω_j for $j \geq j_0$.

Finally, we count the number of the zeros of $g(z)$ inside Ω_j. Note that all the zeros of $\sin((2m/n)\pi z)$ in Ω_j are

$$\frac{n}{2m}\left(m\left(j+\frac{1}{2}\right)+\frac{2p+1}{2}\right), \quad p = 0\,,\ 1, \ldots, m-1\,,$$

while those of $\sin(2(1-m/n)\pi z)$ in Ω_j are

$$\frac{n}{2(n-m)}\left((n-m)\left(j+\frac{1}{2}\right)+\frac{2q+1}{2}\right), \quad q = 0\,,\ 1, \ldots, n-m-1\,.$$

Therefore, the function $g(z)$ admits exactly n zeros inside Ω_j for $j \geq j_0$, and so does $\Phi_+(\mu)$. Since the zeros of $\Phi_+(\mu)$ in Ω_j are real, we conclude that $\Phi_+(\mu)$ possesses exactly n zeros inside the interval $(nj/2+n/4,\ n(j+1)/2+n/4)$ for any $j \geq j_0$. In a similar fashion, we infer that $\Phi_-(\mu)$ has exactly n zeros in the interval $(nj/2+n/4,\ n(j+1)/2+n/4)$ for all $j \geq j_0$ □

We sharpen the above lemma in the following Lemmas 3–5.

LEMMA 3. *There exists $j_3 \in \mathbf{N}$ such that the function $\Phi_+(\mu)$ has a unique zero in the interval $(nj/2+n/4+(l-1)/2,\ nj/2+n/4+l/2)$ for $j \geq j_3$ and $l = 1, 2, \ldots, n/2-1, n/2+2, \ldots, n$. Furthermore, the function $\Phi_+(\mu)$ admits exactly two zeros in the interval $(n(j+1)/2-1/2,\ n(j+1)/2+1/2)$ for $j \geq j_3$.*

PROOF. Notice that $n-m \in 2\mathbf{N}-1$. First, we discuss the case where $\beta_1\beta_2 < 0$ and $n \in 2^{\alpha+1}(2\mathbf{N}-1)$ with $\alpha \in \mathbf{N}$. Let $j \geq j_0$. We fix $k \in \{1, 3, 5, \ldots, n-1\}$. We have

$$\Phi_+\left(\frac{nj}{2}+\frac{n}{4}+\frac{k-1}{2}\right) = -\beta_1\beta_2\left(\frac{nj}{2}+\frac{n}{4}+\frac{k-1}{2}\right)^2 \cos^2\frac{(k-1)m}{n}\pi \ \geq 0\,, \tag{13}$$

$$\Phi_+\left(\frac{nj}{2}+\frac{n}{4}+\frac{k}{2}\right) = -4+\beta_1\beta_2\left(\frac{nj}{2}+\frac{n}{4}+\frac{k}{2}\right)^2 \cos^2\frac{km}{n}\pi \ < 0\,. \tag{14}$$

On (13) the equality holds if and only if $k = n/2+1$. By the intermediate value theorem, we claim that the function $\Phi_+(\mu)$ has at least one zero in $(nj/2+n/4+(l-1)/2,\ nj/2+n/4+l/2)$ for $l = 1, 2, \ldots, n/2-1, n/2+2, \ldots, n$.

Next, we investigate the number of the zeros of $\Phi_+(\mu)$ inside $(n(j+1)/2-1/2,\ n(j+1)/2+1/2)$. By a simple calculation, we have

$$\Phi_+\left(\frac{n(j+1)}{2}\right)=0\,, \tag{15}$$

$$\Phi_+'\left(\frac{n(j+1)}{2}\right)=-(\beta_1+\beta_2)\pi n(j+1)\,. \tag{16}$$

So, for a sufficiently small positive number ε, we get

$$\Phi_+\left(\frac{n(j+1)}{2}-\varepsilon\right)>0 \quad \text{if} \quad \beta_1+\beta_2>0\,,$$
$$\Phi_+\left(\frac{n(j+1)}{2}+\varepsilon\right)>0 \quad \text{if} \quad \beta_1+\beta_2<0\,.$$

Combining these with (14) and the intermediate value theorem, we see that $\Phi_+(\mu)$ has at least two zeros inside $(n(j+1)/2-1/2,\ n(j+1)/2+1/2)$. Using Lemma 2 and the above discussion, we obtain the assertion of Lemma 3 in the case where $\beta_1\beta_2<0$ and $n\in 2^{\alpha+1}(2\mathbf{N}-1)$ with $\alpha\in\mathbf{N}$. Likewise, we get the conclusion of Lemma 3 in the other cases. □

LEMMA 4. *Let $j\geq j_0$ and $\beta_1\beta_2<0$. Then, the function $\Phi_-(\mu)$ has exactly one root in the interval $(nj/2+n/4+(l-1)/2,\ nj/2+n/4+l/2)$ for $l=1,2,\dots,2n$.*

PROOF. Note that $n-m\notin 2\mathbf{N}$. First, we consider the case where $n\in 2^{\alpha+1}(2\mathbf{N}-1)$ with $\alpha\in\mathbf{N}$. We fix $k\in\{1,3,5,\dots,n-1\}$. Since

$$\Phi_-\left(\frac{nj}{2}+\frac{n}{4}+\frac{k-1}{2}\right)=4-\beta_1\beta_2\left(\frac{nj}{2}+\frac{n}{4}+\frac{l-1}{2}\right)^2\cos^2\frac{(k-1)m}{n}\pi\ >0\,,$$
$$\Phi_-\left(\frac{nj}{2}+\frac{n}{4}+\frac{k}{2}\right)=\beta_1\beta_2\left(\frac{nj}{2}+\frac{n}{4}+\frac{k}{2}\right)^2\cos^2\frac{km}{n}\pi\ <0\,,$$

we claim by the intermediate value theorem that the function $\Phi_-(\mu)$ has at least one zero inside $(nj/2+n/4+(l-1)/2,\ nj/2+n/4+l/2)$ for $l=1,2,\dots,n$. Combining this with Lemma 2, we get the assertion of Lemma 4 in the case where $n\in 2^{\alpha+1}(2\mathbf{N}-1)$ with $\alpha\in\mathbf{N}$. In a similar way, we get the conclusion of Lemma 4 in the case where $n\in 2(2\mathbf{N}-1)$. □

LEMMA 5. *Let $\beta_1\beta_2>0$. Then, there exists $j_4\in\mathbf{N}$ such that the function $\Phi_-(\mu)$ has a unique zero in the interval $(nj/2+n/4+(l-1)/2,\ nj/2+n/4+l/2)$ for $j\geq j_4$ and $l=1,2,\dots,n/2-1,n/2+2,\dots,n$. Moreover, the function $\Phi_-(\mu)$ possesses exactly two zeros in the interval $(n(j+1)/2-1/2,\ n(j+1)/2+1/2)$ for $j\geq j_4$.*

PROOF. Recall that $n-m\notin 2\mathbf{N}$. We discuss the case where $n\in 2^{\alpha+1}(2\mathbf{N}-1)$ with $\alpha\in\mathbf{N}$. Let $j\geq j_0$. We fix $k\in\{1,3,5,\dots,n/2-1,n/2+3,\dots,n-1\}$. Since there exists

$j_4 \in \mathbf{N}$ such that

$$\Phi_-\left(\frac{nj}{2}+\frac{n}{4}+\frac{k-1}{2}\right)=4-\beta_1\beta_2\left(\frac{nj}{2}+\frac{n}{4}+\frac{k-1}{2}\right)^2\cos^2\frac{(k-1)m}{n}\pi<0\,,$$

$$\Phi_-\left(\frac{nj}{2}+\frac{n}{4}+\frac{k}{2}\right)=\beta_1\beta_2\left(\frac{nj}{2}+\frac{n}{4}+\frac{k}{2}\right)^2\cos^2\frac{km}{n}\pi>0$$

for $j \geq j_4$, we infer by the intermediate value theorem that the function $\Phi_-(\mu)$ has at least one zero in $(nj/2+n/4+(l-1)/2,\ nj/2+n/4+l/2)$ for $l = 1, 2, \ldots, n/2-1, n/2+3, \ldots, n$ and $j \geq j_4$.

Next, we discuss the number of the zeros of $\Phi_-(\mu)$ inside $(n(j+1)/2-1/2,\ n(j+1)/2+1/2)$. By a simple calculation, we have

$$\Phi_-\left(\frac{n(j+1)}{2}\right)=4\,, \tag{17}$$

$$\Phi'_-\left(\frac{n(j+1)}{2}\right)=-(\beta_1+\beta_2)\pi n(j+1)\,.$$

Moreover, we obtain

$$\Phi_-\left(\frac{n(j+1)}{2}-\frac{1}{2}\right)>4$$

and

$$\Phi_-\left(\frac{n(j+1)}{2}+\frac{1}{2}\right)>4 \tag{18}$$

for sufficiently large j. First we discuss the case where $\beta_1+\beta_2>0$. Let $\alpha_1 \leq \alpha_2$ be the first two zeros of $\Phi_-(\mu)$ inside $(n(j+1)/2,\ \infty)$. Combining (17), $\Phi'_-(n(j+1)/2)<0$, and Proposition 1(d) of [10], we have $\Phi_-(\mu) \leq 4$ for $\mu \in [n(j+1)/2,\ \alpha_2]$. This together with (18) yields that $\Phi_-(\mu)$ has at least two zeros in the interval $(n(j+1)/2,\ n(j+1)/2+1/2)$. In a similar fashion we claim that $\Phi_-(\mu)$ admits at least two zeros inside $(n(j+1)/2-1/2,\ n(j+1)/2)$ for $\beta_1+\beta_2<0$. Using Lemma 2 and the above discussion, we obtain the assertion of Lemma 5 for $n \in 2^{\alpha+1}(2\mathbf{N}-1)$ with $\alpha \in \mathbf{N}$. In a similar manner, we conclude the assertion of Lemma 5 in the case where $n \in 2(2\mathbf{N}-1)$. □

We further need the following implication.

LEMMA 6. *There exists* $j_5 \in \mathbf{N}$ *such that the function* $D(\cdot)\mp 2$ *has exactly* $(nj+n/2+1)$ *zeros in the interval* $\bigl(-\infty,\ (nj/2+n/4)^2\bigr)$ *for* $j \geq j_5$.

PROOF. First, we prove the assertion for $D(\lambda)-2$. We recall (7) and (8). Let $j \geq j_0$. We put

$$F(\lambda)=f(\sqrt{\lambda})=-4\sin^2\pi\sqrt{\lambda}-(\beta_1+\beta_2)\sqrt{\lambda}\sin 2\pi\sqrt{\lambda}\,,$$

$$G(\lambda) = g(\sqrt{\lambda}) = \beta_1\beta_2\lambda \sin\frac{2m}{n}\pi\sqrt{\lambda}\sin 2\left(1-\frac{m}{n}\right)\pi\sqrt{\lambda}\,,$$

$$\Lambda_j = \left\{\lambda \in \mathbf{C}\,;\ \ |\mathrm{Re}\,\lambda| < \left(\frac{nj}{2}+\frac{n}{4}\right)^2 - \frac{|\mathrm{Im}\,\lambda|^2}{4\left(\frac{nj}{2}+\frac{n}{4}\right)^2},\quad |\mathrm{Im}\,\lambda| < 2\left(\frac{nj}{2}+\frac{n}{4}\right)^2\right\},$$

$$S_1 = \left\{z \in \mathbf{C}\,;\ \ \mathrm{Re}\,z = \frac{nj}{2}+\frac{n}{4},\ \ 0 \le \mathrm{Im}\,z \le \frac{nj}{2}+\frac{n}{4}\right\},$$

$$S_2 = \left\{z \in \mathbf{C}\,;\ \ |\mathrm{Re}\,z| \le \frac{nj}{2}+\frac{n}{4},\ \ \mathrm{Im}\,z = \frac{nj}{2}+\frac{n}{4}\right\},$$

$$S_3 = \left\{z \in \mathbf{C}\,;\ \ \mathrm{Re}\,z = -\left(\frac{nj}{2}+\frac{n}{4}\right),\ \ 0 \le \mathrm{Im}\,z \le \frac{nj}{2}+\frac{n}{4}\right\}.$$

We show the inequality $|F(\lambda)| < |G(\lambda)|$ on $\partial\Lambda_j$. The function $z = \sqrt{\lambda}$ maps $\partial\Lambda_j$ to $S_1 \cup S_2 \cup S_3$, bijectively. In Lemma 2 we have proved that $|f(z)| < |g(z)|$ on the two lines $\mathrm{Im}\,z = nj/2 + n/4$ and $\mathrm{Re}\,z = nj/2 + n/4$. Since $f(z)$ and $g(z)$ are odd functions of z, we obtain $|f(z)| < |g(z)|$ on $S_1 \cup S_2 \cup S_3$. Hence, we get $|F(\lambda)| < |G(\lambda)|$ on $\partial\Lambda_j$. It then follows that the functions $D(\lambda) - 2$ and $G(\lambda)$ have the same number of zeros inside Λ_j.

Next, we count the number of the zeros of $G(\lambda)$ inside Λ_j. Notice that all the zeros of $\sin((2m/n)\pi\sqrt{\lambda})$ in Λ_j are

$$\left(\frac{np}{2m}\right)^2, \quad p = 0, 1, \dots, mj + (m-1)/2\,,$$

while those of $\sin(2(1 - m/n)\pi\sqrt{\lambda})$ in Λ_j are

$$\left(\frac{nq}{2(n-m)}\right)^2, \quad q = 0\,,\ 1, \dots, (n-m)j + (n-m-1)/2\,.$$

We note that 0 is a double zero of $G(\lambda)$. Hence the function $D(\lambda)-2$ has exactly $(nj+n/2+1)$ zeros inside Λ_j. Combining this with the fact

$$\lim_{\lambda\to-\infty}|D(\lambda)| = +\infty\,,$$

we claim that $D(\lambda) - 2$ has exactly $(nj + n/2 + 1)$ zeros in the interval $(-\infty,\ (nj/2+n/4)^2)$ for sufficiently large $j \in \mathbf{N}$. Likewise, we infer that $D(\lambda) + 2$ has exactly $(nj + n/2 + 1)$ zeros inside the interval $(-\infty,\ (nj/2 + n/4)^2)$ for sufficiently large $j \in \mathbf{N}$. $\square$

We are now ready to prove Theorem 1. First, we show the statement (ii).

PROOF OF THEOREM 1(ii). Let us discuss the case where $\beta_1 + \beta_2 > 0$. By the proof of Lemma 3 we observe that the two zeros of $\Phi_+(\mu)$ in the interval $(n(j + 1)/2 - 1/2,\ n(j + 1)/2 + 1/2)$ are written in the form

$$\mu_j^+ = \frac{n(j+1)}{2}\,, \quad \mu_j^- = \frac{n(j+1)}{2} - r_j^-$$

with $0 < r_j^- < 1/2$. By Lemmas 3 and 6 we have

$$
\begin{aligned}
|G_{n(j+1)+1}| &= (\mu_j^+)^2 - (\mu_j^-)^2 \\
&= n(j+1)r_j^- - (r_j^-)^2 .
\end{aligned} \tag{19}
$$

Let us show that

$$
r_j^- \to 0 \quad \text{as} \quad j \to \infty . \tag{20}
$$

Seeking a contradiction, we assume that $\{r_j^-\}_{j=1}^\infty$ does not tend to zero. Then there would exist a subsequence $\{j(l)\}_{l=1}^\infty$ of $\{j\}_{j=1}^\infty$ and a number $\delta \in (0, 1/2)$ such that $\delta < r_{j(l)}^-$ for all $l \in \mathbf{N}$. Then we have

$$
\sin \frac{2m}{n}\pi r_{j(l)}^- \geq \min\left(\sin \frac{2m}{n}\pi\delta,\ \sin \frac{m}{n}\pi \right) > 0 ,
$$

$$
\sin 2\left(1 - \frac{m}{n}\right)\pi r_{j(l)}^- \geq \min\left(\sin 2\left(1 - \frac{m}{n}\right)\pi\delta,\ \sin \frac{m}{n}\pi \right) > 0 .
$$

Since $\Phi_+(\mu_{j(l)}^-) = 0$, we arrive at

$$
\begin{aligned}
&\left(\frac{n(j(l)+1)}{2} + r_{j(l)}^- - \frac{(\beta_1 + \beta_2)\sin 2\pi r_{j(l)}^-}{2\beta_1\beta_2 \sin \frac{2m}{n}\pi r_{j(l)}^- \sin 2\left(1 - \frac{m}{n}\right)\pi r_{j(l)}^-} \right)^2 \\
&\quad = \frac{4\sin^2 \pi r_{j(l)}^-}{\beta_1\beta_2 \sin \frac{2m}{n}\pi r_{j(l)}^- \sin 2\left(1 - \frac{m}{n}\right)\pi r_{j(l)}^-} \\
&\qquad + \left(\frac{(\beta_1 + \beta_2)\sin 2\pi r_{j(l)}^-}{2\beta_1\beta_2 \sin \frac{2m}{n}\pi r_{j(l)}^- \sin 2\left(1 - \frac{m}{n}\right)\pi r_{j(l)}^-} \right)^2 .
\end{aligned} \tag{21}
$$

Hence the left-hand side of (21) tends to $+\infty$ as $l \to \infty$, while the right-hand side of (21) is bounded. Because we have found a contradiction, we obtain $r_j^- \to 0$ as $j \to \infty$.

Next we analyze more precisely the asymptotic behavior of r_j^- as $j \to \infty$. Recall (15) and (16). Since

$$
\Phi_+''\left(\frac{n(j+1)}{2} \right) = -8\pi^2 - 4(\beta_1 + \beta_2)\pi + 2\beta_1\beta_2 m(n-m)\pi^2 (j+1)^2 ,
$$

$$
\sup_{1 \leq \mu} \mu^{-2} |\Phi_+'''(\mu)| < +\infty ,
$$

we have by the Taylor theorem

$$
\Bigg| \Phi_+(\mu) + (\beta_1 + \beta_2)\pi n(j+1)\left(\mu - \frac{n(j+1)}{2} \right)
$$

$$+\left\{4\pi^2+2(\beta_1+\beta_2)\pi-\beta_1\beta_2 m(n-m)\pi^2(j+1)^2\right\}\left(\mu-\frac{n(j+1)}{2}\right)^2\Bigg|$$

$$\le C(j+1)^2\left|\mu-\frac{n(j+1)}{2}\right|^3 \tag{22}$$

for $\mu\in(n(j+1)/2-1/2, n(j+1)/2+1/2)$ and $j\in\mathbf{N}$, where C is a constant independent of μ and j. Because $\Phi_+(\mu_j^-)=0$, we have

$$\left|-(\beta_1+\beta_2)\pi n(j+1)+\left\{4\pi^2+2(\beta_1+\beta_2)\pi-\beta_1\beta_2 m(n-m)\pi^2(j+1)^2\right\}r_j^-\right| \tag{23}$$

$$\le C(j+1)^2(r_j^-)^2\,.$$

Since $r_j^-\to 0$ as $j\to\infty$, there exists $j_6\in\mathbf{N}$ such that

$$C\big(j+1\big)^2(r_j^-)\le\frac{1}{2}\big(4\pi^2+2(\beta_1+\beta_2)\pi-\beta_1\beta_2 m(n-m)\pi^2(j+1)^2\big)$$

for $j\ge j_6$. Inserting this into (23), we get

$$r_j^-\le\frac{2(\beta_1+\beta_2)n(j+1)}{4\pi+2(\beta_1+\beta_2)-\beta_1\beta_2 m(n-m)\pi(j+1)^2}\,.$$

Thus, there exists $C_0>0$ such that

$$0<r_j^-\le C_0(j+1)^{-1} \tag{24}$$

for all $j\ge j_6$. By substituting (24) for (23), we get

$$\frac{(\beta_1+\beta_2)n(j+1)-CC_0^2}{4\pi+2(\beta_1+\beta_2)-\beta_1\beta_2 m(n-m)\pi(j+1)^2}$$

$$\le r_j^-\le\frac{(\beta_1+\beta_2)n(j+1)+CC_0^2}{4\pi+2(\beta_1+\beta_2)-\beta_1\beta_2 m(n-m)\pi(j+1)^2}\,.$$

Hence we obtain

$$n(j+1)r_j^-=\frac{-(\beta_1+\beta_2)n^2}{\beta_1\beta_2 m(n-m)\pi}+O(j^{-1})$$

as $j\to\infty$. Therefore we conclude by (19) that

$$|G_{n(j+1)+1}|=\frac{-(\beta_1+\beta_2)n^2}{\beta_1\beta_2 m(n-m)\pi}+O(j^{-1})$$

as $j\to\infty$. Thus we have proved the statement (ii) for $\beta_1+\beta_2>0$. In a similar manner, we obtain (ii) for $\beta_1+\beta_2<0$. It follows by Theorem 2 of [10] that $|G_{nj+1}|=0$ for all $j\in\mathbf{N}$, provided $\beta_1+\beta_2=0$. Hence, (ii) also holds for $\beta_1+\beta_2=0$. □

The proof of the statement (iii) is slightly complicated than that of (ii).

PROOF OF THEOREM 1 (iii). Suppose that $\beta_1 + \beta_2 > 0$. It follows by the proof of Lemma 5 that, for $j \geq j_4$, the function $\Phi_-(\mu)$ admits exactly two zeros inside $(n(j+1)/2, n(j+1)/2 + 1/2)$, which we denote by $\tau_j^- < \tau_j^+$. Using Lemmas 4 and 6, we obtain

$$|G_{n(j+1)+1}| = (\tau_j^+)^2 - (\tau_j^-)^2 .$$

Put $s_j^\pm = \tau_j^\pm - n(j+1)/2$. We have $0 < s_j^- < s_j^+ < 1/2$. As in the proof of (20), we have

$$s_j^\pm \to 0 \quad \text{as} \quad j \to \infty . \tag{25}$$

Since $\Phi_-(\tau_j^\pm) = 0$, we have by (22)

$$\begin{aligned} &\Big| -4 + (\beta_1 + \beta_2)\pi n(j+1)s_j^\pm \\ &\qquad + \big\{4\pi^2 + 2(\beta_1+\beta_2)\pi - \beta_1\beta_2 m(n-m)\pi^2(j+1)^2\big\}(s_j^\pm)^2\Big| \\ &\quad \leq C(j+1)^2(s_j^\pm)^3 . \end{aligned} \tag{26}$$

Because $s_j^\pm \to 0$ as $j \to \infty$, there exists $j_7 \in \mathbf{N}$ such that

$$C(j+1)^2 s_j^\pm \leq \frac{1}{2}\big\{-4\pi^2 - 2(\beta_1+\beta_2)\pi + \beta_1\beta_2 m(n-m)\pi^2(j+1)^2\big\}$$

for $j \geq j_7$. This together with (26) implies that

$$\begin{aligned} &\frac{1}{2}\big\{-4\pi^2 - 2(\beta_1+\beta_2)\pi + \beta_1\beta_2 m(n-m)\pi^2(j+1)^2\big\}(s_j^\pm)^2 \\ &\qquad - (\beta_1+\beta_2)\pi n(j+1)s_j^\pm + 4 \leq 0 , \\ &\frac{3}{2}\big\{-4\pi^2 - 2(\beta_1+\beta_2)\pi + \beta_1\beta_2 m(n-m)\pi^2(j+1)^2\big\}(s_j^\pm)^2 \\ &\qquad - (\beta_1+\beta_2)\pi n(j+1)s_j^\pm + 4 \geq 0 . \end{aligned}$$

By the former inequality we have

$$0 < s_j^\pm \leq C_1(j+1)^{-1} ,$$

where C_1 is a constant independent of j. Then we infer by (26) that

$$\begin{aligned} &\Big| -4 + (\beta_1 + \beta_2)\pi n(j+1)s_j^\pm \\ &\qquad + \big\{4\pi^2 + 2(\beta_1+\beta_2)\pi - \beta_1\beta_2 m(n-m)\pi^2(j+1)^2\big\}(s_j^\pm)^2\Big| \\ &\quad \leq \frac{CC_1^3}{j+1} . \end{aligned} \tag{27}$$

Put

$$\gamma_j = -\frac{(\beta_1+\beta_2)\pi n(j+1)}{2(4\pi^2 + 2(\beta_1+\beta_2)\pi - \beta_1\beta_2 m(n-m)\pi^2(j+1)^2)} .$$

Then we claim by (22) that

$$\Phi_-\left(\frac{n(j+1)}{2}+\gamma_j\right)=\frac{-(\beta_1+\beta_2)^2n^2+16\beta_1\beta_2m(n-m)}{2\beta_1\beta_2m(n-m)}+O(j^{-1})$$

as $j\to\infty$. Notice that $(\beta_1+\beta_2)^2\geq 4\beta_1\beta_2>0$, where the equality holds if and only if $\beta_1=\beta_2$. So we get

$$\Phi_-\left(\frac{n(j+1)}{2}+\gamma_j\right)\leq -\frac{2(n-2m)^2}{m(n-m)}+O(j^{-1})\,. \tag{28}$$

The first term of the right-hand side of (28) is equal to 0 if and only if $(m,n)=(1,2)$. Hence we claim by (**A.2**) that

$$\Phi_-\left(\frac{n(j+1)}{2}+\gamma_j\right)<0$$

for sufficiently large j. Combining this with (17) and (18), we get

$$s_j^-<\gamma_j<s_j^+\,. \tag{29}$$

We denote by $\alpha_{1,j}^{\pm}<\alpha_{2,j}^{\pm}$ the zeros of

$$4-(\beta_1+\beta_2)\pi n(j+1)x-\left\{4\pi^2+2(\beta_1+\beta_2)\pi-\beta_1\beta_2m(n-m)\pi^2(j+1)^2\right\}x^2\pm\frac{CC_0^3}{j+1}\,.$$

Using (27) and (29), we obtain

$$\alpha_{1,j}^-\leq s_j^-\leq\alpha_{1,j}^+\,,\quad \alpha_{2,j}^+\leq s_j^+\leq\alpha_{2,j}^-\,.$$

Therefore we get

$$n(j+1)s_j^{\pm}=\frac{(\beta_1+\beta_2)\pi n^2\pm n\sqrt{(\beta_1+\beta_2)^2\pi^2n^2-16\beta_1\beta_2m(n-m)\pi^2}}{2\beta_1\beta_2m(n-m)\pi^2}+O(j^{-1})$$

as $j\to\infty$. Hence we conclude that

$$|G_{n(j+1)+1}|=\frac{n\sqrt{(\beta_1+\beta_2)^2\pi^2n^2-16\beta_1\beta_2m(n-m)\pi^2}}{\beta_1\beta_2m(n-m)\pi^2}+O(j^{-1})$$

as $j\to\infty$, which proves the statement (iii) for $\beta_1+\beta_2>0$. Likewise, we get (iii) for $\beta_1+\beta_2<0$. □

Finally, we show the statement (i).

PROOF OF THEOREM 1 (i). We show that the zeros of $\Phi_+(\mu)/\mu^2$ are asymptotically equal to that of $\sin\kappa\mu\sin(2\pi-\kappa)\mu$. We put

$$\tilde{\Phi}_+(\mu)=\frac{1}{\mu^2}\Phi_+(\mu)=\beta_1\beta_2\sin\kappa\mu\sin(2\pi-\kappa)\mu-\frac{\beta_1+\beta_2}{\mu}\sin 2\pi\mu-\frac{4}{\mu^2}\sin^2\pi\mu\,,$$

$$\Psi_+(\mu) = \sin\kappa\mu \sin(2\pi - \kappa)\mu \,.$$

We see that the zeros of $\Psi_+(\mu)$ in the interval $(nj/2, n(j+1)/2)$ are

$$\frac{nj}{2} + \frac{n}{2m}k \quad (k = 1, 2, \dots, m-1)$$

and

$$\frac{nj}{2} + \frac{n}{2(m-n)}l \quad (l = 1, 2, \dots, n-m-1)\,.$$

We recall (2), (3) and (4). Now, we prove that there exists a constant $C > 0$ and $j_7 \in \mathbf{N}$ such that $\tilde{\Phi}_+(\mu)$ has a unique zero in the interval $(nj/2 + c_k - C/j,\ nj/2 + c_k + C/j)$ for all $j \geq j_7$ and $k = 1, 2, \dots, n-2$.

First, we consider the neighborhood of $nj/2 + a_k$ for $k = 1, 2, \dots, m-1$. By the Taylor theorem, we get

$$\sin\kappa\left(\frac{nj}{2} + \frac{n}{2m}k + \eta\right) = \kappa\eta + O(|\eta|^3)\,,$$

$$\sin(2\pi - \kappa)\left(\frac{nj}{2} + \frac{n}{2m}k + \eta\right) = (-1)^j \sin(2\pi - \kappa)\frac{n}{2m}k + O(|\eta|)$$

as $\eta \to 0$, where the error terms are uniform with respect to $j \in \mathbf{N}$. Notice that

$$\left|\frac{\sin 2\pi\mu}{\mu}\right| \leq \frac{2}{n(j-1)} \quad \text{and} \quad \left|\frac{\sin^2 \pi\mu}{\mu^2}\right| \leq \frac{4}{n^2(j-1)^2}$$

for $\mu \geq n(j-1)/2$. Thus we obtain

$$\begin{aligned} &\tilde{\Phi}_+\left(\frac{nj}{2} + \frac{n}{2m}k + \eta\right) \\ &\quad = \beta_1\beta_2(\kappa\eta + O(|\eta|^3))\left((-1)^j \sin(2\pi - \kappa)\frac{n}{2m}k + O(|\eta|)\right) + O(j^{-1}) \\ &\quad = (-1)^j \beta_1\beta_2\kappa\eta \sin(2\pi - \kappa)\frac{n}{2m}k + O(\eta^2) + O(j^{-1})\,, \end{aligned} \tag{30}$$

as $\eta \to 0$ and $j \to \infty$, where the first error term in (30) is uniform with respect to $j \in \mathbf{N}$ and the last error term in (30) is uniform with respect to $\eta \in (-1, 1)$. Hence, there exists a constant $C_1 > 0$ and $j_7 \in \mathbf{N}$ such that $\tilde{\Phi}_+(\mu)$ has a unique zero in the interval $(nj/2 + a_k - C_1/j,\ nj/2 + a_k + C_1/j)$ for all $j \geq j_7$. In the exactly same way, we claim that there exists a constant $C_2 > 0$ and $j_8 \in \mathbf{N}$ such that $\tilde{\Phi}_+(\mu)$ admits a unique zero inside $(nj/2 + b_l - C_2/j,\ nj/2 + b_l + C_2/j)$ for all $j \geq j_8$ and $l = 1, 2, \dots, n-m-1$. Hence the zeros of $\tilde{\Phi}_+(\mu)$ in $(nj/2 + d_1/2,\ n(j+1)/2 - d_{n-1}/2)$ are written in the form

$$\rho_k^j = \frac{nj}{2} + c_k + O(j^{-1})\,, \quad k = 1, 2, \dots, n-2\,.$$

This together with (20), (25) and Lemmas 3, 6 implies that

$$\begin{aligned}|G_{nj+1+2k}| &= (\rho_{2k}^{j})^2 - (\rho_{2k-1}^{j})^2 \\ &= nj(c_{2k} - c_{2k-1}) + O(1)\end{aligned}$$

as $j \to \infty$ for $k = 1, 2, \dots, n/2 - 1$.

Next we denote the zeros of Φ_- in the interval $(nj/2 - 1/2,\ nj/2 + 1/2)$ by $\xi_j^- < \xi_j^+$. As in the proof of Theorem 1 (iii), we have $|\xi_j^{\pm} - nj/2| \to 0$ as $j \to \infty$. Combining this with the above arguments, we obtain

$$|G_{nj+2+2k}| = nj(c_{2k+1} - c_{2k}) + O(1)$$

as $j \to \infty$ for $k = 0, 1, \dots, n/2 - 1$. Hence, we have the statement (i). □

3. Proof of Theorem 1 for odd n

We give the proof of Theorem 1 for odd n with omitting details, since it is a minor modification of that for even n; we need the following claim instead of Lemma 2.

LEMMA 7. *If n is odd, then the function $\Phi_{\pm}(\mu)$ admits exactly $2n$ zeros in the interval $(nj/2 + 1/2,\ n(j+2)/2 + 1/2)$ for sufficiently large j.*

PROOF. For $j \in \mathbf{N}$, we put

$$V_j = \left\{ z \in \mathbf{C}\,;\ \ \operatorname{Re} z = \frac{nj}{2} + \frac{1}{2} \right\}.$$

Recall (5)–(8). Let us prove that $|f(z)| < |g(z)|$ on V_j for sufficiently large j. Let $z = nj/2 + 1/2 + i\tau$, $\tau \in \mathbf{R}$. By (10) we have

$$\begin{aligned}\left|\sin \frac{2m}{n}\pi z\right|^2 &= \frac{1}{4}\left(e^{\frac{4m}{n}\pi|\tau|} - 2\cos\frac{2m}{n}\pi + e^{-\frac{4m}{n}\pi|\tau|}\right) \\ &\geq \frac{1}{2}\left(1 - \cos\frac{2m}{n}\pi\right) > 0\,, \\ \left|\sin 2\left(1 - \frac{m}{n}\right)\pi z\right|^2 &= \frac{1}{4}\left(e^{4(1-\frac{m}{n})\pi|\tau|} - 2\cos\frac{2m}{n}\pi + e^{-4(1-\frac{m}{n})\pi|\tau|}\right) \\ &\geq \frac{1}{2}\left(1 - \cos\frac{2m}{n}\pi\right) > 0\,.\end{aligned}$$

If

$$\left|2\cos\frac{2m}{n}\pi\right| \leq \frac{1}{2}e^{\frac{4m}{n}\pi|\tau|}$$

and

$$\left|2\cos 2\left(1-\frac{m}{n}\right)\pi\right| \leq \frac{1}{2}e^{4\left(1-\frac{m}{n}\right)\pi|\tau|},$$

then we get

$$\left|\sin\frac{2m}{n}\pi z\right|^2 \geq \frac{1}{8}e^{\frac{4m}{n}\pi|\tau|},$$

$$\left|\sin 2\left(1-\frac{m}{n}\right)\pi z\right|^2 \geq \frac{1}{8}e^{4\left(1-\frac{m}{n}\right)\pi|\tau|},$$

and hence

$$\frac{(e^{\pi\tau}+e^{-\pi\tau})^2}{|\sin\frac{2m}{n}\pi z||\sin 2\left(1-\frac{m}{n}\right)\pi z|} \leq \frac{4e^{2\pi|\tau|}}{\frac{1}{8}e^{2\pi|\tau|}} = 32\,.$$

So, we obtain

$$\sup_{j\in\mathbf{N}}\sup_{z\in V_j}\frac{(e^{\pi\operatorname{Im} z}+e^{-\pi\operatorname{Im} z})^2}{|\sin\frac{2m}{n}\pi z||\sin 2\left(1-\frac{m}{n}\right)\pi z|} < \infty\,,$$

and thus

$$\left|\frac{f(z)}{g(z)}\right| < 1$$

for sufficiently large j in view of (9), (11) and (12). By this estimate and the arguments employed in the proof of Lemma 2, we get the conclusion of Lemma 7. □

Using Lemma 7 and mimicking the methods in the previous section after Lemma 2, we get the assertion of Theorem 1 for odd n.

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