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Apollonius Points and Anharmonic Ratios

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Abstract. We give a characterization of Möbius transformation by use of Apollonius points introduced by Haruki and Rassias [2]. Our result is stronger than theirs.

1. Introduction

In their paper [2], Haruki and Rassias introduced a concept of *Apollonius points* for three distinct points z_1, z_2 and z_3 in the complex plane. $z \in \mathbf{C}$ is called an Apollonius point of z_1, z_2, z_3 if

$$|z_1 - z_2| \cdot |z_3 - z| = |z_2 - z_3| \cdot |z_1 - z| = |z_3 - z_1| \cdot |z_2 - z| .$$

It is easy to see that this equation is equivalent to

$$[z_1, z_2; z_3, z] = \frac{1 \pm \sqrt{3}i}{2} , \tag{1.1}$$

where the left hand side is the anharmonic ratio of z_1, z_2, z_3 and z. Namely, by definition,

$$[z_1, z_2; z_3, z] = \frac{z_1 - z_3}{z_3 - z_2} \cdot \frac{z_2 - z}{z - z_1} .$$

Thus there are generally two Apollonius points for z_1, z_2 and z_3; one inside the circle through z_1, z_2 and z_3, and the other outside the circle.

Haruki and Rassias have proved that *a complex analytic univalent function $w = f(z)$ which preserves Apollonius points must be a Möbius transformation*. Here we say that f preserves Apollonius points if $f(z)$ is an Apollonius point of $f(z_1)$, $f(z_2)$, $f(z_3)$ whenever z is an Apollonius point of z_1, z_2, z_3. We extend this result and will prove the following.

THEOREM. *Let $U \subset \mathbf{C}$ be a domain and $f : U \to \mathbf{C}$ be a C^1-mapping (may not necessarily be complex analytic). If f preserves Apollonius points, then f is a Möbius transformation or its conjugate.*

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2. Functions which preserve an anharmonic ratio

In this section we will prove the following theorem from which together with (1.1) Theorem in Introduction follows immediately.

THEOREM 2.1. *Let* $\lambda \in \mathbf{C} \setminus \mathbf{R}$ *be not a real number. Suppose* $f : U \to \mathbf{C}$ *is a* C^1*-mapping such that* $[f(z_1), f(z_2); f(z_3), f(z_4)] = \lambda$ *if* $[z_1, z_2; z_3, z_4] = \lambda$. *Then* f *is a Möbius transformation.*

The proof of Theorem 2.1 is divided into two steps. One is the following.

PROPOSITION 2.2. *Let* $\lambda \in \mathbf{C} \setminus \mathbf{R}$ *be not a real number. Suppose* $f : U \to \mathbf{C}$ *is a* C^1*-mapping such that* $[f(z_1), f(z_2); f(z_3), f(z_4)] = \lambda$ *if* $[z_1, z_2; z_3, z_4] = \lambda$. *Then* f *is complex analytic.*

The latter half is the following.

PROPOSITION 2.3. *Suppose* $\lambda \in \mathbf{C} \setminus \{0, 1\}$, *and* $f : U \to \mathbf{C}$ *is a complex analytic function such that* $[f(z_1), f(z_2); f(z_3), f(z_4)] = \lambda$ *if* $[z_1, z_2; z_3, z_4] = \lambda$. *Then* f *is a Möbius transformation.*

PROOF OF PROPOSITION 2.2. Choose $a, b, c, d \in \mathbf{C}$ such that $a, b, c \in \mathbf{R}$ and $[a, b; c, d] = \lambda$. The condition that λ is not real means that d is not real. Let $z \in U$ and $t \in \mathbf{C} \setminus \{0\}$ be small enough so that $z + ta, z + tb, z + tc, z + td \in U$. We remark that $[z + ta, z + tb; z + tc, z + td] = \lambda$. From the Taylor development,

$$f(z + ta) = f(z) + \partial_z f(z)ta + \bar{\partial}_z f(z)\bar{t}\bar{a} + o(t)\,.$$

Hence we have

$$\begin{aligned}
&[f(z + ta), f(z + tb); f(z + tc), f(z + td)] \\
&\quad = \frac{\partial_z f(z)t(a - c) + \bar{\partial}_z f(z)\bar{t}(\bar{a} - \bar{c})}{\partial_z f(z)t(c - b) + \bar{\partial}_z f(z)\bar{t}(\bar{c} - \bar{b})} \cdot \frac{\partial_z f(z)t(b - d) + \bar{\partial}_z f(z)\bar{t}(\bar{b} - \bar{d})}{\partial_z f(z)t(d - a) + \bar{\partial}_z f(z)\bar{t}(\bar{d} - \bar{a})} + o(t)\,.
\end{aligned}$$

Since a, b and c are real, we obtain

$$\begin{aligned}
&[f(z + ta), f(z + tb); f(z + tc), f(z + td)] \\
&= \frac{(\partial_z f(z)t + \bar{\partial}_z f(z)\bar{t})(a - c)}{(\partial_z f(z)t + \bar{\partial}_z f(z)\bar{t})(c - b)} \cdot \frac{(\partial_z f(z)t + \bar{\partial}_z f(z)\bar{t})b - (\partial_z f(z)td + \bar{\partial}_z f(z)\bar{t}\bar{d})}{(\partial_z f(z)td + \bar{\partial}_z f(z)\bar{t}\bar{d}) - (\partial_z f(z)t + \bar{\partial}_z f(z)\bar{t})a} + o(t) \\
&= \left[a, b; c, \frac{\partial_z f(z)td + \bar{\partial}_z f(z)\bar{t}\bar{d}}{\partial_z f(z)t + \bar{\partial}_z f(z)\bar{t}}\right] + o(t)\,.
\end{aligned}$$

From the assumption we see that the first term must converge as t goes to 0 and hence be equal to $\lambda = [a, b; c, d]$. That is, we have

$$\frac{\partial_z f(z)td + \bar{\partial}_z f(z)\bar{t}\bar{d}}{\partial_z f(z)t + \bar{\partial}_z f(z)\bar{t}} = d\,.$$

This implies $\bar{\partial}_z f(z) = 0$ because $d \neq \bar{d}$. Thus f satisfies the Cauchy-Riemann equation. □

PROOF OF PROPOSITION 2.3. Choose $a, b, c, d \in \mathbf{C}$ such that $[a, b; c, d] = \lambda$. The condition $\lambda \neq 1$ implies $a \neq b$ and $c \neq d$. The formula (11) of Ahlfors [1] says that for a complex analytic function f

$$[f(z+ta), f(z+tb); f(z+tc), f(z+td)] = [a, b; c, d]\left(1 + \frac{1}{6}(a-b)(c-d)Sf(z)t^2 + o(t^2)\right),$$

where Sf is the Schwarzian derivative of f defined as

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

Therefore $[f(z+ta), f(z+tb); f(z+tc), f(z+td)] = \lambda$ yields $Sf(z) = 0$. This implies that f is a linear fractional function. □

References

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