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On the Exponents of 2-Multiarrangements

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Abstract. In this paper we study the exponents of 2-multiarrangements. More precisely, we compose a basis for $D(\mathcal{A}, k)$ in the case where $\mathcal{A}$ consists of three lines using **Q**-polynomials $\binom{X}{\lambda}$. Here $\binom{X}{\lambda}$ is the *generalized binomial coefficient* of the partition λ.

1. Introduction

Let V be an ℓ-dimensional vector space ($\ell > 0$) over a field of characteristic zero $\mathbf{K}$. Let $\mathcal{A}$ be a central hyperplane arrangement in V, that is, $\mathcal{A}$ is a finite set of codimension one subspaces of V. For simplicity, we call $\mathcal{A}$ an ℓ-arrangement. A pair $(\mathcal{A}, k)$ consisting of an ℓ-arrangement $\mathcal{A}$ and a **multiplicity** $k : \mathcal{A} \to \mathbf{N} = \mathbf{Z}_{\geq 0}$ is called an ℓ-**multiarrangement** in V. This term was introduced by G. Ziegler in [6]. We can regard any arrangement $\mathcal{A}$ as a multiarrangement with the constant multiplicity $k(H) = 1$ for all $H \in \mathcal{A}$. The restriction of an arrangement $\mathcal{A}$ to one of its hyperplanes is a typical example: Fix $H \in \mathcal{A}$ and define a multiarrangement $(\mathcal{A}^H, k)$ in H by $\mathcal{A}^H := \{H' \cap H \mid H' \in \mathcal{A} \setminus \{H\}\}$ and $k(X) := \#\{H' \in \mathcal{A} \setminus \{H\} \mid H' \cap H = X\}$.

Let V^* be the dual space of V and $S = \mathbf{K}[V]$ be the algebra of all polynomial functions on V which is equal to $\mathbf{K}[x_1, \ldots, x_\ell]$ for any basis $(x_1, \ldots, x_\ell)$ for V^*. The algebra S is naturally graded by $S = \bigoplus_{q \geq 0} S_q$ where S_q is the $\mathbf{K}$-vector space consisting of zero and all homogeneous polynomials of degree q. It is convenient to define $S_q = 0$ for $q < 0$. For each hyperplane H, we choose a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Let $(\mathcal{A}, k)$ be an ℓ-multiarrangement. Define a homogeneous polynomial $Q(\mathcal{A}, k) \in S$ by

$$Q(\mathcal{A}, k) := \prod_{H \in \mathcal{A}} {\alpha_H}^{k(H)} .$$

We call $Q(\mathcal{A}, k)$ the defining polynomial of the multiarrangement $(\mathcal{A}, k)$.

A $\mathbf{K}$-derivation of S is a $\mathbf{K}$-linear map $\theta : S \to S$ such that

$$\theta(fg) = \theta(f)g + f\theta(g) \quad (f, g \in S) .$$

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Let $\mathrm{Der}_{\mathbf{K}}(S)$ be the S-module of all $\mathbf{K}$-derivations of S. A non-zero $\mathbf{K}$-derivation θ is called a homogeneous derivation of degree q if $\theta(V^*) \subseteq S_q$. Let $\mathrm{Der}_{\mathbf{K}}(S)_q$ denote the $\mathbf{K}$-vector space consisting of zero and all homogeneous derivations of degree q. For each ℓ-multiarrangement $(\mathcal{A}, k)$, define an S-submodule $D(\mathcal{A}, k)$ of $\mathrm{Der}_{\mathbf{K}}(S)$ by

$$D(\mathcal{A}, k) := \{\theta \in \mathrm{Der}_{\mathbf{K}}(S) \mid \theta(\alpha_H) \in {\alpha_H}^{k(H)} S \text{ for any } H \in \mathcal{A}\} .$$

An element of $D(\mathcal{A}, k)$ is called an $(\mathcal{A}, k)$**-derivation**. For each $q \in \mathbf{Z}$, put $D(\mathcal{A}, k)_q := D(\mathcal{A}, k) \cap \mathrm{Der}_{\mathbf{K}}(S)_q$. Then $D(\mathcal{A}, k) = \bigoplus_{q \in \mathbf{Z}} D(\mathcal{A}, k)_q$. The S-module $D(\mathcal{A}, k)$ is graded by the direct sum decomposition. An ℓ-multiarrangement $(\mathcal{A}, k)$ is said to be **free** if $D(\mathcal{A}, k)$ is a free S-module. Then the degrees $\exp(\mathcal{A}, k) := [d_1, \ldots, d_\ell]$ of a homogeneous basis for $D(\mathcal{A}, k)$ are called the **exponents** of $(\mathcal{A}, k)$. For a given (multi)arrangement, it is important to examine its freeness. The following theorem is fundamental:

THEOREM 1.1 (G. Ziegler [6, Corollary 7]). *Every 2-multiarrangement is free.*

As for 3-arrangements, M. Yoshinaga [5, Theorem 3.2] showed the following:

THEOREM 1.2 (M. Yoshinaga [5, Theorem 3.2]). *Let $\mathcal{A}$ be a 3-arrangement which contains a hyperplane H. Put $\chi_0(\mathcal{A}, t) := (t-1)^{-1}\chi(\mathcal{A}, t)$, where $\chi(\mathcal{A}, t)$ is the characteristic polynomial of $\mathcal{A}$. Let $[d_1, d_2]$ be the exponents of the restricted multiarrangement $(\mathcal{A}^H, k)$. Then the dimension of the cokernel of the restriction mapping $\mathrm{res}^1_H : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}^H, k)$ is finite and is given by*

$$\chi_0(\mathcal{A}, 0) - d_1 \cdot d_2 .$$

By this theorem, we can characterize the freeness of 3-arrangements. Moreover, we can explicitly write the characteristic polynomial $\chi(\mathcal{A}, t) = \sum_{X \in L_{\mathcal{A}}} \mu(X) t^{\dim X}$ of a 3-arrangement $\mathcal{A}$ with $H \in \mathcal{A}$ as

$$\chi(\mathcal{A}, t) = (t-1)\{(t-d_1)(t-d_2) + \dim_{\mathbf{K}} \mathrm{coker}(\mathrm{res}^1_H)\} ,$$

where $\exp(\mathcal{A}^H, k) = [d_1, d_2]$.

Because of this theorem, the exponents $[d_1, d_2]$ of 2-multiarrangements are important in order to study the freeness of a 3-arrangement. We give some known examples of 2-multiarrangements and their exponents:

EXAMPLE 1.3. Let $(\mathcal{A}, k)$ be a 2-multiarrangement.

(1) If $\mathcal{A} = \{H\}$, then $\exp(\mathcal{A}, k) = [k(H), 0]$.

(2) If $\mathcal{A} = \{H_1, H_2\}$ $(H_1 \neq H_2)$, then $\exp(\mathcal{A}, k) = [k(H_1), k(H_2)]$.

EXAMPLE 1.4 (L. Solomon-H. Terao [4, §5. Examples 1], S. Yuzvinsky). Let $(\mathcal{A}, k)$ be a 2-multiarrangement with $\#\mathcal{A} \geq 2$ and $1 \leq k(H) \leq 2$ for any $H \in \mathcal{A}$. Then

$$\exp(\mathcal{A}, k) = \begin{cases} [n-1, \varepsilon+1] & \text{if } \varepsilon < n , \\ [n, n] & \text{if } \varepsilon = n \end{cases} ,$$

where $n = \#\mathcal{A}$ and $\varepsilon = \#\{H \in \mathcal{A} \mid k(H) = 2\}$.

In this paper we give explicitly a homogeneous basis for $D(\tilde{\mathcal{A}})$ using the *generalized binomial coefficients* $\binom{X}{\lambda} \in \mathbf{Q}[X]$ for any 2-multiarrangement $\tilde{\mathcal{A}}$ consisting of three lines. We give the definition of $\binom{X}{\lambda}$ in Definition 2.1, in Section two. We will essentially use the fact that a special value of the well-known *Schur function* can be written as a *generalized binomial coefficient* (Lemma 2.5). From this observation, we can describe the *generalized binomial coefficient* as the determinant of a matrix whose entries are the usual binomial coefficients (Theorem 2.8).

Let $\ell = \dim_{\mathbf{K}} V = 2$. To state our main theorem, we prepare some notations. For each triple of natural numbers $k = (k_1, k_2, k_3) \in \mathbf{N}^3$, define $|k| := k_1 + k_2 + k_3$ and

$$\mathbf{Z}_k := \left\{ q \in \mathbf{Z} \;\middle|\; \frac{|k|-1}{2} \le q \le k_1 + k_2 - 1 \right\}.$$

Put $r_{k,q} := k_1 + k_2 - q - 1$ and $s_{k,q} := k_1 + k_3 - q - 1$ for each $k \in \mathbf{N}^3$ and $q \in \mathbf{Z}$. In addition, define $\mathbf{N}_0^3 := \{k = (k_1, k_2, k_3) \in \mathbf{N}^3 \mid \max\{k_1, k_2\} \le k_3\}$. Let $\Sigma = (x, y)$ be a $\mathbf{K}$-basis for V^* and $(k, q) \in \mathbf{N}_0^3 \times \mathbf{Z}$ with $q \in \mathbf{Z}_k$. Define a homogeneous derivation $\theta_\Sigma(k, q)$ of degree q by

$$\theta_\Sigma(k,q) := \left(\sum_{j=1}^{q-k_1+1} \binom{k_3}{\lambda_{k,q}^{(j)}} x^{q+1-j} y^{j-1} \right) \frac{\partial}{\partial x} + (-1)^{r_{k,q}} \left(\sum_{j=k_2+1}^{|k|-q} \binom{k_3}{\lambda_{k,q}^{(j)}} x^{q+1-j} y^{j-1} \right) \frac{\partial}{\partial y},$$

where $\lambda_{k,q}^{(j)}$ are the following partitions:

$$\lambda_{k,q}^{(j)} := \begin{cases} (k_3 - j + 1, \underbrace{s_{k,q}+1, \ldots, s_{k,q}+1}_{r_{k,q}}) & j = 1, \ldots, q - k_1 + 1, \\ (\overbrace{s_{k,q}, \ldots, s_{k,q}}^{r_{k,q}}, |k| - q - j) & j = k_2 + 1, \ldots, |k| - q. \end{cases}$$

For each $\mathbf{K}$-basis $\Sigma = (x, y)$ for V^*, define a 2-arrangement $\mathcal{A}_\Sigma$ by

$$\mathcal{A}_\Sigma := \{\ker(x),\ \ker(y),\ \ker(x+y)\}.$$

Moreover for any $k \in \mathbf{N}^3$, we assume that $\mathcal{A}_{\Sigma,k}$ is the 2-multiarrangement on $\mathcal{A}_\Sigma$ with the multiplicity defined by $\ker(x) \mapsto k_1$, $\ker(y) \mapsto k_2$, $\ker(x+y) \mapsto k_3$. Note that we can express every 2-multiarrangement consisting of three lines as $\mathcal{A}_{\Sigma,k}$ for some $\mathbf{K}$-basis Σ for V^* and $k \in \mathbf{N}_0^3$.

The main result of this paper is the following:

THEOREM 1.5. *Let $\tilde{\mathcal{A}}$ be a 2-multiarrangement consisting of three lines, and write $\tilde{\mathcal{A}} = \mathcal{A}_{\Sigma,k}$ for some basis $\Sigma = (x, y)$ for V^* and $k \in \mathbf{N}_0^3$.*

If $k_1 + k_2 - 1 \leq k_3$, then

$$\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y},\ x^{k_1} y^{k_2}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)\right)$$

is a homogeneous basis for $D(\tilde{\mathcal{A}})$, where $f = \sum_{i=k_1}^{k_3} \binom{k_3}{i} x^i y^{k_3-i}$, $g = \sum_{i=0}^{k_1-1} \binom{k_3}{i} x^i y^{k_3-i}$.

If $k_3 < k_1 + k_2 - 1$, then

$$\left(\theta_\Sigma\left(k, \frac{|k|}{2}\right),\ \theta_\Sigma\left(k', \frac{|k|}{2}\right)\right) \qquad \textit{if } |k| \textit{ is even},$$

$$\left(\theta_\Sigma\left(k, \frac{|k|-1}{2}\right),\ \theta_\Sigma\left(k, \frac{|k|+1}{2}\right)\right) \quad \textit{if } |k| \textit{ is odd}$$

is a homogeneous basis for $D(\tilde{\mathcal{A}})$, where $k' = k + (0, 0, 1) \in \mathbf{N}_0^3$.

Throughout this paper, we use the following notation:

- $\# X$ or $|X|$: The cardinal number of a finite set X.
- $\mathbf{P}(q) := \{1, \ldots, q\}$, where $q \in \mathbf{Z}$. ($q \leq 0 \Rightarrow \mathbf{P}(q) = \emptyset$.)
- $[A]_{ij}$: The (i, j)-entry of a matrix A.
- ${}^t A$: The transpose matrix of a matrix A: $[{}^t A]_{ij} = [A]_{ji}$.
- $\mathbf{K}^n := \{ {}^t(a_1, \ldots, a_n) \mid a_i \in \mathbf{K} \}$. (The n-dimensional "column" vector space.)
- $\ker A := \{ \boldsymbol{u} \in \mathbf{K}^n \mid A\boldsymbol{u} = \mathbf{0} \}$ for an $m \times n$ $\mathbf{K}$-matrix A.
- For each (m, n)-type matrix $A = (a_{ij})$, $\alpha = \{i_1 < \cdots < i_p\} \subseteq \mathbf{P}(m)$ and $\beta = \{j_1 < \cdots < j_q\} \subseteq \mathbf{P}(n)$, we define

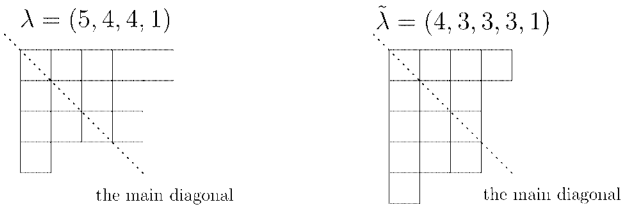
$$A[\alpha, \beta] := \begin{pmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_q} \\ \vdots & & \vdots \\ a_{i_p j_1} & \cdots & a_{i_p j_q} \end{pmatrix}.$$

2. Preliminaries for Generalized Binomial Coefficients

In this section we define the *generalized binomial coefficients* following I. G. Macdonald [1]. Furthermore, we describe some properties of them. In particular, the relation between the *Schur functions* and the *generalized binomial coefficients* is important (Lemma 2.5). This relation leads us to the expression for each *generalized binomial coefficient* as the determinant of a matrix consisting of the (usual) binomial coefficients (Theorem 2.8). The theorem plays a central role in this paper.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ be a partition. In other words, (1) $\lambda_1 \geq \lambda_2 \geq \cdots$ are non-negative integers, (2) there exists a positive integer $N \in \mathbf{Z}_{>0}$ such that $\lambda_n = 0$ for all $n \in \mathbf{Z}_{>0}$ whenever $n \geq N$. Regard a finite sequence $(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbf{N}^n$ of non-negative integers with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ as a partition $(\mu_1, \ldots, \mu_n, 0, 0, \ldots)$. Define the **weight** $|\lambda|$ and the

length $\ell(\lambda)$ by $|\lambda| := \sum_i \lambda_i$ and $\ell(\lambda) := \#\{i \in \mathbf{Z}_{>0} \mid \lambda_i \neq 0\}$. Moreover, define the **Young diagram** $\mathbf{Y}(\lambda)$ of λ by $\mathbf{Y}(\lambda) := \{(i, j) \in {\mathbf{Z}_{>0}}^2 \mid j \leq \lambda_i\}$. Sometimes we express the Young diagram of a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ by drawing the left-justified array of squares with λ_i squares in the i-th row. For each $i \geq 1$, define $\tilde{\lambda}_i := \#\{j \in \mathbf{Z}_{>0} \mid i \leq \lambda_j\} \in \mathbf{N}$. In particular, $\tilde{\lambda}_1 = \ell(\lambda)$. Then $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$ is also a partition. We call this partition the **conjugate** of λ. (e.g. $\lambda = (5, 4, 4, 1) \Rightarrow \tilde{\lambda} = (4, 3, 3, 3, 1)$.) By definition, $\mathbf{Y}(\tilde{\lambda}) = \{(j, i) \mid (i, j) \in \mathbf{Y}(\lambda)\}$. In other words, $\mathbf{Y}(\tilde{\lambda})$ is the diagram which is obtained by reflecting $\mathbf{Y}(\lambda)$ with respect to the main diagonal. In particular, it follows that $\tilde{\tilde{\lambda}} = \lambda$.



Define the **hook-length function** of λ $h_\lambda : {\mathbf{Z}_{>0}}^2 \to \mathbf{Z}$ by $h_\lambda(i, j) := \lambda_i - j + \tilde{\lambda}_j - i + 1$ ($\neq 0$). For each $P = (i_0, j_0) \in \mathbf{Y}(\lambda)$, $h_\lambda(P)$ expresses the number of points of the intersection $\mathbf{Y}(\lambda)$ and the hook H_P which has the right angle at P:

$$H_P = \{(i_0, j) \in {\mathbf{Z}_{>0}}^2 \mid j \geq j_0\} \cup \{(i, j_0) \in {\mathbf{Z}_{>0}}^2 \mid i \geq i_0\}\,.$$

(e.g. If $\lambda = (5, 4, 4, 1)$ and $P = (1, 2)$, then $h_\lambda(P) = 6$. See Figure 2.1.)

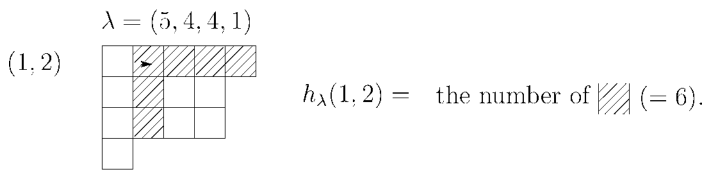


FIGURE 2.1: The hook-length function h_λ

Now we are ready to state the following.

DEFINITION 2.1. Let λ be a partition. Define a $\mathbf{Q}$-coefficient polynomial $\binom{X}{\lambda}$ by

$$\binom{X}{\lambda} := \prod_{(i,j)\in\mathbf{Y}(\lambda)} \frac{X - c(i, j)}{h_\lambda(i, j)}\,,$$

where $c(i, j) = j - i$. We call the polynomial $\binom{X}{\lambda}$ the **generalized binomial coefficient** (corresponding to λ).

EXAMPLE 2.2. (1) Let $\lambda = (5, 4, 4, 1)$ and $m = 7$. Then $h_\lambda(P)$ and $c(P)$ are as follows:

$$h_\lambda(P):\ \begin{array}{|c|c|c|c|c|}\hline 8&6&5&4&1\\ \hline 6&4&3&2\\ \cline{1-4} 5&3&2&1\\ \cline{1-4} 1\\ \cline{1-1}\end{array} \qquad m - c(P):\ \begin{array}{|c|c|c|c|c|}\hline 7&6&5&4&3\\ \hline 8&7&6&5\\ \cline{1-4} 9&8&7&6\\ \cline{1-4} 10\\ \cline{1-1}\end{array}$$

Each number in the square at P expresses $h_\lambda(P)$ and $m - c(P)$ respectively. Computing $\binom{m}{\lambda}$, we get

$$\binom{m}{\lambda} = \frac{10 \times 9 \times 8^2 \times 7^3 \times 6^3 \times 5^2 \times 4 \times 3}{(8 \times 6 \times 5 \times 4) \times (6 \times 4 \times 3 \times 2) \times (5 \times 3 \times 2)} = 30870\,.$$

(2) If $\lambda = (n, 0, 0, \dots)$ ($n \in \mathbf{N}$), then

$$\binom{X}{\lambda} = \frac{X(X-1)\cdots(X-n+1)}{n!} = \binom{X}{n},$$

which is usually called the binomial coefficient. In other words, regarding a natural number as a special partition, we can regard the generalized binomial coefficient as the usual one which is extended to every partition. This is the reason why we call $\binom{X}{\lambda}$ the generalized binomial coefficient.

LEMMA 2.3. *Let $r \in \mathbf{Q}$ and $\lambda = (\lambda_i)_{i \geq 1}$ be a partition. If $\lambda_1 \leq r$, then $\binom{r}{\lambda} > 0$.*

PROOF. Since $h_\lambda(i, j) > 0$ and

$$c(i, j) = j - i \leq \lambda_i - i \leq \lambda_1 - i < \lambda_1$$

for any $(i, j) \in \mathbf{Y}(\lambda)$, it follows that $\binom{r}{\lambda} > 0$. □

Fix a positive integer n. For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{N}^n$, we write ${X_1}^{\lambda_1} \cdots {X_n}^{\lambda_n} = X^\lambda$, where $X_1, \dots, X_n$ are variables over $\mathbf{Z}$. Define a polynomial $a_\lambda(X_1, \dots, X_n) \in \mathbf{Z}[X_1, \dots, X_n]$ by

$$a_\lambda = a_\lambda(X_1, \dots, X_n) := \det({X_j}^{\lambda_i})_{1 \leq i, j \leq n}\,.$$

If we substitute X_j for X_i in the polynomial a_λ, then $a_\lambda = 0$ for any (i, j) with $1 \leq i < j \leq n$. This means that a_λ is divisible in $\mathbf{Z}[X_1, \dots, X_n]$ by each of the differences $X_i - X_j$ $(1 \leq i < j \leq n)$ and hence by their product $a_\delta = \prod_{i<j}(X_i - X_j)$, where $\delta := (n-1, \dots, 2, 1, 0)$.

DEFINITION 2.4. Let λ be a partition of length $\leq n$. (Then we can regard $\lambda \in \mathbf{N}^n$.) Define the **Schur function** corresponding to λ by

$$S_\lambda = S_\lambda(X_1, \dots, X_n) := \frac{a_{\lambda+\delta}}{a_\delta}\,.$$

Then S_λ is a symmetric function for any partition λ with $\ell(\lambda) \leq n$.

A special value of the function S_λ can be expressed as a generalized binomial coefficient. The following lemma expresses this fact:

LEMMA 2.5 (cf. I. G. Macdonald [p. 45 Example 4]). *Let λ be a partition such that $\ell(\lambda) \leq n$. Then*

$$S_\lambda(1, 1, \ldots, 1) = \binom{n}{\tilde{\lambda}}.$$

For each integer $r \geq 0$, the r-th **elementary symmetric function** $e_r(X_1, \ldots, X_n) \in \mathbf{Z}[X_1, \ldots, X_n]$ is the sum of all products of r distinct variables X_i so that

$$e_r = e_r(X_1, \ldots, X_n) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} X_{i_1} \cdots X_{i_r}.$$

Define $e_r = 0$ for any $r < 0$. The following lemma is the basic proposition to connect the Schur function with the elementary symmetric functions:

LEMMA 2.6 (cf. I. G. Macdonald [p. 41 (3.5)]). *Let λ be a partition of length $\leq n$. Then*

$$S_\lambda = \det(e_{\tilde{\lambda}_i - i + j})_{1 \leq i, j \leq m},$$

for any positive integer m with $\lambda_1 \leq m$.

From Lemmas 2.5 and 2.6, we get the following:

LEMMA 2.7. *Let λ be a partition of length $\leq n$. Then*

$$\binom{n}{\tilde{\lambda}} = \det\left(\binom{n}{\tilde{\lambda}_i + c(i, j)}\right)_{1 \leq i, j \leq m}$$

for any positive integer m with $\lambda_1 \leq m$.

This lemma holds for arbitrary n. Therefore, we have the following theorem:

THEOREM 2.8 (cf. I. G. Macdonald [p. 45 Examples 4]). *Let $\lambda = (\lambda_i)_{i \geq 1}$ be a partition and m be a positive integer. If $\ell(\lambda) \leq m$, then*

$$\binom{X}{\lambda} = \det\left(\binom{X}{\lambda_i + c(i, j)}\right)_{1 \leq i, j \leq m}.$$

3. Proof of Theorem 1.5

In this section we will prove Theorem 1.5. First we prepare two criteria for the freeness of multiarrangements. We recall that $\mathrm{Der}_{\mathbf{K}}(S)$ is the S-module of all $\mathbf{K}$-derivations of the symmetric algebra $S = \mathbf{K}[V]$. For simplicity, write $\mathrm{Der}_S := \mathrm{Der}_{\mathbf{K}}(S)$. Let $(x_1, \ldots, x_\ell)$ be a $\mathbf{K}$-basis for V^*. For given derivations $\theta_1, \ldots, \theta_\ell \in \mathrm{Der}_S$, define the coefficient matrix (with respect to the basis $(x_1, \ldots, x_\ell)$ for V^*) $M = M(\theta_1, \ldots, \theta_\ell)$ by $[M]_{ij} = \theta_j(x_i)$. By definition, we can write $\theta_j = \sum_i [M]_{ij}\partial_i$, where ∂_i is the usual derivation $\frac{\partial}{\partial x_i}$. Then we have the following criterion:

THEOREM 3.1 (Ziegler's criterion [6]). *Let $\theta_1, \dots, \theta_\ell$ be $(\mathcal{A}, k)$-derivations. Then they form a basis for $D(\mathcal{A}, k)$ if and only if* $\det M(\theta_1, \dots, \theta_\ell) \doteq Q(\mathcal{A}, k)$.

Here and elsewhere $\doteq$ stands for equality up to a nonzero constant multiple: $f \doteq g \Leftrightarrow f = cg$ for some $c \in \mathbf{K}^*$ $(f, g \in S)$. This criterion is the "multi-version" of Saito's criterion [2, Theorem 4.19], [3, p. 270]. The following theorem can easily be derived from Ziegler's criterion:

THEOREM 3.2. *Let $\theta_1, \dots, \theta_\ell \in D(\mathcal{A}, k)$ be homogeneous and linearly independent over S. Then $(\theta_1, \dots, \theta_\ell)$ is a basis for $D(\mathcal{A}, k)$ if and only if*

$$\sum_{j=1}^{\ell} \deg \theta_j = \sum_{H \in \mathcal{A}} k(H)\,.$$

Next we list some basic properties of 2-multiarrangements.

LEMMA 3.3. *Let $(\mathcal{A}, k)$ be a 2-multiarrangement and put $|k| = \sum_{H\in\mathcal{A}} k(H)$.*
(1) *If $q_0 = \min\{q \in \mathbf{Z} \mid D(\mathcal{A}, k)_q \neq 0\}$, then* $\exp(\mathcal{A}, k) = [q_0, |k| - q_0]$.
(2) *Let $H \in \mathcal{A}$ and $q \in \mathbf{Z}$. If $q < \min\{k(H), |k| - k(H)\}$, then $D(\mathcal{A}, k)_q = 0$.*
(3) *If $(|k| - 1)/2 \le q \in \mathbf{Z}$, then $D(\mathcal{A}, k)_q \neq 0$.*
(4) *Let $H \in \mathcal{A}$. If $(|k| - 1)/2 \le k(H)$, then* $\exp(\mathcal{A}, k) = [k(H), |k| - k(H)]$.

PROOF. (1) Write $\exp(\mathcal{A}, k) = [d_1, d_2]$ ($d_1 \le d_2$). Then it follows that $d_1 = q_0$, since the Poincaré series $\mathrm{Poin}(D(\mathcal{A}, k), t) = \sum_{q\in\mathbf{Z}}(\dim_{\mathbf{K}} D(\mathcal{A}, k)_q)t^q$ is equal to $(t^{d_1} + t^{d_2})/(1 - t)^2$. Moreover by Theorem 3.2, $d_2 = |k| - q_0$. Thus we have $\exp(\mathcal{A}, k) = [q_0, |k| - q_0]$.

(2) We choose coordinates (x, y) so that $x = \alpha_H$. Then $\theta(x) \in S_q \cap x^{k(H)}S = x^{k(H)}S_{q-k(H)}$. Since $q < k(H)$, $\theta(x) = 0$. Next we show that $\theta(y) = 0$. Let $Q = Q(\mathcal{A}, k)/\alpha_H{}^{k(H)}$. Since $\theta(x) = 0$, it follows that $\theta(y)\frac{\partial \alpha_{H'}}{\partial y} \in \alpha_{H'}{}^{k(H')}S$ for any $H' \in \mathcal{A} \setminus \{H\}$. Since the polynomials $\alpha_{H'}{}^{k(H')}$ are relatively prime and $\frac{\partial \alpha_{H'}}{\partial y} \neq 0$, we have $\theta(y) \in QS$. On the other hand, $\theta(y) \in S_q$. By the assumption, we obtain $\theta(y) = 0$.

(3) Suppose that there is an integer $q \ge (|k| - 1)/2$ such that $D(\mathcal{A}, k)_q = 0$. Write $\exp(\mathcal{A}, k) = [d_1, d_2]$. Then since $q + 1 \le d_1, d_2$, we have $|k| + 1 \le 2q + 2 \le d_1 + d_2$. It follows from Theorem 3.2 that $|k| + 1 \le |k|$. This is a contradiction.

(4) Put $m := \min\{q \in \mathbf{Z} \mid D(\mathcal{A}, k)_q \neq 0\}$ and $Q := Q(\mathcal{A}, k)/\alpha_H{}^{k(H)}$. Let (x, y) be a basis for V^* where $x = \alpha_H$. Since $(|k|-1)/2 \le k(H)$, it follows from (3) that $D(\mathcal{A}, k)_{k(H)} \neq 0$. Thus we have $m \le \min\{k(H), |k| - k(H)\}$ because $Q\frac{\partial}{\partial y} \in D(\mathcal{A}, k)$ and $\deg Q\frac{\partial}{\partial y} = |k| - k(H)$. On the other hand, from (2) and by the definition of m, $\min\{k(H), |k| - k(H)\} \le m$. Thus we have $m = \min\{k(H), |k| - k(H)\}$. From (1), we can conclude that $\exp(\mathcal{A}, k) = [k(H), |k| - k(H)]$. □

LEMMA 3.4. *Let* $M = \bigoplus_{n\in\mathbf{Z}} M_n$ *be a free graded* S*-module with a homogeneous basis* (δ_1, δ_2) *such that* $\deg\delta_1 \le \deg\delta_2$*. Put* $p := \deg\delta_1$*,* $q := \deg\delta_2$ *and* $d := q - p$*. If* $\theta_1 \in M_p$*,* $\theta_2 \in M_q$ *and* $x^d\theta_1, x^{d-1}y\theta_1, \dots, y^d\theta_1, \theta_2$ *are linearly independent over* $\mathbf{K}$*, then* (θ_1, θ_2) *is a basis for* M*, where* (x, y) *is a* $\mathbf{K}$*-basis for* V^**.*

PROOF. Since (δ_1, δ_2) is a basis for M, there exist $a, b \in \mathbf{K}$, $f \in S_{-d}$ and $g \in S_d$ such that $\theta_1 = a\delta_1 + f\delta_2$, $\theta_2 = g\delta_1 + b\delta_2$. Define a matrix A by

$$A = \begin{pmatrix} a & g \\ f & b \end{pmatrix}.$$

Then $(\theta_1, \theta_2) = (\delta_1, \delta_2)A$. Consider the following two cases.

Case 1: $d = 0$. In this case, $f, g \in \mathbf{K}$. In other words, A is a $\mathbf{K}$-matrix. It follows that $\det A \in \mathbf{K}^* = \mathbf{K} \setminus \{0\}$, since θ_1, θ_2 are linearly independent over $\mathbf{K}$. Thus (θ_1, θ_2) is a basis for M.

Case 2: $d > 0$. In this case, $f = 0$ because $f \in S_{-d}$. Write $g = \sum_{i=0}^{d} a_i x^{d-i} y^i$ with $a_i \in \mathbf{K}$ and let

$$A' = \left(\begin{array}{ccc|c} a & \cdots & 0 & a_0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a & a_d \\ \hline 0 & \cdots & 0 & b \end{array}\right)$$

Then $(x^d\theta_1, x^{d-1}y\theta_1, \dots, y^d\theta_1, \theta_2) = (x^d\delta_1, x^{d-1}y\delta_1, \dots, y^d\delta_1, \delta_2)A'$. It follows from the assumption that $a^{d+1}b = \det A' \ne 0$ and hence $\det A = ab \ne 0$. Thus we can conclude that (θ_1, θ_2) is a basis for M. □

We retain the notation of Section one. Now we start preparing for the proof of Theorem 1.5. Fix a $\mathbf{K}$-basis $\Sigma = (x, y)$ for V^* and $k = (k_1, k_2, k_3) \in \mathbf{N}_0^3$. For each $q \in \mathbf{Z}$, put $r_q := r_{k,q} = k_1 + k_2 - q - 1$, $s_q := s_{k,q} = k_1 + k_3 - q - 1$ and $t_q := q - k_3 + 1$. Moreover for each $q \in \mathbf{Z}$ with $q \ge k_3$, define a $(q + 1, t_q)$-type matrix M_q by

$$M_q := \left(\binom{k_3}{k_3 + c(i, j)}\right)_{\substack{1\le i\le q+1 \\ 1\le j\le t_q}}.$$

Here, when $m < n$ or $n < 0$, the value of the binomial coefficient $\binom{m}{n}$ is set to zero ($m, n \in \mathbf{N}$) and $c(i, j) = j - i$. Then it follows that

$$(X^q, X^{q-1}Y, \dots, Y^q)M_q = (X + Y)^{k_3}(X^{q-k_3}, X^{q-k_3-1}Y, \dots, Y^{q-k_3}), \tag{3.1}$$

for any $q \in \mathbf{Z}$ with $q \ge k_3$. For each $q \in \mathbf{Z}$, put $\alpha_q := \mathbf{P}(q-k_1+1)$ and $\beta_q := \mathbf{P}(q+1)\setminus\mathbf{P}(k_2)$. ($\alpha_q$ and β_q are subsets of $\mathbf{P}(q + 1)$.) When $q \ge k_3$, define

$$A_q := M_q[\alpha_q, \mathbf{P}(t_q)], \quad B_q := M_q[\beta_q, \mathbf{P}(t_q)].$$

In other words, A_q (resp. B_q) is the matrix consisting of the first (resp. last) $q - k_1 + 1$ (resp. $q - k_2 + 1$) rows of M_q. Furthermore define $\boldsymbol{f}_q := (x^q, x^{q-1}y, \ldots, x^{k_1}y^{q-k_1})A_q$, $\boldsymbol{g}_q := (x^{q-k_2}y^{k_2}, \ldots, xy^{q-1}, y^q)B_q$ and a $\mathbf{K}$-linear mapping $\rho_q : \mathbf{K}^{t_q} \longrightarrow (\mathrm{Der}_S)_q$ by

$$\rho_q(\boldsymbol{u}) := \boldsymbol{f}_q\boldsymbol{u}\frac{\partial}{\partial x} + \boldsymbol{g}_q\boldsymbol{u}\frac{\partial}{\partial y}$$

($\boldsymbol{u} \in \mathbf{K}^{t_q}$ is a column vector), for each $q \in \mathbf{Z}$ with $q \geq k_3$.

LEMMA 3.5. *The* $\mathbf{K}$*-linear mapping* ρ_q *is injective for all* $q \in \mathbf{Z}$ *such that* $q \geq k_3$.

PROOF. Since $[A_q]_{ii} = 1$ for all i, $[A_q]_{ij} = 0$ for all (i, j) with $j > i$ and $q - k_1 + 1 \geq q - k_3 + 1 = t_q$, it follows that $\ker A_q = 0$. Thus we have

$$\begin{aligned} \rho_q(\boldsymbol{u}) = 0 &\Rightarrow \boldsymbol{f}_q\boldsymbol{u} = 0 \\ &\Rightarrow A_q\boldsymbol{u} = \mathbf{0} \\ &\Rightarrow \boldsymbol{u} = \mathbf{0}\,, \end{aligned}$$

for any $\boldsymbol{u} \in \mathbf{K}^{t_q}$. This completes the proof. □

For each $q \in \mathbf{Z}$, put $\gamma_q := \mathbf{P}(q+1) \setminus (\alpha_q \cup \beta_q) \subseteq \mathbf{P}(q+1)$. If $k_3 \leq q < k_1 + k_2 - 1$, then $\gamma_q = \{q - k_1 + 2, \ldots, k_2\} \neq \emptyset$. Therefore we can define a (r_q, t_q)-type matrix C_q by

$$C_q := M_q[\gamma_q, \mathbf{P}(t_q)] = \left(\binom{k_3}{s_q + c(i, j)} \right)_{\substack{1 \leq i \leq r_q \\ 1 \leq j \leq t_q}}.$$

Then it follows that

$$M_q = \begin{pmatrix} A_q \\ \hline C_q \\ \hline B_q \end{pmatrix}.$$

Moreover, define a subspace W_q of $\mathbf{K}^{t_q}$ by

$$W_q := \begin{cases} \mathbf{K}^{t_q} & \text{if } q = k_1 + k_2 - 1\,, \\ \ker C_q & \text{if } q < k_1 + k_2 - 1 \end{cases}$$

for each $q \in \mathbf{Z}$ with $k_3 \leq q \leq k_1 + k_2 - 1$.

LEMMA 3.6. $\rho_q(W_q) = D(\mathcal{A}_{\Sigma,k})_q$ *for all* $q \in \mathbf{Z}$ *with* $k_3 \leq q \leq k_1 + k_2 - 1$. *In particular, it follows from Lemma* 3.5 *that* W_q *and* $D(\mathcal{A}_{\Sigma,k})_q$ *are isomorphic as* $\mathbf{K}$*-vector spaces.*

PROOF. First we show that $\rho_q(W_q) \subseteq D(\mathcal{A}_{\Sigma,k})_q$. Let $\boldsymbol{u} \in W_q$ and put $\theta := \rho_q(\boldsymbol{u})$. Then we have

$$\begin{aligned} \theta(x) &= \boldsymbol{f}_q\boldsymbol{u} = (x^q, x^{q-1}y, \ldots, x^{k_1}y^{q-k_1})A_q\boldsymbol{u} \in x^{k_1}S\,, \\ \theta(y) &= \boldsymbol{g}_q\boldsymbol{u} = (x^{q-k_2}y^{k_2}, \ldots, xy^{q-1}, y^q)B_q\boldsymbol{u} \in y^{k_2}S\,, \end{aligned}$$

$$\theta(x+y) = \boldsymbol{f}_q\boldsymbol{u} + \boldsymbol{g}_q\boldsymbol{u} = (x^q, x^{q-1}y, \ldots, y^q)M_q\boldsymbol{u}$$
$$= (x+y)^{k_3}(x^{q-k_3}, x^{q-k_3-1}y, \ldots, y^{q-k_3})\boldsymbol{u} \in (x+y)^{k_3}S\,.$$

The last equality follows from (3.1). Thus $\theta \in D(\mathcal{A}_{\Sigma,k})$. Since this holds for any $\boldsymbol{u} \in W_q$, we can conclude that $\rho_q(W_q) \subseteq D(\mathcal{A}_{\Sigma,k})_q$. Next we show that $\rho_q(W_q) \supseteq D(\mathcal{A}_{\Sigma,k})_q$. Let $\theta \in D(\mathcal{A}_{\Sigma,k})_q$. Then we get

$$\theta(x) \in x^{k_1}S \cap S_q = \bigoplus_{i=k_1}^{q} \mathbf{K}x^i y^{q-i}\,, \tag{3.2}$$

$$\theta(y) \in y^{k_2}S \cap S_q = \bigoplus_{i=0}^{q-k_2} \mathbf{K}x^i y^{q-i}\,. \tag{3.3}$$

Since $\theta(x+y) \in (x+y)^{k_3}S_{q-k_3}$, there exists $\boldsymbol{u} \in \mathbf{K}^{t_q}$ such that

$$\theta(x+y) = (x^q, x^{q-1}y, \ldots, y^q)M_q\boldsymbol{u}\,. \tag{3.4}$$

By (3.2), (3.3) and (3.4), we have $\theta(x) = \boldsymbol{f}_q\boldsymbol{u}$, $\theta(y) = \boldsymbol{g}_q\boldsymbol{u}$. In other words $\theta = \rho_q(\boldsymbol{u})$. Moreover we get $C_q\boldsymbol{u} = \mathbf{0}$, if $q < k_1 + k_2 - 1$. Thus $\theta \in \rho_q(W_q)$. Since this holds for any θ, we can conclude that $D(\mathcal{A}_{\Sigma,k})_q \subseteq \rho_q(W_q)$. □

The next result follows from Lemma 3.6.

LEMMA 3.7. *If* $k_3 \le k_1 + k_2$, *then* $\exp(\mathcal{A}_{\Sigma,k}) = [\lfloor \frac{|k|}{2} \rfloor,\ \lceil \frac{|k|}{2} \rceil]$. *Here* $\lfloor a \rfloor = \max\{m \in \mathbf{Z} \mid m \le a\}$ *and* $\lceil a \rceil = \min\{m \in \mathbf{Z} \mid a \le m\}$ *for any* $a \in \mathbf{R}$.

PROOF. Since $(|k| - 1)/2 \le \lfloor |k|/2 \rfloor$, it follows from Lemma 3.3 (3) that $D(\mathcal{A}_{\Sigma,k})_{\lfloor \frac{|k|}{2} \rfloor} \neq 0$. Next we show that $D(\mathcal{A}_{\Sigma,k})_q = 0$ for any integer q with $q < \lfloor |k|/2 \rfloor$. Let q be an integer which satisfies $q < \lfloor |k|/2 \rfloor$. (Then $t_q \le r_q$.) If $q < k_3$, then it follows from Lemma 3.3 (2) that $D(\mathcal{A}_{\Sigma,k})_q = 0$ since $k_3 \le k_1 + k_2$. Thus we may assume that $k_3 \le q$, namely, $t_q \ge 1$. Define a partition λ by $\lambda = (s_q, \ldots, s_q) \in \mathbf{N}^{t_q}$. Then it follows from Theorem 2.8 that

$$\det C_q[\mathbf{P}(t_q), \mathbf{P}(t_q)] = \binom{k_3}{\lambda}\,. \tag{3.5}$$

On the other hand, since $k_3 \ge s_q$, it follows from Lemma 2.3 that $\binom{k_3}{\lambda} > 0$. From this inequality and (3.5), we have $\det C_q[\mathbf{P}(t_q), \mathbf{P}(t_q)] \neq 0$ and hence $W_q = \ker C_q = 0$. By Lemma 3.6, we get $D(\mathcal{A}_{\Sigma,k})_q = 0$. Thus we can conclude from Lemma 3.3 (1) that $\exp(\mathcal{A}_{\Sigma,k}) = [\lfloor |k|/2 \rfloor,\ \lceil |k|/2 \rceil]$. □

Any 2-multiarrangement consisting of three lines is of the form $\mathcal{A}_{\Sigma,k}$ for some $\mathbf{K}$-basis Σ for V^* and $k \in \mathbf{N}_0^3$. Thus we can completely determine the exponents $\exp(\mathcal{A}, k)$ for all 2-multiarrangements $(\mathcal{A}, k)$ with $|\mathcal{A}| = 3$ from Lemmas 3.3 (4) and 3.7.

THEOREM 3.8. *Let* $(\mathcal{A}, k)$ *be a* 2*-multiarrangement with* $|\mathcal{A}| = 3$. *Put* $|k| := \sum_{H \in \mathcal{A}} k(H)$ *and* $m := \max\{k(H) \mid H \in \mathcal{A}\}$. *Then*

$$\exp(\mathcal{A}, k) = \begin{cases} [m,\ |k| - m] & \text{if } \frac{|k|-1}{2} \leq m\,, \\ [\lfloor \frac{|k|}{2} \rfloor, \lceil \frac{|k|}{2} \rceil] & \text{if } m \leq \frac{|k|}{2}\,. \end{cases}$$

We proceed to the proof of Theorem 1.5. Define two $\mathbf{K}$-linear mappings

$$\varphi_q, \psi_q : \mathbf{K}^{t_q} \Rightarrow \mathbf{K}^{t_q+1}$$

by $\varphi_q(\boldsymbol{u}) := \binom{\boldsymbol{u}}{0}$, $\psi_q(\boldsymbol{u}) := \binom{0}{\boldsymbol{u}}$, for $q \in \mathbf{Z}$ with $k_3 \leq q < k_1 + k_2 - 1$.

LEMMA 3.9. *The following diagrams are commutative*:

$$\begin{array}{ccc} W_q & \xrightarrow{\varphi_q|_{W_q}} & \mathbf{K}^{t_q+1} \\ {\scriptstyle \rho_q|_{W_q}}\downarrow & & \downarrow{\scriptstyle \rho_{q+1}} \\ (\mathrm{Der}_S)_q & \xrightarrow[x\cdot]{} & (\mathrm{Der}_S)_{q+1} \end{array}\ , \quad \begin{array}{ccc} W_q & \xrightarrow{\psi_q|_{W_q}} & \mathbf{K}^{t_q+1} \\ {\scriptstyle \rho_q|_{W_q}}\downarrow & & \downarrow{\scriptstyle \rho_{q+1}} \\ (\mathrm{Der}_S)_q & \xrightarrow[y\cdot]{} & (\mathrm{Der}_S)_{q+1} \end{array}\ .$$

In particular, it follows from Lemmas 3.5 *and* 3.6 *that* $\varphi_q(W_q) \cap \psi_q(W_q) \subseteq W_{q+1}$.

PROOF. Let $\boldsymbol{x}^{(i)}$ be the i-th row of the matrix C_q. Then we have

$$A_{q+1} = \left(\begin{array}{c|c} A_q & * \\ \hline \boldsymbol{x}^{(1)} & \binom{k_3}{k_1} \end{array}\right) = \left(\begin{array}{c|c} 1 & {}^t\mathbf{0} \\ \hline * & A_q \end{array}\right),$$

$$B_{q+1} = \left(\begin{array}{c|c} B_q & * \\ \hline {}^t\mathbf{0} & 1 \end{array}\right) = \left(\begin{array}{c|c} \binom{k_3}{k_2} & \boldsymbol{x}^{(t_q)} \\ \hline * & B_q \end{array}\right).$$

Let $\boldsymbol{u} \in W_q = \ker C_q$ and put $\bar{\boldsymbol{u}} := \varphi_q(\boldsymbol{u})$. It follows from the above expressions that $A_{q+1}\bar{\boldsymbol{u}} = \binom{A_q\boldsymbol{u}}{0}$ and $B_{q+1}\bar{\boldsymbol{u}} = \binom{B_q\boldsymbol{u}}{0}$. Thus we have

$$\begin{aligned} \boldsymbol{f}_{q+1}\bar{\boldsymbol{u}} &= (x^{q+1}, x^q y, \ldots, x^{k_1+1}y^{q-k_1}, x^{k_1}y^{q+1-k_1})A_{q+1}\bar{\boldsymbol{u}} \\ &= x \cdot \boldsymbol{f}_q\boldsymbol{u}\,, \\ \boldsymbol{g}_{q+1}\bar{\boldsymbol{u}} &= (x^{q+1-k_2}y^{k_2}, \ldots, xy^q, y^{q+1})B_{q+1}\bar{\boldsymbol{u}} \\ &= x \cdot \boldsymbol{g}_q\boldsymbol{u}\,, \end{aligned}$$

and hence $\rho_{q+1}(\bar{\boldsymbol{u}}) = x \cdot \rho_q(\boldsymbol{u})$. Since this holds for any $\boldsymbol{u} \in W_q$, the left diagram is commutative. Similarly, we can show that the right diagram is commutative. □

Let $q \in \mathbf{Z}_k$. (Then $0 \leq r_q < t_q$.) For $j \in \alpha_q \cup \beta_q$, put

$$\Delta_q^{(j)} := \det M_q[\gamma_q \cup \{j\}, \mathbf{P}(r_q + 1)]\,.$$

Here we recall the partitions $\lambda_{k,q}^{(j)}$ and the derivation $\theta_\Sigma(k,q)$ (see Section one). By Theorem 2.8, it follows that

$$\Delta_q^{(j)} = \begin{cases} \binom{k_3}{\lambda_{k,q}^{(j)}} & \text{if } \ j \in \alpha_q \cup (\beta_q \cap \mathbf{P}(|k| - q))\,, \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$\theta_\Sigma(k,q) = \left(\sum_{j\in\alpha_q} \Delta_q^{(j)} x^{q+1-j} y^{j-1}\right)\frac{\partial}{\partial x} + (-1)^{r_q}\left(\sum_{j\in\beta_q} \Delta_q^{(j)} x^{q+1-j} y^{j-1}\right)\frac{\partial}{\partial y}\,. \tag{3.6}$$

The following result is the key lemma for the proof of Theorem 1.5:

LEMMA 3.10. *$\theta_\Sigma(k,q) \in D(\mathcal{A}_{\Sigma,k}) \setminus D(\mathcal{A}_{\Sigma,k'})$, where $k' = k + (0,0,1) \in \mathbf{N}_0^3$.*

PROOF. First we claim that $\theta_\Sigma(k,q) \in D(\mathcal{A}_{\Sigma,k})$. If $q = k_1 + k_2 - 1$, then $\Delta_q^{(j)} = [M_q]_{j,1}$. Putting $\boldsymbol{u}_q = {}^t(1,0,\ldots,0)$, we have $\theta_\Sigma(k,q) = \rho_q(\boldsymbol{u}_q)$ from (3.6). By Lemma 3.6, it follows that $\theta_\Sigma(k,q) \in D(\mathcal{A}_{\Sigma,k})$ because $W_q = \mathbf{K}^{t_q}$ in this case. When $q \neq k_1 + k_2 - 1$, put $C'_q := M_q[\gamma_q, \mathbf{P}(r_q)]$ and $C_q^{(i)} := M[\gamma_q, (\mathbf{P}(r_q) \setminus \{i\}) \cup \{r_q + 1\}]$ for any $i = 1, \ldots, r_q$. Define a vector $\boldsymbol{u}_q \in \mathbf{K}^{t_q}$ by

$$\boldsymbol{u}_q := {}^t(\det C_q^{(1)}, -\det C_q^{(2)}, \ldots, (-1)^{r_q-1}\det C_q^{(r_q)}, (-1)^{r_q}\det C'_q, \overbrace{0,\ldots,0}^{t_q-r_q-1})\,,$$

then $\boldsymbol{u}_q \in W_q = \ker C_q$ and $\theta_\Sigma(k,q) = \rho_q(\boldsymbol{u}_q)$. Thus we can conclude that $\theta_\Sigma(k,q) \in D(\mathcal{A}_{\Sigma,k})$. Next we show that $\theta_\Sigma(k,q) \notin D(\mathcal{A}_{\Sigma,k'})$. From (3.1),

$$[\rho_q(\boldsymbol{u})](x+y) = (x+y)^{k_3}(x^{q-k_3}, x^{q-k_3-1}y, \ldots, y^{q-k_3})\boldsymbol{u}$$

for any $\boldsymbol{u} \in W_q$. Thus we have the following:

$$(*)\ \text{For } \boldsymbol{u} \in W_q,\ \rho_q(\boldsymbol{u}) \in D(\mathcal{A}_{\Sigma,k'}) \Leftrightarrow (x^{q-k_3}, x^{q-k_3-1}y, \ldots, y^{q-k_3})\boldsymbol{u} \in (x+y)S$$
$$\Leftrightarrow (1,-1,\ldots,(-1)^{t_q-1})\boldsymbol{u} = 0\,.$$

If $q = k_1 + k_2 - 1$, then $\boldsymbol{u}_q = {}^t(1,0,\ldots,0)$. It follows from $(*)$ that $\theta_\Sigma(k,q) = \rho_q(\boldsymbol{u}_q) \notin D(\mathcal{A}_{\Sigma,k'})$. In $q \neq k_1 + k_2 - 1$ case, define partitions μ_i $(i = 1, 2, \ldots, r_q)$ by

$$\mu_i := (\overbrace{s_q+1,\ldots,s_q+1}^{r_q-i+1}, \overbrace{s_q,\ldots,s_q}^{i-1})\,.$$

Then $\det C_q^{(i)} = \binom{k_3}{\mu_i}$ by Theorem 2.8. Since $k_3 \geq s_q + 1$, it follows from Lemma 2.3 that $\det C_q^{(i)} = \binom{k_3}{\mu_i} > 0$. Similarly, if $\mu := (\overbrace{s_q, \dots, s_q}^{r_q})$, then $\det C'_q = \binom{k_3}{\mu} > 0$. Thus we have

$$(1, -1, \dots, (-1)^{t_q - 1})\boldsymbol{u}_q = \sum_{i=1}^{r_q} \det C_q^{(i)} + \det C'_q > 0\,.$$

We can conclude from $(*)$ that $\theta_\Sigma(k, q) = \rho_q(\boldsymbol{u}_q) \notin D(\mathcal{A}_{\Sigma,k'})$. □

Now we prove Theorem 1.5.

PROOF OF THEOREM 1.5. Case 1 : $k_1 + k_2 - 1 \leq k_3$. Put

$$\theta_1 := f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}, \quad \theta_2 := x^{k_1} y^{k_2}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right),$$

where $f = \sum_{i=k_1}^{k_3} \binom{k_3}{i} x^i y^{k_3 - i}$ and $g = \sum_{i=0}^{k_1 - 1} \binom{k_3}{i} x^i y^{k_3 - i}$. By definition, $\theta_1(x) = f \in x^{k_1} S$. Since $k_1 + k_2 - 1 \leq k_3$, it follows that $k_2 \leq k_3 - i$ for each $i = 0, \dots, k_1 - 1$, and hence $\theta_1(y) = g \in y^{k_2} S$. Moreover $\theta_1(x + y) = f + g = (x + y)^{k_3}$. Thus we can conclude that $\theta_1 \in D(\mathcal{A}_{\Sigma,k})$. Since $\theta_2(x + y) = 0$, it is standard to see that θ_2 is a $\mathcal{A}_{\Sigma,k}$-derivation. Here compute $\det M(\theta_1, \theta_2)$:

$$\det M(\theta_1, \theta_2) = x^{k_1} y^{k_2} \begin{vmatrix} f & -1 \\ g & 1 \end{vmatrix} = x^{k_1} y^{k_2} (x + y)^{k_3}\,.$$

It follows from Ziegler's criterion 3.1 that (θ_1, θ_2) is a basis for $D(\mathcal{A}_{\Sigma,k})$.

Case 2: $k_3 < k_1 + k_2 - 1$. By Lemma 3.10, it follows that

(i) $\theta_\Sigma(k, q), \theta_\Sigma(k', q) \in D(\mathcal{A}_{\Sigma,k})$ are linearly independent over $\mathbf{K}$, for any $q \in \mathbf{Z}_{k'} \subseteq \mathbf{Z}_k$.

For any $q \in \mathbf{Z}_k$ with $q + 1 \in \mathbf{Z}_k$, $\varphi_q(\boldsymbol{u}_q)$, $\psi_q(\boldsymbol{u}_q)$ and $\boldsymbol{u}_{q+1} \in \mathbf{K}^{t_q + 1}$ are linearly independent over $\mathbf{K}$, where $\boldsymbol{u}_q$ is the vector defined in the proof of Lemma 3.10. By the injectivity of ρ_{q+1} (Lemma 3.5) and Lemma 3.9, we obtain the following:

(ii) $x \cdot \theta_\Sigma(k, q)$, $y \cdot \theta_\Sigma(k, q)$, $\theta_\Sigma(k, q + 1)$ are linearly independent over $\mathbf{K}$, for any $q \in \mathbf{Z}_k$ such that $q + 1 \in \mathbf{Z}_k$.

When $|k|$ is even, we apply (i) to $q = \frac{|k|}{2} \in \mathbf{Z}_{k'}$. Then from Lemmas 3.4 and 3.7, $(\theta_\Sigma(k, \frac{|k|}{2}),\ \theta_\Sigma(k', \frac{|k|}{2}))$ is a homogeneous basis for $D(\mathcal{A}_{\Sigma,k})$. When $|k|$ is odd, we apply (ii) to $q = \frac{|k|-1}{2}$. Then from Lemmas 3.4 and 3.7, $(\theta_\Sigma(k, \frac{|k|-1}{2}),\ \theta_\Sigma(k, \frac{|k|+1}{2}))$ is a homogeneous basis for $D(\mathcal{A}_{\Sigma,k})$. In both cases, we can prove Theorem 1.5. □

4. Some Examples

We will give some examples. Let $\Sigma = (x, y)$ be a $\mathbf{K}$-basis for V^* and $k = (k_1, k_2, k_3) \in \mathbf{N}_0^3$.

EXAMPLE 4.1. Suppose that $k_3 = k_1 + k_2 - 2$ (e.g. $k = (3, 3, 4), (3, 4, 5), (4, 4, 6), \ldots$). Then $\lfloor \frac{|k|}{2} \rfloor = \lceil \frac{|k|}{2} \rceil = k_3 + 1$ and $r_{k,k_3+1} = 0$. From Theorem 1.5,

$$\theta_1 := \left(\sum_{j=0}^{k_2-1} \binom{k_3}{j} x^{k_3+1-j} y^j \right) \frac{\partial}{\partial x} + \left(\sum_{j=k_2}^{k_3} \binom{k_3}{j} x^{k_3+1-j} y^j \right) \frac{\partial}{\partial y},$$

$$\theta_2 := \left(\sum_{j=0}^{k_2-1} \binom{k_3+1}{j} x^{k_3+1-j} y^j \right) \frac{\partial}{\partial x} + \left(\sum_{j=k_2}^{k_3+1} \binom{k_3+1}{j} x^{k_3+1-j} y^j \right) \frac{\partial}{\partial y}$$

is a homogeneous basis for $D(\mathcal{A}_{\Sigma,k})$. On the other hand, put

$$\theta_2' := \left(\sum_{j=0}^{k_2-2} \binom{k_3}{j} x^{k_3-j} y^{j+1} \right) \frac{\partial}{\partial x} + \left(\sum_{j=k_2-1}^{k_3} \binom{k_3}{j} x^{k_3-j} y^{j+1} \right) \frac{\partial}{\partial y}.$$

Then $\theta_1 + \theta_2' = \theta_2$. Moreover putting $f := \sum_{j=0}^{k_2-2} \binom{k_3}{j} x^{k_3-j} y^j$ and $g := \sum_{j=k_2-1}^{k_3} \binom{k_3}{j} x^{k_3-j} y^j$, we have

$$\theta_1 = x\left\{ \left(f + \binom{k_3}{k_1-1} x^{k_1-1} y^{k_2-1} \right) \frac{\partial}{\partial x} + \left(g - \binom{k_3}{k_1-1} x^{k_1-1} y^{k_2-1} \right) \frac{\partial}{\partial y} \right\}$$

$$\theta_2' = y\left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right).$$

Thus

$$\det M(\theta_1, \theta_2) = \det M(\theta_1, \theta_2') = \binom{k_3}{k_1-1} x^{k_1} y^{k_2} \begin{vmatrix} 1 & f \\ -1 & g \end{vmatrix} \doteq Q(\mathcal{A}_{\Sigma,k}).$$

This also shows that (θ_1, θ_2) is a basis for $D(\mathcal{A}_{\Sigma,k})$ thanks to Ziegler's criterion 3.1.

EXAMPLE 4.2. The case $k = (4, 4, 4)$: Then, $|k|/2 = 6$, $r_{k,6} = r_{k',6} = 1$, $s_{k,6} = 1$, $s_{k',6} = 2$ and hence

$$\lambda_j := \lambda_{k,6}^{(j)} = \begin{cases} (5-j, 2) & \text{if } j = 1, 2, 3 \\ (1, 6-j) & \text{if } j = 5, 6 \end{cases}, \quad \mu_j := \lambda_{k',6}^{(j)} = \begin{cases} (6-j, 3) & \text{if } j = 1, 2, 3 \\ (2, 7-j) & \text{if } j = 5, 6, 7 \end{cases},$$

where $k' = k + (0, 0, 1)$. By Theorem 1.5, it follows that $\theta_1 = \theta_\Sigma(k, 6)$, $\theta_2 = \theta_\Sigma(k', 6)$ is a basis for $D(\mathcal{A}_{\Sigma,k})$. Now see Figure 4.1 in page 15. The figure expresses $4 - c(P)$, the hook-length $h_{\lambda_j}(P)$ (at $P \in \mathbf{Y}(\lambda_j)$) and $\binom{4}{\lambda_j}$. Thus we have the following explicit expression for $\theta_1 = \theta_\Sigma(k, 6)$:

$$\theta_1 = 2\left\{(3x^6 + 10x^5y + 10x^4y^2)\frac{\partial}{\partial x} - (5x^2y^4 + 2xy^5)\frac{\partial}{\partial y}\right\}.$$

Similarly, we get the explicit expression for $\theta_2 = \theta_\Sigma(k', 6)$ (see Figure 4):

$$\theta_2 = 10\left\{(x^6 + 4x^5y + 5x^4y^2)\frac{\partial}{\partial x} - (5x^2y^4 + 4xy^5 + y^6)\frac{\partial}{\partial y}\right\}.$$

Compute $\theta_1(x+y)$, $\theta_2(x+y)$ and the determinant of the coefficient matrix $\det M(\theta_1, \theta_2)$:

$$\begin{aligned}\theta_1(x+y) &= 2x(3x-2y)(x+y)^4\,,\\ \theta_2(x+y) &= 10(x-y)(x+y)^5\,,\\ \det M(\theta_1,\theta_2) &= -200x^4y^4(x+y)^4\,.\end{aligned}$$

Therefore, we know that (θ_1, θ_2) is a basis for $D(\mathcal{A}_{\Sigma,k})$ thanks to Ziegler's criterion 3.1.

EXAMPLE 4.3. The case $k = (5,5,5)$: Then, $\lfloor\frac{|k|}{2}\rfloor = 7$, $\lceil\frac{|k|}{2}\rceil = 8$, $r_{k,7} = s_{k,7} = 2$, $r_{k,8} = s_{k,8} = 1$ and hence

$$\lambda_j := \lambda_{k,7}^{(j)} = \begin{cases}(6-j,3,3) & \text{if } j = 1,2,3\\ (2,2,8-j) & \text{if } j = 6,7,8\end{cases},\quad \mu_j := \lambda_{k,8}^{(j)} = \begin{cases}(6-j,2) & \text{if } j = 1,2,3,4\\ (1,7-j) & \text{if } j = 6,7\end{cases}.$$

By Theorem 1.5, $\theta_1 := \theta_\Sigma(k,7)$, $\theta_2 := \theta_\Sigma(k,8)$ is a homogeneous basis for $D(\mathcal{A}_{\Sigma,k})$. Explicitly, θ_1 and θ_2 are expressed as follows (see Figure 4 and 4):

$$\begin{aligned}\theta_1 &= 25\left\{(2x^7 + 7x^6y + 7x^5y^2)\frac{\partial}{\partial x} + (7x^2y^5 + 7xy^6 + 2y^7)\frac{\partial}{\partial y}\right\},\\ \theta_2 &= 5\left\{(2x^8 + 9x^7y + 15x^6y^2 + 10x^5y^3)\frac{\partial}{\partial x} - (3x^3y^5 + x^2y^6)\frac{\partial}{\partial y}\right\}.\end{aligned}$$

Compute $\theta_1(x+y)$, $\theta_2(x+y)$ and the determinant of the coefficient matrix $\det M(\theta_1, \theta_2)$:

$$\begin{aligned}\theta_1(x+y) &= 25(x+y)^5(2x^2 - 3xy + 2y^2)\,,\\ \theta_2(x+y) &= 5x^2(2x-y)(x+y)^5\,,\\ \det M(\theta_1,\theta_2) &= -2500x^5y^5(x+y)^5\,.\end{aligned}$$

Therefore, we verify that (θ_1, θ_2) is a basis for $D(\mathcal{A}_{\Sigma,k})$ thanks to Ziegler's criterion.

j	1	2	3	5	6
$4 - c(P)$	4 3 2 1 5 4	4 3 2 5 4	4 3 5 4	4 5	4
$h_{\lambda_j}(P)$	5 4 2 1 2 1	4 3 1 2 1	3 2 2 1	2 1	1
$\binom{4}{\lambda_j}$	6	20	20	10	4

FIGURE 4.1: The Young diagrams $\mathbf{Y}(\lambda_{k,6}^{(j)})$

j	1	2	3	5	6	7
$5 - c(P)$	5 4 3 2 1 6 5 4	5 4 3 2 6 5 4	5 4 3 6 5 4	5 4 6 5	5 4 6	5 4
$h_{\mu_j}(P)$	6 5 4 2 1 3 2 1	5 4 3 1 3 2 1	4 3 2 3 2 1	3 2 2 1	3 1 1	2 1
$\binom{5}{\mu_j}$	10	40	50	50	40	10

FIGURE 4.2: The Young diagrams $\mathbf{Y}(\lambda_{k',6}^{(j)})$

j	1	2	3	5	6	7
$5 - c(P)$	5 4 3 2 1 6 5 4	5 4 3 2 6 5 4	5 4 3 6 5 4	5 4 6 5	5 4 6	5 4
$h_{\mu_j}(P)$	6 5 4 2 1 3 2 1	5 4 3 1 3 2 1	4 3 2 3 2 1	3 2 2 1	3 1 1	2 1
$\binom{5}{\mu_j}$	10	40	50	50	40	10

FIGURE 4.3: The Young diagrams $\mathbf{Y}(\lambda_{k,7}^{(j)})$

j	1	2	3	5	6	7
$5 - c(P)$	5 4 3 2 1 6 5 4	5 4 3 2 6 5 4	5 4 3 6 5 4	5 4 6 5	5 4 6	5 4
$h_{\mu_j}(P)$	6 5 4 2 1 3 2 1	5 4 3 1 3 2 1	4 3 2 3 2 1	3 2 2 1	3 1 1	2 1
$\binom{5}{\mu_j}$	10	40	50	50	40	10

FIGURE 4.4: The Young diagrams $\mathbf{Y}(\lambda_{k,8}^{(j)})$

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