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Maximal Determinant Knots

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Abstract. The Kauffman bracket approach is used to give estimates on the size of the determinant (and this way also on the coefficients of the Jones and Alexander polynomial) of a link of given crossing number, or equivalently on the number of spanning trees of planar graphs with given number of edges. Properties of the knots and links with maximal determinant for given crossing number are investigated.

1. Introduction

If Δ_L denotes the (1-variable) Alexander polynomial of a link $L \hookrightarrow S^3$ [Al], then $\det(L) = |\,\Delta_L(-1)\,|$ is the order of the homology group $H_1(D_L)$ (over $\mathbf{Z}$) of the double branched cover D_L of S^3 over L (or 0 if this group is infinite) and carries the name "determinant" because of its expression (up to sign) as the determinant of a Seifert [Ro, p. 213] or Goeritz [GL] matrix. This group carries much interesting information on the link (in particular on sliceness [Ro], chirality [HK, St], and unknotting number estimates [We]).

In [St4] we began the investigation of the question how much the coefficients of the various link polynomials can grow on knots and links of given number of crossings, and showed how via the Kauffman state models [Ka2, Ka3] the problem for the Jones [J] and Alexander [Al] polynomial to be equivalent to this for the determinant. We also found that the maximum will be realized by alternating knots/links. The quest for better estimates of this maximum and studying the properties of the links attaining it will make a substantial part of this paper. In this regard, several questions (suggested by computational results) will be formulated, and partially solved.

Most of our results admit direct graph theoretic translations. They will imply certain properties of planar graphs with maximal number of spanning trees for a given number of edges. Graphs with the maximal number of spanning trees have been independently studied by Kelmans and Chelnokov [K, K2, KC]. Their results (and methods) are quite different, since they consider graphs with a fixed number of vertices and relatively high number of edges, most of which are therefore not planar.

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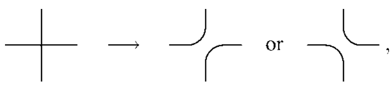
2. The determinant of alternating diagrams

2.1. Estimates for the determinant. Via the relation $\Delta(-1) = V(-1)$ to the Jones polynomial V (see [J2, §12]) the determinant provides a bridge between the classical Alexander polynomial and its successors [BLM, F&, Ka, J], whose nature is rather combinatorial, and it is one of the little topologically understandable information encoded in these more recent invariants. On the other hand, this opens combinatorial approaches for calculating the determinant.

One such approach, which is particularly nice for alternating diagrams, was given by Krebes [Kr] using the Kauffman bracket/state model for the Jones polynomial. (Alternating diagrams are those, in which any strand passes crossings alternatingly over-under.)

If D is an alternating link diagram, then consider $\hat{D} \subset \mathbf{R}^2$, the (image of) the associated immersed plane curve(s). Then $\det(D)$ is equal to the number of ways to splice the crossings (self-intersections) of $\hat{D}$



(1)

so that the resulting collection of disjoint circles has only one component (a single circle; such choices of splicings are called in [Kr] *monocyclic states*).

To any alternating link diagram one can associate its *checkerboard graph* (see [Ka, DH, Kr, St, Th]). In general the checkerboard graph is a planar graph (that is, equipped with a planar embedding), with possibly multiple edges. It is defined up to duality (corresponding to the switch between black and white regions in the checkerboard coloring). Any such graph is the checkerboard graph of some alternating link diagram. The operations in (1) correspond to contraction and deletion of an edge in the checkerboard graph.

In [St4, theorem 3.2], we showed via the skein relation for the Jones polynomial that for a diagram D of $c(D)$ crossings, $n(D)$ components, and maximal bridge length $d(D)$ (see [Ki] for latter's definition), we have

$$\big| V(D) \big|_1 := \sum_{2k \in \mathbf{Z}} \big| [V(D)]_{t^k} \big| \leq 5^{c(D)-d(D)} 2^{n(D)-1},$$

where $[V]_{t^k}$ is the coefficient of t^k in V. Simple experiments reveal that this bound is not particularly sharp.

The first observation towards an improvement was that using Krebes's approach, we have

LEMMA 2.1. *With the above notation,*

$$\big| V(D) \big|_1 \leq 2^{c(D)-1}.$$

This inequality is of more practical use, since D may have several components, and $d(D)$ is in general small compared to $c(D)$.

PROOF. Let D' be the alternating diagram obtained from D by changing crossings. Then by Kauffman's bracket, we have

$$|\,V(D)\,|_1 \;\le\; \det(D')\,.$$

If we resolve any $c(D')-1$ crossings in D' in some arbitrary way, then for the last one there is at most one splitting so as the circle picture to have only one component, so that the result follows. □

Although this lemma already gives (at least in practice) a better estimate, we can push it even a little further. In the theorem below an *arborescent diagram* is one with Conway basic polyhedron 1^* [Co], or alternatively, a diagram whose checkerboard graph is series-parallel. The following constant will be of major importance throughout the paper.

DEFINITION 2.1. Let $\delta \approx 1.83929$ be the inverse of

$$\delta^{-1} \;=\; -\frac{1}{3}-\frac{2}{3\sqrt[3]{17+3\sqrt{33}}}+\frac{\sqrt[3]{17+3\sqrt{33}}}{3}\approx 0.543689\,,$$

the real positive zero of $f(x)=x^3+x^2+x-1$.

THEOREM 2.1. 1) *There exists a constant $C>0$ such that for any link diagram D of $c(D)$ crossings*

$$\det(D) \;\le\; C\cdot\delta^{c(D)}\,. \tag{2}$$

2) *If D is an arborescent diagram, then*

$$\det(D)\le F_{c(D)+1}\,, \tag{3}$$

with F_i denoting the Fibonacci numbers (*defined by $F_1=1$, $F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n>2$*). *Moreover, the inequality is sharp, that is, there are relevant diagrams for which equality holds.*

PROOF. We start with the second part. Let

$$d_n^a \;:=\; \max\{\det(D)\;:\;D \text{ arborescent of } n \text{ crossings}\}\,.$$

An arborescent diagram always has a clasp, a fragment of the type


,

whose resolution (switching one of the crossings, and eliminating the two crossings by a Reidemeister II move) preserves arborescency. When splicing one of the crossings in the clasp, one of the two resulting diagrams has a kink, so that only one of the splicings of the second crossing can give a circle picture with only one component.

Thus

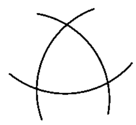
$$d_n^a \;\le\; d_{n-1}^a+d_{n-2}^a\,,$$

which, together with the trivial correctness for $c(D) = 1, 2$ by induction establishes the inequality (3). The sharpness of the inequality follows from considering the rational links $L_n = C(\underbrace{1, 1, \dots, 1}_{n \text{ times}})$ (here we use Conway's notation [Co]). To see that $\det(L_n) = F_{n+1}$ is an easy calculation with iterated fractions.

The argument for the first part is analogous. Let[1]

$$d_n^\infty := \max\{\det(D) : D \text{ link diagram of } n \text{ crossings}\}. \tag{4}$$

Then either D has a clasp, or a triangle



$$. \tag{5}$$

Then the above argument modifies to show that

$$d_n^\infty \le d_{n-1}^\infty + d_{n-2}^\infty + d_{n-3}^\infty \qquad (n > 2), \tag{6}$$

and thus d_n^∞ can be estimated by (a multiple of) Tribonacci numbers (whose defining recursion is the equality in (6) and hence) whose rate of growth is δ. $\square$

REMARK 2.1. 1) We have the explicit expression

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \tag{7}$$

so that for arborescent diagrams (2) holds with the smaller base $\dfrac{\sqrt{5}+1}{2} \approx 1.61803$ instead of $\delta \approx 1.83929$.

2) The constant C in (2), the way that it comes from the estimate (6), can be certainly effectively calculated, but it does not appear appropriate to do so. The standard way is to apply partial fraction decomposition to the generating (rational) function, obtaining an expression involving the (powers of the) zeros of the denominator polynomial. (Such formula is rather unpleasant since for a cubic equation these zeros are difficult to express.) Moreover, C can be successively improved by noting that (6) will hardly be sharp in general. Writing down the first values of d_n^∞ we have

n	0	1	2	3	4	5	6
d_n^∞	1	1	2	3	5	8	16

We see that (6) is sharp for $n = 6$, but not for $n < 6$ (because a diagram of $n < 6$ crossings has a clasp, so that we have the simplified recursion $d_n^\infty \le d_{n-1}^\infty + d_{n-2}^\infty$), and it will certainly not be for high n. Thus one can start the iteration on the right of (6) with higher and higher

[1] The unnaturally appearing superscript ∞ is used for conformity with notation which will be introduced later.

values of n and (proportionally) smaller initial data, obtaining a sequence of constants C with decreasing numerical value but increasing arithmetical complexity ... However, it is worth remarking that, because of connected sums, in every case $C = 1$ must validate (2).

2.2. Links with maximal determinant. Again it appears appropriate to make an experiment how good the bound is compared to the actual values of d_n. In [St4, §4] we made such an experiment, where we replaced $c(D)$ by span $V(D) - 1$. (Here the *span* of V is the difference between minimal and maximal power of t in the monomials of V.) This led to a picture dominated by non-alternating knots of small span V. Thus here we consider only alternating knots and links of given crossing number.

For what follows it will be helpful to make some definitions.

TABLE 1. The knots K_n for $n \leq 16$ and some of their data (from left to right): crossing number, knot identifier, determinant, fiberedness, clasp-freeness, flypefreeness, achirality, invertibility, signature, existence of alternating braid representation, braid index.

ncr	knot-id	det	fibered	claspf	flypef	achir	invert	σ	al braid	bind
3	1	3	✔		✔		✔	2	✔	2
4	1	5	✔		✔	$+/-$	✔	0	✔	3
5	2	7			✔		✔	2		3
6	3	13	✔		✔	$+/-$	✔	0	✔	3
7	7	21	✔				✔	0	✔	4
8	18	45	✔	✔	✔	$+/-$	✔	0	✔	3
9	40	75	✔	✔	✔		✔	2	✔	4
10	123	121	✔	✔	✔	$+/-$	✔	0	✔	3
11	266	209	✔	✔	✔		✔	0	✔	4
12	868	377	✔	✔	✔	$-$		0		5
13	3478	663	✔	✔	✔		✔	2	✔	4
14	17895	1145	✔	✔	✔	$-$		0	✔	5
15	82477	2037	✔	✔	✔		✔	0	✔	4
16	361172	3581	✔	✔	✔	$-$		0	✔	5

DEFINITION 2.2. Let $S \subset \mathbf{N}$. Then define

$$d_n^S = \max\{\det(D) : n(D) \in S,\ c(D) = n\},$$

where $n(D)$ is the number of components of D, and let K_n^S be a link attaining the maximum. Set

$$\begin{array}{lll} K_n^i := K_n^{\{i\}}, & K_n^\infty := K_n^{\mathbf{N}}, & K_n := K_n^1, \\ d_n^i := d_n^{\{i\}}, & d_n^\infty := d_n^{\mathbf{N}}, & d_n := d_n^1 \end{array}$$

(and compare to (4)).

This definition already contains a question.

QUESTION 2.1. Is K_n^S unique for all S and n?

In all special cases I checked it was so. However, it is not clear in general. For what follows let us avoid any possible ambiguity by choosing one fixed maximizing link for each n and S. The properties of K_n^S we will state below will be valid *whatever* choice of K_n^S is made, that is, they hold for *all* knots/links that could be chosen as K_n^S.

With this understanding, we point out the following important fact remarked in [St4].

THEOREM 2.2. *K_n^S is alternating for each n and S.*

As tabulation (up to crossing numbers sufficing to give some more concrete picture) are available only for knots, we made a more serious calculation only for $S = \{1\}$. The knots K_n for $n \leq 16$ are listed in Table 1, together with the indication of (lack of) some specific properties and, beside their determinants d_n, some other classical invariants. (The genera are not included; their behaviour will be clarified later.) The last 6 knots, which are not given in Rolfsen's tables [Ro, appendix], are drawn on Figure 1. They are numbered according to the tables in [HT]. (See [HTW] for an account on the tabulation of knots.)

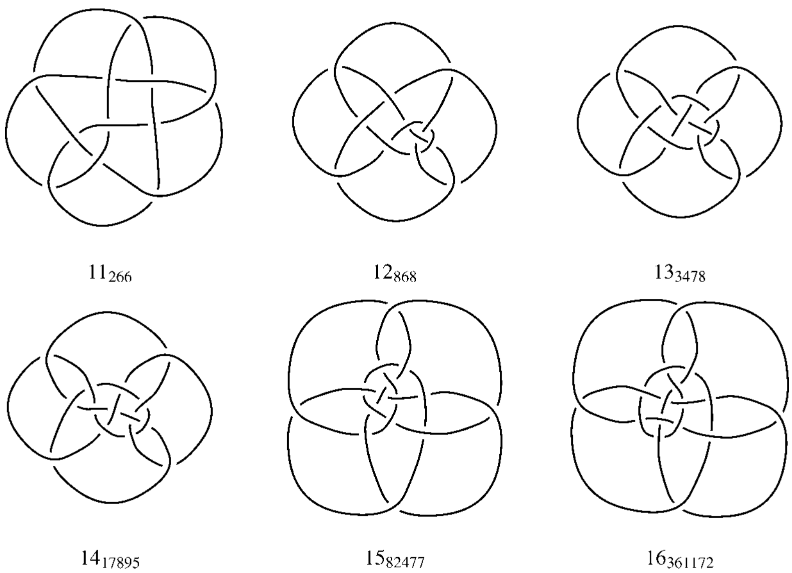


FIGURE 1. The knots of 11 to 16 crossings with maximal determinant.

The meaning of the properties "flype-free" and "clasp-free" is as follows (for the definition of flypes, see [MT, MT2]).

DEFINITION 2.3. A knot or link is called *flype-free*, if there is no essential flype applicable on its alternating diagram, that is, by [MT, MT2], it has only one alternating diagram (modulo moves in S^2).

DEFINITION 2.4. A knot or link is called *clasp-free,* if there is no (possibly trivial) sequence of flypes making any of its alternating diagrams to have a clasp. This is equivalent to being a polyhedral link in the sense of Conway [Co].

Table 1 reveals several striking coincidences and leads to some (possibly too optimistic, but at least to some extent justifiable) conjectures. Although sufficient experimental data is not available for links, it appears that similar phenomena occur there as well.

CONJECTURE 2.1.
1) *K_n is fibered for $n \neq 5$.*
2) *K_n is clasp-free for $n \geq 8$.*
3) *K_n is flype-free for $n \neq 7$.*
4) *K_n is invertible for odd n and $-$achiral for even n.*
5) *$\sigma(K_n) \in \{-2, 0, 2\}$, where σ is Murasugi's signature.*
6) *K_n is* (*the closure of*) *an alternating braid except for $n = 5, 12$.*
7) *K_n is prime.*
8) *K_n is unique* (*up to mirroring and orientation*).

Note that there is some causality between the various properties. For, example alternating braids are fibered. On the other hand, evidence for other such relations from the common knot tables can be misleading. It may appear at first glance, for example, that clasp-free alternating knots are fibered, too. However, this is not always true. The simplest example of a clasp-free alternating non-fibered knot is 13_{4695}.

In the following we initiate the investigation of some of the observed phenomena – flype-freeness, clasp-freeness and primality, and give some relations between properties of d_n and such of K_n. We defer the discussion of the braid index of K_n towards the end of the paper, after braids are considered in more detail.

2.3. Properties of maximal determinant links. We will be able to obtain the most precise results on clasp-freeness. We begin with the following statement.

THEOREM 2.3. *Let $S = \{1\}$ or $S = \infty$. Then K_n^S is clasp-free for infinitely many values of n. More specifically, for $S = \infty$, every subset of* $\mathbf{N}$ *of the form $\{x,\ x + 2,\ x + 4, \ldots,\ x + 160\}$ contains at least two such n.*

The proof of theorem 2.3 initiates from some more conditional, but still self-contained properties of K_n related to such of d_n.

PROPOSITION 2.1.

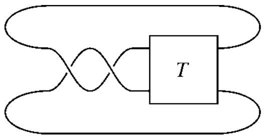
a) *If $d_n > \max(3d_{n-2}, d_{n-1} + 2d_{n-3})$, then K_n is clasp-free.*

b) *If for $S = \{1\}$ or $S = \infty$ we have $d_n^S > d_l^S d_{n-l}^S$ for any $1 < l < n - 1$, then K_n^S is prime.*

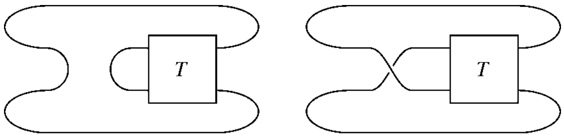
c) *If for $S = \infty$ we have $d_n^S > 3d_l^S d_{n-l-1}^S$ for any $1 < l < n-2$, then K_n^S is flype-free.*

d) *If for $S = \infty$, $d_n^S > \min(3d_{n-2}^S, d_{n-1}^S + 2d_{n-3}^S)$, then K_n^S is clasp-free.*

PROOF. a) Assume K_n has a clasp, i.e.

$K_n =$  .

Then splicing of the one crossing in the clasp gives a knot and a 2 component link.



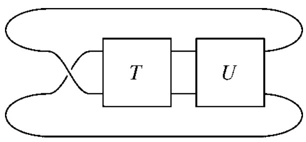
(a) (b) (9)

CASE 1. The diagram (a) in (9) represents the knot and (b) the (2 component) link. Then (a) contributes at most d_{n-2} to d_n. The diagram (b) has a mixed crossing (unless it is split in which case it has zero determinant), whose two splicings give again knots, so the contribution is at most $2d_{n-2}$.

CASE 2. The diagram (b) is the knot and (a) the link. Then (b) contributes at most d_{n-1} and (a) contributes after splicing a mixed crossing at most $2d_{n-3}$.

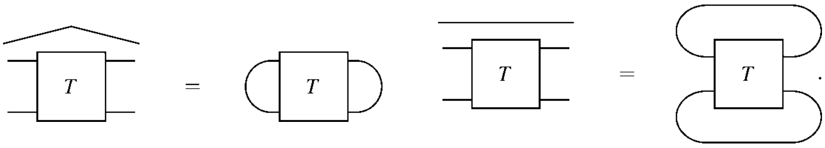
b) This is straightforward from the multiplicativity of the determinant under connected sum and the result of Menasco [Me].

c) Assume that K_n is not flype-free, in particular a diagram of $K = K_n$ is of the form



(10)

with $c(T), c(U) > 1$. Let the two possible closures of a tangle be denoted as follows:



With this notation (10) can be written as

$$K = \overline{1, T, U}\,,$$

where '1' is the 1-tangle and the comma operator denotes tangle sum in the Conway [Co] sense. Let

$$\mathrm{Kr}(T) := \frac{\det(\overline{T})}{\det(\widehat{T})} \in \tilde{\mathbf{Q}} = \mathbf{Z} \times \mathbf{Z}/(p,q) \sim (-p,-q)$$

be Krebes's invariant [Kr]. It provides a convenient formalism to calculate determinants of certain tangle sums. By Krebes's calculus we have with the "fraction" addition $\oplus$ on $\tilde{\mathbf{Q}}$,

$$\frac{\det(K_n)}{*} = \mathrm{Kr}(1,T,U) = \frac{\pm 1}{1} \oplus \frac{\pm\det(\overline{T})}{\det(\widehat{T})} \oplus \frac{\pm\det(\overline{U})}{\det(\widehat{U})}\,.$$

Comparing the numerators we obtain

$$d_n = \det(K_n) = \pm\big(\det(\overline{T}) + \det(\widehat{T})\big)\cdot\det(\widehat{U}) \pm \det(\widehat{T})\det(\overline{U}) \le 3d_{n-l-1}d_l\,,$$

with $l = c(T)$.

d) Use the inequality $d_n^S \le d_{n-1}^S + d_{n-2}^S$, following from resolving a crossing in a clasp, and

$$d_{n-1}^S \le 2d_{n-2}^S\,,$$

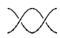
following from splicing any arbitrary crossing in an $n-1$ crossing link diagram. □

We now prove several, mostly unconditional, inequalities between the d_n^S.

LEMMA 2.2.

1. *$d_{n+1}^1 \ge d_n^1$.*
2. *$d_n^k \le 2d_{n-1}^{k-1}$ for $k > 1$.*
3. *$d_n^1 \le d_{n-1}^1 + d_{n-1}^2$.*
4. *$d_{n+3}^\infty \le 7d_n^\infty$.*
5. *$d_{n+2}^1 \le 5d_n^1$.*
6. *$d_{n+3}^1 \le 11d_n^1$.*
7. *$d_{n+1}^\infty \le 2d_n^\infty$.*
8. *If K_{n+2}^∞ is not clasp-free, then $d_{n+2}^\infty \le 3d_n^\infty$.*
9. *If K_{n+3}^∞ is not clasp-free, then $d_{n+3}^\infty \le 6d_n^\infty$.*

PROOF.

1. Replace a crossing  in K_n^1 by a clasp  such that the resulting diagram is again a diagram of a knot K'. Then K' has $n+1$ crossings and that $\det(K') \ge \det(K)$ follows by splicing one of the 2 crossings in the clasp.

2. This follows directly from splicing any mixed crossing in K_n^k. (If such does not exist, then K_n^k is split, and has zero determinant, which is impossible.)

3. This follows directly from splicing any crossing in K_n^1.

4. This follows by splicing the 3 crossings on the edges of a triangle (5) in K^{∞}_{n+3}. One of the resulting 8 diagrams has a split loop.

5. We have from parts 1, 2 and 3

$$d^1_{n+2} \overset{3}{\leq} d^1_{n+1} + d^2_{n+1} \overset{3,\,2}{\leq} d^1_n + d^2_n + 2d^1_n \overset{2,1}{\leq} 5d^1_n\,.$$

6. Follows as part 5, but applying parts 2 and 3 once more, before applying part 1.
7. This is trivial.
8. This follows from proposition 2.1, part d).
9. Use parts 7 and 8.

□

We now come to the proof of theorem 2.3.

PROOF OF THEOREM 2.3. Let K be a knot or link of $n = c(K)$ crossings. Then $d^{\infty}_n \geq \det(K)$, and $d^{\infty}_{n+k} \geq d^{\infty}_k \cdot \det(K)$ for any k because of connected sums with K. Therefore, $d^{\infty}_{k+2n} \geq \det(K)^2 d^{\infty}_k$. Assume now l of the links $K^{\infty}_{k+2}, K^{\infty}_{k+4}, \dots, K^{\infty}_{k+2n}$ have a clasp. Then by parts 7 and 8 of lemma 2.2 we have

$$d^{\infty}_{k+2n} \leq 3^l 4^{n-l} d^{\infty}_k\,.$$

On the other hand

$$d^{\infty}_{k+2n} \geq \det(K)^2 d^{\infty}_k\,.$$

Thus $\det(K)^2 \leq 3^l 4^{n-l}$, or

$$\sqrt[n]{\det(K)} \;\leq\; \sqrt[2n]{3^l 4^{n-l}} \;=\; 2\left(\frac{\sqrt{3}}{2}\right)^{l/n}\,.$$

Thus

$$l \;\leq\; n \cdot \frac{\ln\sqrt[n]{\det(K)} - \ln 2}{\ln\sqrt{3} - \ln 2}\,. \tag{11}$$

It is clear that this to give a non-trivial estimate, one must have $\sqrt[n]{\det(K)} > \sqrt{3}$. This, unfortunately, is not the case for knots of $\leq$ 16 crossings, and we need to look at more complicated examples. Luckily, however, the determinant can be computed via the Seifert matrix in polynomial time. A package for this using braid representations was written by S. Orevkov for MATHEMATICA™ [Wo]. Using it I found the closed 81 crossing alternating 10-string braid

$$K \;=\; \widehat{\;}\Big(\big(\sigma_1\sigma_3\sigma_5\sigma_7\sigma_9\sigma_2^{-1}\sigma_4^{-1}\sigma_6^{-1}\sigma_8^{-1}\big)^9\Big) \tag{12}$$

(with σ_i being, as usual, the Artin generators), where $\det(K) = 24743382596536452489$, and hence $\mu_K := \det(K) \cdot 3^{-c(K)/2} \approx 1.17503$.

Putting this into (11) gives a right hand-side of integer part 79, so the result follows for $S=\infty$.

Now let $S=\{1\}$. If for almost all n the knot K_n had a clasp, then

$$d_n \;\le\; \max\left(3d_{n-2}, d_{n-1}+2d_{n-3}\right)\;,$$

coming from part a) in proposition 2.1, shows $d_n \le C\cdot\sqrt{3}^{\,n}$. (The zero of $2x^3+x-1$ on $[0,\infty)$ close to $1/2$ is higher than $1/\sqrt{3}$, so that the higher rate of growth comes from the first alternative in the maximum.) This contradicts the existence of the above example (12). □

We remark that the inequality $d_{a+b}\ge d_a d_b$ implies that $\tilde d := \lim\limits_{n\to\infty}\sqrt[n]{d_n}$ exists and that

$$\lim_{n\to\infty}\sqrt[n]{d_n} \;=\; \sup_K \sqrt[c(K)]{\det(K)}\,,$$

where the supremum is taken over all (alternating) knots K. Thus we have

COROLLARY 2.1. $\sqrt{3} < \sqrt[81]{2.474338\cdot 10^{19}} \approx 1.7355032 \le \lim\limits_{n\to\infty}\sqrt[n]{d_n} \le \delta\,.$ □

We should also point out that the lower bound for $\tilde d$ can be successively improved by finding knots K with higher value of $\sqrt[c(K)]{\det(K)}$, and for this purpose calculating the determinant of appropriate more and more complicated knots. This will be done in §4, thus improving theorem 2.3 for $S=\infty$, and showing that at least $2/9$ of all K_n^∞ are clasp-free.

QUESTION 2.2. Is $\tilde d=\delta$, or $\tilde d_\infty := \lim\limits_{n\to\infty}\sqrt[n]{d_n^\infty}=\delta$?

REMARK 2.2. If we have a knot K of k crossings with $\mu_K>1$ and know d_n for $n<k$, then we can obtain an explicit (upper) estimate depending on $\varepsilon>0$ of the smallest number n_0 with $\sqrt[n]{d_n}>\sqrt{3}+\varepsilon$ for any $n>n_0$. If (this estimate on) n_0 is sufficiently small, it can be used to prove the clasp-freeness of K_n for almost all n. There is little hope to be able to proceed this way, though. Indeed, it does not seem feasible to calculate d_n for n larger than about 20, while $\mu_K>1$ occurs only for rather complicated knots. For example, for

$$K \;=\; \widehat{\;}\Big(\left(\sigma_1\sigma_3\sigma_5\sigma_7\sigma_9\sigma_2^{-1}\sigma_4^{-1}\sigma_6^{-1}\sigma_8^{-1}\right)^7\Big)\,,$$

we have $\mu_K=0.98\ldots$, although it already has crossing number 63.

REMARK 2.3. Similarly to (11), parts 9 and 4 of lemma 2.2 show that

$$l \;\le\; n\cdot\frac{\ln\sqrt[n]{\det(K)}-\ln\sqrt[3]{7}}{\ln\sqrt[3]{6}-\ln\sqrt[3]{7}} \tag{13}$$

for the number l of elements x in sets of the form $\{a,a+3,\ldots,a+3(n-1)\}$ with K_x^∞ clasp-free (and $n=c(K)$). However, the problem to find a knot K with $\sqrt[c(K)]{\det(K)}>\sqrt[3]{6}\approx 1.81$ is still computationally inaccessible, even if $\sqrt[3]{6}<\delta$. So (13) is of little practical use as of now.

In the knot case ($S = \{1\}$), I was unable to prove an analogon of parts 8 and 9 in lemma 2.2, such that '$\sqrt{3}$' in (11) could be replaced by a quantity still below δ. Thus, I have no direct estimate of the frequency of clasp-free K_n so far.

Another property of the K_n follows from the work we have done in [St5], which rewards us with an easy proof of a growth statement for the *genera* $g(K_n)$ of the K_n (see [Ga]).

THEOREM 2.4. $g(K_n) \to \infty$. *More exactly, for any* $\varepsilon > 0$ *we have* $g(K_n) \geq \log_{8+\varepsilon} n$ *for* n *large enough.*

PROOF. That $g(K_n) \to \infty$ follows by [St5, theorem 3.1], because $\det(K)$ grows only polynomially in $c(K)$ for alternating knots K of fixed genus. The specific growth statement comes from an estimate of this polynomial from [St3]. We derived such an estimate in [St3, theorem 3.1]. We showed for K alternating that

$$\det(K) \leq \max_{1 \leq d' \leq d_{g(K)}} \left[\frac{C\, c(K)}{d'} \right]^{d'}, \tag{14}$$

where $C > 1$ is some constant, and $d_{g(K)}$ can be defined by

$$d_g \;:=\; \min \left\{ i \in \mathbf{N} \,:\, \limsup_{n \to \infty} \frac{|\,\mathcal{A}_{n,g}\,|}{n^i} \;=\; 0 \right\}, \tag{15}$$

with

$$\mathcal{A}_{n,g} \;:=\; \{\, K \text{ alternating}, \; g(K) = g, \; c(K) = n \,\}\,. \tag{16}$$

We also use the fact proved in [St5] that

$$d_g = O(8^g)\,. \tag{17}$$

Assume now that there is a sequence $\{n_i\}$ and an $\varepsilon' > 0$ with $g(K_{n_i})/\log_8(n_i) \leq 1 - \varepsilon'$. Then

$$n_i^{\varepsilon''} \;\gg\; 8^{g(K_{n_i})} \tag{18}$$

for any $\varepsilon'' \in\; (1 - \varepsilon', 1)$.

We have that the maximal value of

$$f_n(d') = \left(\frac{n}{d'}\right)^{d'}$$

for $d' \in (0, d]$ is attained for $d' = \min\left\{d, \dfrac{n}{e}\right\}$ (where $e = 2.71828\ldots$), and for $d \leq {}^{n}\!/_{e}$ the function f_n is monotonously growing. Because of (18) for i large enough the former alternative in the maximum applies, and (14) and (17) give

$$\det(K_{n_i}) \leq \left[\frac{C\, n_i}{8^{g(K_{n_i})}} \right]^{C' 8^{g(K_{n_i})}}$$

for some constants C and C'. Using (18) we find

$$\det(K_{n_i}) \leq (C\,n_i)^{C' n_i^{\varepsilon''}}\ . \tag{19}$$

But because of $\varepsilon'' < 1$ we have $C'\,n_i^{\varepsilon''}(\ln n_i + \ln C) \ll C'' n_i$ for any $C'' > 0$. Exponentiating this and using (19) shows that $\{\det(K_{n_i})\}$ grows subexponentially, a contradiction. $\square$

3. Determinants of alternating braids

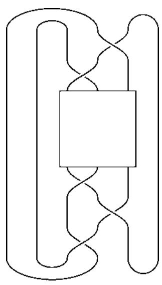
3.1. Alternating 3-braids. Originally the examples $K_8 = 8_{18}$ and $K_{10} = 10_{123}$ suggested to consider for the proof of theorem 2.3 for $S = 1$ more closely the sequence of the closures of the alternating 3-braids $\widehat{(\sigma_1\sigma_2^{-1})^k}$. Although these braids closely fail in giving the desired examples, they can be used to give an estimate for arbitrary alternating 3-braids.

Define certain numbers c_k using Fibonacci numbers:

$$c_k \;=\; F_{2k} + 2\sum_{i=1}^{k-1} F_{2i}\ . \tag{20}$$

LEMMA 3.1. $\det\big(\widehat{(\sigma_1\sigma_2^{-1})^k}\big) = c_k$.

PROOF. Consider the 2 uppermost crossings of $(\sigma_1\sigma_2^{-1})^k$, the ones from the last factor in the power.



Splicing the uppermost one as $\asymp$ gives the rational link $C(\underbrace{1, 1, \dots, 1}_{2k-1})$, whose determinant as we mentioned is F_{2k}. Splicing the uppermost crossing as $)\,($ and the second uppermost one as $\asymp$ gives, after deleting the kink from the lowermost crossing, a rational link $C(\underbrace{1, 1, \dots, 1}_{2k-3})$ with determinant F_{2k-2}. Finally, splicing both crossings as $)\,($, gives $(\sigma_1\sigma_2^{-1})^{k-1}$ and then the result follows by induction from (20). $\square$

COROLLARY 3.1. *If β is an alternating 3-braid, then* $\det(\hat\beta) \leq \left(\frac{\sqrt{5}+1}{2}\right)^{c(\hat\beta)}$, *with the inequality in general sharp up to an additive constant.*

PROOF. We use that any $\beta \in B_3$, except for the ones in the lemma, have a clasp. Splicing one of the crossings in the clasp, we obtain a 3-braid with one crossing less and a rational link. The contribution of the rational link of $c(\hat\beta)-2$ crossings to $\det(\hat\beta)$ is estimated by part 2) in theorem 2.1 from above by

$$\frac{1}{\sqrt5}\left(\frac{2}{\sqrt5+1}\right)\cdot\left(\frac{\sqrt5+1}{2}\right)^{c(\hat\beta)}+C \tag{21}$$

for some fixed constant C. The determinant of the spliced braid is estimated by induction on $c(\hat\beta)$ by

$$\left(\frac{2}{\sqrt5+1}\right)\cdot\left(\frac{\sqrt5+1}{2}\right)^{c(\hat\beta)}.$$

Now

$$\frac{2}{\sqrt5+1}+\frac{1}{\sqrt5}\left(\frac{2}{\sqrt5+1}\right)=\frac{2}{\sqrt5}<1. \tag{22}$$

Therefore, starting the induction for $c(\hat\beta)$ large enough to compensate the constant C in (21) by the strict inequality in (22), and checking the initial cases directly, one is done. □

REMARK 3.1. The links of the form $\widehat{(\sigma_1\sigma_2^{-1})^k}$ have been considered previously, notably in [JP] (at least in the knot case $3 \nmid k$). There it was observed that for odd k (for which the knots are also called “turks head knots”), the braid $\widehat{(\sigma_1\sigma_2^{-1})^k}$ is of the form $\beta\overline{\beta}$, where $\overline{\beta}$ is obtained from $\beta \in B_n$ by the map $\sigma_i^{\pm1}\mapsto\sigma_{n-i}^{\mp1}$. Hence $\widehat{(\sigma_1\sigma_2^{-1})^k}$ is *strongly +achiral*, i. e., admits an embedding into $\mathbf{R}^3$ fixed by the (orientation-reversing) involution $(x,y,z)\mapsto(-x,-y,-z)$, such that this involution additionally preserves the orientation of the knot/link. By the result of [HK] (stated and proved only for knots but true by the same argument also for links[2]), such knots/links have as Alexander module a double $A\oplus A$, so that in particular the Alexander polynomial, and hence the determinant, is a square. This, together with lemma 3.1, shows knot-theoretically that c_k is a square for odd k. The numbers c_k are discussed also in [St].

3.2. Alternating braid powers. For general strand number, the results on 3-braids, and more specifically on the powers of $\sigma_1\sigma_2^{-1}$, generalize followingly.

THEOREM 3.1. *Let $\beta_i \in B_n$ be alternating braids of fixed strand number n.*

1) *Then* $\lambda_{\{\beta_i\}} := \limsup\limits_{i\to\infty} \sqrt[c(\beta_i)]{\det(\hat\beta_i)} \le \delta$.

---

[2] except in the case, when the Alexander module is not (completely) torsion, which is, however, trivial, as then the Alexander polynomial vanishes

2) *Moreover, if $\beta_i = \beta^i$ are powers of some fixed braid β, then $\lambda_\beta := \lambda_{\{\beta^i\}}$ is the norm of a (possibly complex) algebraic number of degree $\leq C_n$, where $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the n-th Catalan number.*

For the proof of theorem 3.1 we need a technical lemma.

LEMMA 3.2. *Let $\lambda_1, \ldots, \lambda_l$ $(l > 1)$ be distinct unit norm complex numbers and $\{a_{j,n}\}_{n=1}^{\infty}$ for $j = 1, \ldots, l$ be sequences with $|a_{j,n}| \geq \varepsilon$ for some $\varepsilon > 0$ and all j, n, and*

$$\frac{a_{j,n+1}}{a_{j,n}} \underset{n\to\infty}{\longrightarrow} 1\,.$$

Then the sequence $s_n := \displaystyle\sum_{j=1}^{l} a_{j,n}\lambda_j^n$ does not converge (in particular, not to 0).

PROOF. Assume $s_n \to s$ for some $s \in \mathbf{C}$. If

$$M_n := \begin{pmatrix} 1 & \cdots & & 1 \\ \frac{a_{1,n+1}}{a_{1,n}}\lambda_1 & \cdots & & \frac{a_{l,n+1}}{a_{l,n}}\lambda_l \\ \frac{a_{1,n+2}}{a_{1,n}}\lambda_1^2 & \cdots & & \frac{a_{l,n+2}}{a_{l,n}}\lambda_l^2 \\ \vdots & & \ddots & \vdots \\ \frac{a_{1,n+l-1}}{a_{1,n}}\lambda_1^{l-1} & \cdots & & \frac{a_{l,n+l-1}}{a_{l,n}}\lambda_l^{l-1} \end{pmatrix},$$

then

$$M_n \begin{pmatrix} a_{1,n}\lambda_1^n \\ \vdots \\ a_{l,n}\lambda_l^n \end{pmatrix} \longrightarrow \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix},$$

and M_n converge to a Vandermonde matrix, which is not singular, so that $||M_n^{-1}||$ is bounded.

Therefore, in particular $\left\{ (a_{1,n}\lambda_1^n, \ldots, a_{l,n}\lambda_l^n) \,:\, n > n_0 \right\}$ must lie in some ε'-ball for n_0 large enough. But $|a_{j,n}| \geq \varepsilon$ shows that these components stay outside of some neighborhood of the origin, and

$$\frac{a_{i,n+1}\lambda_i^{n+1}}{a_{i,n}\lambda_i^n} \longrightarrow \lambda_i \neq 1$$

for some i gives a contradiction for ε' small enough. □

PROOF OF THEOREM 3.1.

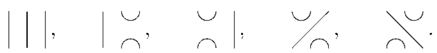
1) This is clearly a consequence of corollary 2.1.

2) Let β_i be n-strand braids and SD_n be the Kauffman algebra of [Ka, definition 3.5] with the special parameter $A = \sqrt{-1}$ (so that a separate loop trivializes). It can be shown

TABLE 2. The table for the pairing $<\,,\,>_3$.

					
	0	0	0	1	0
	0	0	1	0	1
	0	1	0	0	1
	1	0	0	0	0
	0	1	1	0	0

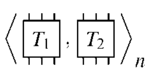
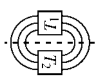
(see [Ka, theorem 4.3]) that SD_n is generated by the C_n loop-free diagrams connecting two rows of $n+n$ points on bottom and on top by n non-intersecting arcs. For example for $n=3$ we have the following 5 elements:

 (23)

The dimension of SD_n is therefore at most C_n. These diagrams also form a basis, as was proved in the study of meanders in theoretical physics and as will be explained next.

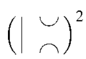
A *meander* (see [DGG]) is a topological type of the union of transversely intersecting circles smoothly embedded in S^2. We require that there is a circle x, such that all circles $y \neq x$ intersect x, but any two circles $y_{1,2} \neq x$ do not intersect each other. We will draw x as a dashed line, and assume it passes through ∞. We call the other circles *loops*.

Define a pairing (or binary quadratic form) on SD_n by

 $=\begin{cases} 1 & \text{if the meander } \end{cases}$  has one loop,
0 else. (24)

For example $<\,,\,>_3$ is given by the Table 2. It suffices to know that $<\,,\,>_n$ is non-degenerate on the above C_n elements. This follows from an explicit expression for its determinant, see formula (5.18) in [DGG].

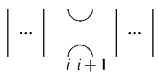
The multiplication in SD_n is given by stacking up, and possibly setting the resulting diagram to 0 if it has a loop. For example

 $= 0.$

Associate to $\beta = \prod_{j=1}^{m} \sigma_{i_j}^{(-1)^{i_j}} \in B_n$ a linear operator ϕ_β

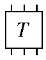
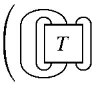
$$SD_n \ni x \overset{\phi_\beta}{\longmapsto} x \prod_{j=1}^{m}(1+s_{i_j}) \in SD_n\,,$$

with

$s_i =$  .

ϕ_β can be decomposed (at least over $\mathbf{C}$) into eigenvalues λ_i and Jordan box spaces V_i. Fix a Jordan basis of ϕ_β. (As ϕ_β has integer coefficients in some rational basis of SD_n, the Jordan basis can be chosen to lie in some degree $\leq C_n$ extension of $\mathbf{Q}$.)

From now on consider only those V_i which are not completely killed by the $\mathbf{C}$-linear extension of the map

$\chi : SD_n \ni$  $\longmapsto \det\Big($  $\Big) \in \mathbb{N} \subset \mathbb{C}.$

Consider the Jordan basis decomposition of $Id = 1 \in SD_n$

$$\Big|\Big|\Big|\cdots\Big| = \sum_{i=1}^{l} x_i + x\,, \quad x_i \in V_i\,,$$

with $\chi(x) = 0$.
Since

$$0 \neq \det(\hat{\beta}) = \det(\widehat{1\cdot\beta}) = \chi(1\cdot\beta) = \chi(\phi_\beta(1))\,,$$

there exists a $1 \leq i_0 \leq l$ with $x_{i_0} \neq 0$ and $\lambda_{i_0} \neq 0$. Among these eigenvalues λ_{i_0}, pick up those $\lambda'_1, \ldots, \lambda'_{l'}$ of maximal norm $\mu := |\lambda'_i|$, and let V'_i be the Jordan box space of λ'_i, and $x'_i = x_{i'}$ for $\lambda_{i'} = \lambda'_i$.

Let $d'_i := \dim V'_i$ for $1 \leq i \leq l'$, and $d' := \max_{i=1}^{l'} d'_i$. Then each x'_i has a contribution to $\det\big(\widehat{\beta^k}\big)$ of the form

$$\sum_{j=1}^{d'_i} c_{i,j} P_{i,j}(k)\,(\lambda'_i)^{k-d'_i+j} = \tilde{P}_i(k)\lambda_i'^{k}\,,$$

for some $c_{i,j} \in \mathbf{C}$ (the coefficients of x'_i in the Jordan basis of V'_i), and polynomials $P_{i,j}(k) \in \mathbf{Q}[k]$ with $\deg P_{i,j} \leq d'_i$, and

$$\tilde{P}_i(k) = \sum_{j=1}^{d'_i} c_{i,j} P_{i,j}(k)\,(\lambda'_i)^{j-d'_i} \in \mathbf{C}[k]\,,$$

with $\deg \tilde{P}_i \leq d'_i$. Without loss of generality, add up all $\tilde{P}_i$ for equal λ'_i, reindex towards the end, and then discard all i for which $\tilde{P}_i = 0$. Then we have

$$\det(\widehat{\beta^k}) = \sum_{i=1}^{l'} \tilde{P}_i(k)\, \lambda_i'^k \quad + \mathrm{O}((\mu - \varepsilon)^k) \tag{25}$$

for some $1 \le l' \le l$, $\lambda_i' \ne \lambda_j'$ for $i \ne j$, and $0 \ne \tilde{P}_i(k) \in \mathbf{C}[k]$ (with $\deg \tilde{P}_i \le d'$). If we show now

$$\limsup_{k\to\infty} \sqrt[k]{\det(\widehat{\beta^k})} \;=\; \limsup_{k\to\infty} \sqrt[k]{\sum_i \tilde{P}_i(k)\, \lambda_i'^k} \;=\; \mu\,,$$

we are through, as λ_i' is the root of a polynomial with rational coefficients of degree C_n.

If $l' = 1$, then the claim is straightforward from (25). Otherwise it follows from lemma 3.2 by rescaling, setting $n = k$ and $a_{j,n} := \tilde{P}_j(n)$. □

The combination of both statements in theorem 3.1 also suggests

COROLLARY 3.2. *Any eigenvalue λ of ϕ_β for any $\beta \in B_n$ satisfies $|\lambda| \le \delta^{c(\beta)}$.*

PROOF. Although a dominating eigenvalue of ϕ_β may have a Jordan space killed by taking the determinant of the usual braid closure, there will often be a (linear combination of) other closure(s) under which not the whole Jordan space is killed (and then for these exotic closures the same argument will apply). That such a closure indeed exists is a consequence of the non-degeneracy of the pairing $<\,,\,>_n$ in (24), as previously explained. □

REMARK 3.2. I owe M. Khovanov the above reference to the paper [DGG]. Table 2 shows easily $<\,,\,>_3$ to be non-degenerate, and I checked it previously by computer for $n = 4, \ldots, 10$. It is also an easy exercise to see that $< T_1,\,.\, > \not\equiv 0$ if T_1 is a single diagram, as one can always find a diagram T_2 with $\widehat{T_1 T_2} = \bigcirc$. However, the argument does not extend in an easy way to arbitrary linear combinations of diagrams, and the proof of non-degeneracy is certainly quite non-trivial.

4. Spanning trees in planar graphs

4.1. Determinant and spanning trees. It the following it will be useful to pass over from alternating diagrams to their checkerboard graphs.

DEFINITION 4.1. Let for a graph G the number of its spanning trees be denoted by $s(G)$.

LEMMA 4.1. *If D is an alternating diagram of link L, and G its checkerboard graph, then* $\det(L) = s(G)$. □

This is a classical result (see e.g. [BZ]), a consequence of Kauffman's state models for the Jones [Ka2] and Alexander polynomial [Ka3]. It was discussed extensively in [St, MS]. In this paper, we mostly use the language of [MS], but do not repeat details to save space. Note, however, that $s(G)$ is written in [MS] as $\Delta_G(1)$, and in [St] as $t(G)$.

In [MS] we observed also that for an n-component link L, $\det(L)$ is odd if and only if L is a knot ($n = 1$), and always $2^{n-1} \mid \det(L)$. Thus now the question on the growth of d_n and d_n^∞ could be reformulated entirely in terms of graph theory:

PROPOSITION 4.1. *d_n^∞ is the maximal number of spanning trees in a planar graph with n edges (multiple edges allowed and counted by multiplicity). d_n is the maximal odd number of spanning trees in a planar graph with n edges.* □

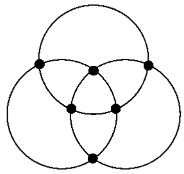
Theorem 2.3 can be interpreted as:

PROPOSITION 4.2. *For infinitely may values of n the planar graphs with n edges and maximal number of spanning trees (or maximal odd number of spanning trees) have no valence-two vertices and no multiple edges.* □

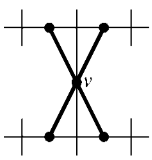
Further properties of the knots K_n are also related to properties of their checkerboard graph. For example, the uniqueness and flype-freeness of K_n imply the uniqueness, up to duality, of the graph G_n with n edges and d_n spanning trees. The achirality and flype-freeness of K_n imply that G_n is self-dual, and the achirality of K_n for itself by the result of [DH] that G_n has a (possibly different) self-dual planar embedding.

4.2. Planar graphs with many spanning trees. We will use now the graph description of the determinant to improve the estimate in theorem 2.3 for $S = \infty$.

Since any link diagram can itself be considered as a planar graph (each crossing being a vertex of valence 4), we can build a new link diagram of which the previous one (regarded as 4-valent graph) is the checkerboard graph. It turns out that this procedure, when iterated, is very good at generating diagrams with high determinant (or graphs with high number of spanning trees), in particular if we start with a clasp-free diagram (4-valent graph with no multiple edges). The simplest such diagram is this of the Borromean rings

$D_0 \quad =$  .

Call this graph D_0. Then one obtains D_{n+1} from D_n by putting a vertex of D_{n+1} to correspond to an edge of D_n and connecting a vertex v of D_{n+1} as follows:



.

(Here the thick lines correspond to edges in D_{n+1} and the thin ones to edges in D_n.) Sometimes D_{n+1} is called the *line graph* of D_n. This procedure doubles the number of edges and vertices. The determinant can still be effectively computed, even for relatively large number of vertices.

LEMMA 4.2. *Let D be an alternating diagram of n crossings. Then in D there is a $1-1$ correspondence between crossings and bridges (all of length 1). Number $n-1$ of the n bridges in D in some way and define a matrix $M=(m_{i,j})_{i,j=1,\dots,n-1}$ by setting for $i,j=1,\dots,n-1$*

$$m_{i,j} := \begin{cases} 2 & i=j\,, \\ -1 & i\neq j \text{ and bridges } i \text{ and } j \text{ meet,} \\ 0 & \text{otherwise}\,. \end{cases}$$

(Here bridges i and j meeting means, without regard the order of i and j, that the crossing overpassed by i is the underpass that marks one of the ends of j.) Then $\det(D)=\det(M)$.

PROOF. This is a classical fact from knot theory. Basically M is a presentation matrix of the Alexander module $\Lambda(t)$ of D specialized at $t=-1$, hence its determinant is the order of this group, which is $\det(D)$. Graph theoretically, this is a variant of the matrix-tree theorem (see [MS]). □

Practical computations were possible for D_n with $n\leq 9$, using MATHEMATICA™, and their result can be briefly summarized in the following table, giving the number of digits of $\det(D_i)$, the CPU time for its calculation, and the number of crossings $c(D_i)$ and components $nc_i=n(D_i)$ of D_i. (Since calculating determinants has cubic complexity, the complexity of $\det(D_n)$ is exponential in n with basis roughly 8. However, in practice the base is about 16, since the number of digits in the integers gets doubled.)

i	# digs $\det(D_i)$	$c(D_i)$	comp. CPU time	nc_i
0	2	6	0	3
1	3	12	0	4
2	6	24	0	6
3	12	48	$0.02''$	8
4	24	96	$0.28''$	12
5	48	192	$4.2''$	16
6	97	384	$1'12''$	24
7	194	768	$19'45''$	32
8	388	1536	$5^h15'40''$	48
9	777	3072	$83^h10'37''$	64

Here D_i is identified with its 4-valent (and not checkerboard) graph, that is, $\det(D_i)=s(D_{i-1})$. (The determinants themselves are clearly too large to print directly, but we will come back to their numerical values in the last section.)

We obtain the following estimates:

LEMMA 4.3. $d^{\infty}_{6\cdot 2^i}\geq\det(D_i)$, $\quad d_{6\cdot 2^i-nc_i+1}\geq\dfrac{\det(D_i)}{2^{nc_i-1}}$ $\quad$ (*where* $nc_i=n(D_i)$).

PROOF. The first claim is trivial. The second one follows by applying $nc_i - 1$ times part 2 in lemma 2.2. □

COROLLARY 4.1. $\tilde{d}_\infty = \sup_k \sqrt[k]{d_k^\infty} \geq \sqrt[6\cdot 2^i]{\det(D_i)}$,

$$\tilde{d} = \sup_k \sqrt[k]{d_k} \geq \sqrt[6\cdot 2^i - nc_i + 1]{\frac{\det(D_i)}{2^{nc_i - 1}}}. \qquad \square$$

With every new value for $\det(D_i)$ we can thus continuously improve the estimate on these suprema (columns 2 and 3 in the table below), and using (11) also the estimates on the number of clasp-free links K_n^∞ in intervals of step 2 of length $6 \cdot 2^i$ (column 4, where l and n refer to (11)). Finally we can show that at least $701/3072$ (or about $2/9$) of all K_n^∞ are clasp-free (in the sense of Banach density).

i	$\tilde{d}_\infty \geq$	$\tilde{d} \geq$	$l/n \leq$
0	1.5874	1.41421	
1	1.64195	1.53746	
2	1.69838	1.62687	
3	1.73436	1.69267	47/48
4	1.75794	1.72884	86/96
5	1.77219	1.75412	161/192
6	1.78064	1.76751	310/384
7	1.78549	1.77699	605/768
8	1.78824	1.78194	1195/1536
9	1.78977	1.78562	2371/3072

The reason for increasingly better estimates is that the step from D_n to D_{n+1} only creates new 4-gons in the graph complement, but no triangles, and the 8 triangles of D_0 become more and more distant as n increases. This heuristic, and some related conjectures, will be explained also in the last section.

5. Some heuristics and problems

As the paper attempts the study of a relatively new type of problem, it is unfortunate, but not surprising, that it opens many more questions than it can answer. At this stage we have only partial progress on conjecture 2.1, and no tools to deal with some of the points raised there. Hoping to whet the interest in further investigations, we conclude by mentioning two further (likely related) problems.

5.1. Braid index. One such problem is that apparently the estimates in part 1) of theorem 3.1 and in corollary 3.2 are not sharp. This is related to the following conjecture:

CONJECTURE 5.1. *Only finitely many K_n have the same braid index $b(K_n)$, or alternatively,* $\liminf_{n\to\infty} b(K_n) = \infty$.

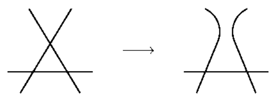
Albeit solving this conjecture appears of considerable difficulty with the present tools, we give some heuristical motivation and approach to it, explaining its relation to some of our previous problems.

The idea behind the conjecture is that the diagrams $\widehat{\beta_i}$ for braids β_i of fixed strand number have either clasps or triangle regions (5) whose distance has an upper bound $k = k_l$ depending only on the strand number l of the β_i. (The distance is here the minimal number of intersections of a path from the one region to the other with the plane curve of the diagram; see also [St2].)

Let

$$\mathcal{D}_l := \{ D : D \text{ is a diagram obtained by splicing crossings in a closed } l\text{-braid diagram} \}.$$

For fixed l, assume we can show that in $\mathcal{D}_l$ any diagram has a clasp, or there are paths of bounded length $\le k$ between triangles passing *only* through 4-gons. This means that, even in the case there is no clasp, the sequence of crossings to splice can be chosen so that we splice the corners of a triangle,



and after k steps we obtain a clasp. Therefore, letting

$$\widehat{d_n} := \max\{ \det(D) : D \text{ is an } n \text{ crossing diagram in } \mathcal{D}_l \}$$

and applying

$$\widehat{d_n} \le \widehat{d}_{n-1} + \widehat{d}_{n-2} + \widehat{d}_{n-3}$$

recursively on each summand on the right, in depth k of the recursion we can in fact use the simpler formula $\widehat{d}_{n'} \le \widehat{d}_{n'-1} + \widehat{d}_{n'-2}$.

Thus $\widehat{d_n} \le \tilde{d}_n$ for a linearly recurrent sequence $\{\tilde{d}_n\}$ with

$$\tilde{d}_n = \sum_{i=1}^{k} a_i \tilde{d}_{n-i},$$

and the Tribonacci numbers T_n (see equation (6) and explanation after it) satisfy

$$T_n = \sum_{i=1}^{k} a'_i T_{n-i}$$

such that $0 \le a_i \le a'_i$ and $a_i < a'_i$ for at least one i. Writing down the generating series of $\tilde{d}_n$ and T_n, the denominator polynomials are $f(x) = \sum_{i=1}^{k} a_i x^i - 1$ and $f_1(x) = \sum_{i=1}^{k} a'_i x^i - 1$

resp. On the positive real line, f and f_1 have unique zeros z_f and z_{f_1}, which are the unique zeros of minimal norm for these functions (use $a_i, a_i' \geq 0$ and apply triangle inequality).

Now $z_{f_1}^{-1} = \limsup_{n\to\infty} \sqrt[n]{T_n} = \delta$ and $z_f^{-1} = \limsup_{n\to\infty} \sqrt[n]{\tilde{d}_n}$, and because $f_1(x) > f(x)$ for $x > 0$, we have $z_f > z_{f_1}$.

Thus there will be a sequence $\{\delta_l\}$ with $\delta_l < \delta_{l+1} < \delta$ and possibly $\delta_l \to \delta$ such that in part 1) of theorem 3.1, 'δ' can be replaced by 'δ_l' for l-strand braids $\{\beta_i\}$. Then we need to answer positively question 2.2. Finally, to control $b(K_n)$, we must prove part 6) of conjecture 2.1 (at least for large n) and apply [Mu].

5.2. Large determinant examples. The next question concerns the examples in §4. The determinants $\det(D_i)$ can be given as follows:

i	prime factorization of $\det(D_i)$
0	2^4
1	$2^7\ 3^1$
2	$2^{12}\ 3^4$
3	$2^{17}\ 3^1\ 5^6\ 7^2$
4	$2^{41}\ 3^2\ 5^{11}\ 7^3$
5	$2^{51}\ 3^{16}\ 5^2\ 11^6\ 13^3\ 23^2\ 37^3\ 127^3$
6	$2^{122}\ 3^{10}\ 5^6\ 7^2\ 11^3\ 13^3\ 17^1\ 19^2\ 31^2\ 43^3\ 421^3\ 4217^3\ 9661^3$
7	$2^{141}\ 3^4\ 7^{10}\ 17^9\ 769^2\ 4241^3\ 22391^3\ 42767^3\ 195863^3\ 483557^2\ 2072131^2\ 6046751^3\ 355243279^3$
8	$2^{315}\ 3^4\ 5^6\ 7^2\ 11^3\ 17^8\ 31^3\ 47^2\ 79^2\ 89^3\ 97^2\ 157^6\ 577^1\ 6271^2\ 20639^3\ 291349^2\ 1159901^3$
	$1579631^3\ 43863223^2\ 323965910452099^3\ 209443904414934601^3\ 3786663141306774259^3$
9	??

(The factorization for $i = 9$ was too hard to obtain, and would be too long anyway.)

Note, that these factorizations are strikingly non-generic—the largest prime factors have only about 1/20 of the number of digits of their product, and almost all primes occur in higher powers. (The only power that can be explained so far, and still to much smaller extent than it occurs, is that of 2; see §4.1.)

QUESTION 5.1. Is there a closed formula for $\det(D_i)$? Can it be used to show that $\sqrt[6\cdot 2^i]{\det(D_i)} \to \delta$?

Note also that the diagrams D_i have, like those of K_n for $n = 12, \ldots, 16$ in figure 1, no ≥ 5-gonal regions (which for clasp-free diagrams is equivalent to having exactly 8 triangles). So one may wonder whether this property holds generally (or at least generically) for K_n^S, in addition to flype- and clasp-freeness.

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