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On the Parity Conjecture for Multiple L-values of Conductor Four

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Abstract. In this paper, we prove that the multiple L-value of conductor 4 can be expressed in terms of lower depth multiple L-values under the condition on the parity of its depth and weight. This can be regarded as a character analogue of what is called the "parity result" for multiple zeta values which was proved by Zagier.

1. Introduction

Let $\mathbf{N}$ be the set of natural numbers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{Z}$ the ring of rational integers, $\mathbf{Q}$ the field of rational numbers, $\mathbf{R}$ the field of real numbers and $\mathbf{C}$ the field of complex numbers.

Let χ be a Dirichlet character. The multiple L-value of depth r and of weight $\sum_{j=1}^{r} k_j$ can be defined by

$$L(k_1, k_2, \ldots, k_r; \chi) = \sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{\chi(n_1)\chi(n_2)\cdots\chi(n_r)}{n_1^{k_1}(n_1+n_2)^{k_2}\cdots(n_1+\cdots+n_r)^{k_r}} \tag{1}$$

for $k_1, \ldots, k_r \in \mathbf{N}$. Arakawa and Kaneko proved some relation formulas for them by considering the shuffle product (see [1]). In particular when χ is the trivial character χ_0, $L(k_1, k_2, \ldots, k_r; \chi_0)$ is the multiple zeta value (also called the Euler-Zagier sum).

In [2], Borwein and Girgensohn conjectured the following fascinating result which is called the parity result or the parity conjecture for multiple zeta values.

PARITY RESULT. *For $r \in \mathbf{N}$ with $r \geq 2$ and $(k_1, \ldots, k_r) \in \mathbf{N}$ with $k_r \geq 2$, $\zeta(k_1, \ldots, k_r)$ can be expressed in terms of lower depth multiple zeta values when its depth and weight are of different parity.*

The case of depth 2 has been already considered by Euler, and the case of depth 3 was proved by Borwein and Girgensohn in [2]. Recently Zagier (with Ihara and Kaneko) gave the proof in the general case (see [4] § 8). More recently the author gave another proof of this result in a different method ([7]).

As a next target, we would like to prove the parity result for multiple L-values. But it seems to be hard. Indeed, Terhune [5] proved a kind of the parity result for another type

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of double L-values. However a complicated calculation is necessary to prove it even for the double L-values. At present, no parity results for general multiple L-values have been known.

The aim of this paper is to prove the parity result for the multiple L-values of general depth attached to the primitive Dirichlet character ψ of conductor 4. Namely $\psi(1) = 1$, $\psi(3) = -1$ and $\psi(2) = \psi(4) = 0$, and

$$\begin{aligned} L(k_1, k_2, \ldots, k_r; \psi) \\ &= \sum_{j_1,\ldots,j_r=0}^{\infty} \frac{(-1)^{j_1} \cdots (-1)^{j_r}}{(2j_1+1)^{k_1}(2j_1+2j_2+2)^{k_2} \cdots (2j_1+\cdots+2j_r+r)^{k_r}} \\ &= \sum_{0 \le m_1 \le \cdots \le m_r} \frac{(-1)^{m_r}}{(2m_1+1)^{k_1}(2m_2+2)^{k_2} \cdots (2m_r+r)^{k_r}} . \end{aligned} \tag{2}$$

Furthermore we let ψ^2 be the non-primitive character such that $\psi^2(n) = \psi(n)^2$ for $n \in \mathbf{Z}$, and consider

$$L(k_1, \ldots, k_r; \psi^2) = \sum_{0 \le m_1 \le \cdots \le m_r} \frac{1}{(2m_1+1)^{k_1} \cdots (2m_r+r)^{k_r}} \tag{3}$$

for $k_1, \ldots, k_r \in \mathbf{N}$ with $k_r \ge 2$.

For $r \in \mathbf{N}$, let Λ_r be the $\mathbf{Q}$-algebra generated by

$$\bigcup_{m=1}^{r} \bigcup_{\chi \in \{\psi, \psi^2\}} \{L(j_1, \ldots, j_m; \chi) \mid (j_1, \ldots, j_m) \in \mathbf{N}^m, \; j_m > 1 \text{ (if } \chi = \psi^2)\} .$$

Note that $\pi \in \Lambda_r$ because of the well-known formula $L(1; \psi) = \pi/4$. Using these notation, we prove the following theorem by the method introduced in our previous work (see [6, 7, 8]).

THEOREM 1. *For $r \in \mathbf{N}$ with $r \ge 2$ and $(k_1, \ldots, k_r) \in \mathbf{N}$ with $k_r \ge 2$, $L(k_1, \ldots, k_r; \psi^2) \in \Lambda_{r-1}$ holds when its depth r and its weight $\sum_{j=1}^{r} k_j$ are of different parity. Furthermore, $L(k_1, \ldots, k_r; \psi) \in \Lambda_{r-1}$ holds when its weight $\sum_{j=1}^{r} k_j$ is odd.*

2. Preliminaries

We make use of the notation and quote some results in[6, 7, 8]. Let $\delta \in \mathbf{R}$ with $\delta > 0$ and $u \in \mathbf{R}$ with $1 \le u \le 1 + \delta$. We define

$$\rho(s; u) = \sum_{m=0}^{\infty} \frac{(-u)^{-m}}{(2m+1)^s} \tag{4}$$

for $s \in \mathbf{C}$. If $u > 1$ then $\rho(s; u)$ is convergent for any $s \in \mathbf{C}$. Note that $\rho(s; 1) = L(s; \psi)$ and $\rho(2j+1; 1)\pi^{-2j-1} \in \mathbf{Q}$ for $j \in \mathbf{N}_0$ (see (47)). Let

$$F(x; u) = \frac{2ue^x}{e^{2x} + u} = \sum_{m=0}^{\infty} \mathcal{E}_m(u) \, \frac{x^m}{m!} \tag{5}$$

for $x \in \mathcal{D}(\pi/2) = \{x \in \mathbf{C} \mid |x| < \frac{\pi}{2}\}$. From [6] Section 2, we have

$$\rho(-j; u) = \frac{1}{2}\mathcal{E}_j(u) \quad (j \in \mathbf{N}_0)\,, \tag{6}$$

$$\mathcal{E}_{2N+1}(1) = 0 \quad (N \in \mathbf{N}_0)\,. \tag{7}$$

Let $\gamma \in \mathbf{R}$ with $0 < \gamma < \pi/2$, and $C_\gamma : z = \gamma e^{it}$ for $0 \le t \le 2\pi$, where $i = \sqrt{-1}$. From (5), we can easily check that

$$\int_{C_\gamma} F(z; u) z^{-n-1} dz = \frac{(2\pi i)\mathcal{E}_n(u)}{n!} \quad (n \in \mathbf{N}_0)\,.$$

Let $M_1(\gamma) = \max |F(z, u)|$ for $(z, u) \in C_\gamma \times [1, 1+\delta]$. Then we obtain

$$\frac{|\mathcal{E}_n(u)|}{n!} \le \frac{M_1(\gamma)}{\gamma^n} \tag{8}$$

for any $n \in \mathbf{N}_0$. This means that (5) is uniformly convergent in the wider sense with respect to $(x, u) \in \mathcal{D}(\pi/2) \times [1, 1+\delta]$. For $\theta \in (-\pi/2, \pi/2) \subset \mathbf{R}$ and $u \in [1, 1+\delta]$, we let

$$\mathcal{G}(\theta; u) = \frac{1}{i\pi}\sum_{j=0}^{\infty} \mathcal{E}_{2j+1}(u)\frac{(i\theta)^{2j+1}}{(2j+1)!}\,; \quad \mathcal{H}(\theta; u) = \frac{1}{\pi}\sum_{j=0}^{\infty} \mathcal{E}_{2j}(u)\frac{(i\theta)^{2j}}{(2j)!}\,. \tag{9}$$

From (6), we see that if $u \in (1, 1+\delta]$ then

$$\begin{aligned} \mathcal{G}(\theta; u) &= \frac{2}{\pi}\sum_{m=0}^{\infty}(-u)^{-m} \sin((2m+1)\theta)\,; \\ \mathcal{H}(\theta; u) &= \frac{2}{\pi}\sum_{m=0}^{\infty}(-u)^{-m} \cos((2m+1)\theta)\,, \end{aligned} \tag{10}$$

where we let $\lambda_m = \{1 + (-1)^m\}/2$ for $m \in \mathbf{Z}$. From (7)–(9), we have

$$\lim_{u \to 1+0} \mathcal{G}(\theta; u) = 0\,. \tag{11}$$

For $s_1, \ldots, s_r \in \mathbf{C}$ and $u \in [1, 1+\delta]$, we let

$$\mathfrak{L}_r(s_1, \ldots, s_r; \psi; u) = \sum_{0 \le m_1 \le \ldots \le m_r} \frac{(-u)^{-m_r}}{(2m_1+1)^{s_1} \cdots (2m_r+r)^{s_r}}\,, \tag{12}$$

$$\mathfrak{L}_r(s_1, \ldots, s_r; \psi^2; u) = \sum_{0 \le m_1 \le \ldots \le m_r} \frac{u^{-m_r}}{(2m_1+1)^{s_1} \cdots (2m_r+r)^{s_r}}\,. \tag{13}$$

We denote the pth derivative of $\sin(X)$ by $\sin^{(p)}(X)$, and further denote $\sin^{(p)}(X)\big|_{X=\alpha}$ by $\sin^{(p)}(\alpha)$ for $\alpha \in \mathbf{R}$. For $a \in \mathbf{N}$, b, $p \in \mathbf{N}_0$, $(k_1, \ldots, k_{r-1}) \in \mathbf{N}^{r-1}$, $u \in [1, 1+\delta]$ and

$\theta \in [-\pi/2, \pi/2]$, we define

$$\begin{aligned}\mathcal{R}_r^p(\theta; k_1, \dots, k_{r-1}; a, b; u) &= \frac{i^{1-p}}{\pi^r} \sum_{\nu=0}^{b} \binom{a-1+b-\nu}{b-\nu} \frac{(-\theta)^\nu}{\nu!} \\ &\times \sum_{0 \le m_1 \le \dots \le m_r} \frac{(-u)^{-m_r} \sin^{(\nu+p)}((2m_r + r)\theta)}{(2m_1+1)^{k_1} \cdots (2m_{r-1}+r-1)^{k_{r-1}} (2m_r + r)^{a+b-\nu}} \,.\end{aligned} \tag{14}$$

Since $\sin^{(\nu+2)} \theta = -\sin^{(\nu)} \theta$ and $i^2 = -1$, we have

$$\mathcal{R}_r^p(\theta; k_1, \dots, k_{r-1}; a, b; u) = \mathcal{R}_r^{p+2}(\theta; k_1, \dots, k_{r-1}; a, b; u) \,. \tag{15}$$

Then we obtain the following lemma. Note that (just as elsewhere in this paper) an empty sum is to be interpreted as zero.

LEMMA 1. *Let $u \in [1, 1+\delta]$. If $a + \lambda_{b+r} \ge 2$, then*

$$\begin{aligned}\mathcal{R}_r^0(\pi/2; k_1, \dots, k_{r-1}; a, b; u) &= \frac{i}{\pi^r} \sum_{\nu=0}^{b} \binom{a-1+b-\nu}{b-\nu} (-1)^{(\nu+r-1)/2} \lambda_{\nu+r+1} \\ &\times \mathfrak{L}_r(k_1, \dots, k_{r-1}, a+b-\nu; \psi^2; u) \frac{(-\pi/2)^\nu}{\nu!} \,.\end{aligned} \tag{16}$$

If $a + \lambda_{b+r+1} \ge 2$, then

$$\begin{aligned}\mathcal{R}_r^1(\pi/2; k_1, \dots, k_{r-1}; a, b; u) &= \frac{1}{\pi^r} \sum_{\nu=0}^{b} \binom{a-1+b-\nu}{b-\nu} (-1)^{(\nu+r)/2} \lambda_{\nu+r} \\ &\times \mathfrak{L}_r(k_1, \dots, k_{r-1}, a+b-\nu; \psi^2; u) \frac{(-\pi/2)^\nu}{\nu!} \,.\end{aligned} \tag{17}$$

In particular, for $\mu \in \{0, 1\}$,

$$\begin{aligned}&\mathcal{R}_r^\mu(\pi/2; k_1, \dots, k_{r-1}; a, 0; u) \\ &\quad = \begin{cases} 0 & (p \equiv r \pmod 2)\,; \\ \dfrac{i^{1-\mu}}{\pi^r} L(k_1, \dots, k_{r-1}, a; \psi^2) & (p \not\equiv r \pmod 2)\,, \end{cases}\end{aligned} \tag{18}$$

and

$$(i\pi)^{\mu-1} \mathcal{R}_r^\mu(\pi/2; k_1, \dots, k_{r-1}; a, b; 1) \in \frac{1}{\pi^{r+1-\mu}} \Lambda_r \,. \tag{19}$$

PROOF. We can easily check that

$$\sin^{(\nu)}\left((2m+r)\frac{\pi}{2}\right) = (-1)^{m+(\nu+r-1)/2} \lambda_{\nu+r+1}$$

for $\nu, m \in \mathbf{N}_0$. From (14), we have the assertion.

Now we prepare some lemmas similar to those in [7] Section 2 as follows. From (2.16) in [7], we have the formal relation

$$\sum_{\nu=0}^{b}\binom{a-1+b-\nu}{b-\nu}\frac{(-\theta)^{\nu}}{\nu!}\frac{\sin^{(\nu+p)}(\theta x)}{x^{a+b-\nu}} = i^{p-1}\sum_{N=0}^{\infty}\binom{a-1+b-N}{b}\frac{(i\theta)^N}{N!}\lambda_{p+1+N}\,x^{-a-b+N}\,. \tag{20}$$

LEMMA 2. *With the above notation and for $u \in (1, 1+\delta]$,*

$$\mathcal{R}_r^p(\theta; k_1, \ldots, k_{r-1}; a, b; u) = \frac{1}{\pi^r}\sum_{N=0}^{\infty}\binom{a-1+b-N}{b} \times \mathfrak{L}_r(k_1, \ldots, k_{r-1}, a+b-N; \psi; u)\lambda_{p+1+N}\frac{(i\theta)^N}{N!}\,. \tag{21}$$

In particular, for $c \in \mathbf{N}_0$ we have

$$\mathcal{R}_r^{p+c}(\theta; k_1, \ldots, k_{r-1}; a+c, b; u) = \frac{1}{\pi^r}\sum_{m=-c}^{\infty}(-1)^b\binom{m-a}{b} \times \mathfrak{L}_r(k_1, \ldots, k_{r-1}, a+b-m; \psi; u)\lambda_{p+1+m}\frac{(i\theta)^{m+c}}{(m+c)!}\,. \tag{22}$$

PROOF. By (12), (14) and (20), we obtain (21). (22) can be proved by replacing p with $p+c$, a with $a+c$ and putting $N = m+c$ in (21), and by using the well-known relation

$$\binom{-X}{j} = (-1)^j\binom{X+j-1}{j}.$$

LEMMA 3. *With the above notation and for $u \in (1, 1+\delta]$,*

$$i\mathcal{R}_r^{p+1}(\theta; k_1, \ldots, k_{r-1}; a, b; u)\mathcal{G}(\theta; u) + \mathcal{R}_r^p(\theta; k_1, \ldots, k_{r-1}; a, b; u)\mathcal{H}(\theta; u) = \frac{2}{\pi^{r+1}}\sum_{m=0}^{\infty}\left\{\sum_{\nu=0}^{b}(-1)^{\nu}\binom{m}{\nu}\binom{a-1+b-\nu}{b-\nu} \times \mathfrak{L}_{r+1}(k_1, \ldots, k_{r-1}, a+b-\nu, \nu-m; \psi; u)\right\}\lambda_{p+1+m}\frac{(i\theta)^m}{m!}\,. \tag{23}$$

PROOF. By (10), (12), (14) and using the well-known relations

$$\sin^{(k+1)}\alpha\cdot\sin\beta + \sin^{(k)}\alpha\cdot\cos\beta = \sin^{(k)}(\alpha+\beta)$$

and

$$\sin^{(p)}(\theta) = i^{p-1}\sum_{n=0}^{\infty}\lambda_{p+1+n}\frac{(i\theta)^n}{n!}\,,$$

we can verify that the left-hand side of (23) equals to

$$
\begin{aligned}
&\frac{2}{i^{p-1}}\sum_{\nu=0}^{b}\binom{a-1+b-\nu}{b-\nu}\frac{(-\theta)^{\nu}}{\nu!}\\
&\quad\times\sum_{m_1<\cdots<m_r}\frac{(-u)^{-m_{r+1}}\sin^{(\nu+p)}((2m_{r+1}+r+1)\theta)}{(2m_1+1)^{k_1}\cdots(2m_{r-1}+r-1)^{k_{r-1}}(2m_r+r)^{a+b-\nu}}\\
&=\frac{2}{i^{p-1}}\sum_{\nu=0}^{b}\binom{a-1+b-\nu}{b-\nu}\frac{(-\theta)^{\nu}}{\nu!}\\
&\quad\times i^{\nu+p-1}\sum_{n=0}^{\infty}\mathfrak{L}_{r+1}(k_1,\ldots,k_{r-1},a+b-\nu,-n;\psi;u)\lambda_{\nu+p+1+n}\frac{(i\theta)^n}{n!}\,.
\end{aligned}
$$

From the binomial theorem, this equals to

$$
\begin{aligned}
&2\sum_{m=0}^{\infty}\sum_{\nu=0}^{b}\binom{m}{\nu}(-1)^{\nu}\binom{a-1+b-\nu}{b-\nu}\\
&\quad\times\mathfrak{L}_{r+1}(k_1,\ldots,k_{r-1},a+b-\nu,\nu-m;\psi;u)\lambda_{p+1+m}\frac{(i\theta)^m}{m!}\,.
\end{aligned}
$$

Thus we have the assertion.

Let $a\in\mathbf{N}$, $b\in\mathbf{N}_0$ and $(k_1,\ k_2,\ldots,k_{r-1})\in\mathbf{N}^{r-1}$ and $u\in(1,1+\delta]$. For $m\in\mathbf{Z}$, we define

$$
\begin{aligned}
\mathcal{A}_m(k_1,\ldots,k_{r-1};a,b;u)=&\frac{2}{\pi^{r+1}}\sum_{\nu=0}^{b}(-1)^{\nu}\binom{m}{\nu}\binom{a-1+b-\nu}{b-\nu}\\
&\times\mathfrak{L}_{r+1}(k_1,\ldots,k_{r-1},a+b-\nu,\nu-m;\psi;u)\,.
\end{aligned}
\tag{24}
$$

In particular when $m\leq -1$, we can define

$$
\mathcal{A}_m(k_1,\ldots,k_{r-1};a,b;1)=\lim_{u\to 1+0}\mathcal{A}_m(k_1,\ldots,k_{r-1};a,b;u)\,. \tag{25}
$$

Lemma 3 states that

$$
\begin{aligned}
&i\mathcal{R}_r^{p+1}(\theta;k_1,\ldots,k_{r-1};a,b;u)\mathcal{G}(\theta;u)+\mathcal{R}_r^{p}(\theta;k_1,\ldots,k_{r-1};a,b;u)\mathcal{H}(\theta;u)\\
&\quad=\sum_{m=0}^{\infty}\mathcal{A}_m(k_1,\ldots,k_{r-1};a,b;u)\lambda_{p+1+m}\frac{(i\theta)^m}{m!}\,.
\end{aligned}
\tag{26}
$$

LEMMA 4. *With the above notation and for $c \in \mathbf{N}_0$,*

$$\sum_{\nu=0}^{b} \binom{a-1+b-\nu}{b-\nu} \mathcal{R}_{r+1}^{p+c+1}(\theta; k_1, \ldots, k_{r-1}, a+b-\nu; c, \nu; u) \\ = \frac{1}{2} \sum_{m=-c}^{\infty} \mathcal{A}_m(k_1, \ldots, k_{r-1}; a, b; u) \lambda_{p+m} \frac{(i\theta)^{m+c}}{(m+c)!} . \tag{27}$$

PROOF. By applying (22) to the left-hand side of (27) and using (24), we obtain the asserted formula.

PROPOSITION 1. *With the above notation, $\mathcal{R}_r^p(\theta; k_1, \ldots, k_{r-1}; a, b; u)$ is defined and holomorphic for all $\theta \in \mathcal{D}(\pi/2)$ when $u \in [1, 1+\delta]$. Furthermore, for any $\gamma \in \mathbf{R}$ with $0 < \gamma < \pi/2$, there exists a constant $\mathfrak{M}_r(\gamma)(> 0)$ independent of u such that*

$$\frac{|\mathcal{A}_m(k_1, \ldots, k_{r-1}; a, b; u)|}{m!} \le \frac{\mathfrak{M}_r(\gamma)}{\gamma^m} \quad (m \in \mathbf{N}_0,\ u \in (1, 1+\delta]) . \tag{28}$$

In particular

$$\liminf_{m\to\infty} \left\{ \frac{|\mathcal{A}_m(k_1, \ldots, k_{r-1}; a, b; u)|}{m!} \right\}^{-1/m} \ge \frac{\pi}{2} \quad (u \in (1, 1+\delta]) . \tag{29}$$

PROOF. We prove this proposition by induction on $r \in \mathbf{N}$. When $r = 1$, by (21), we have

$$\mathcal{R}_1^p(\theta; ; a, b; u) \\ = \frac{1}{\pi} \sum_{N=0}^{\infty} \binom{a-1+b-N}{b} \rho(a+b-N; u) \lambda_{p+1+N} \frac{(i\theta)^N}{N!} \tag{30}$$

for $u \in (1, 1+\delta]$. From (8), the right-hand side of (30) is uniformly convergent with respect to $(\theta, u) \in [-\gamma, \gamma] \times [1, 1+\delta]$ for any $\gamma \in \mathbf{R}$ with $0 < \gamma < \pi/2$. Hence (30) holds for $u \in [1, 1+\delta]$ when $\theta \in (-\pi/2, \pi/2)$. Namely, for $u \in [1, 1+\delta]$, $\mathcal{R}_1^p(\theta; ; a, b; u)$ is defined and holomorphic for all $\theta \in \mathcal{D}(\pi/2)$ and continuous for all $(\theta, u) \in \mathcal{D}(\pi/2) \times [1, 1+\delta]$. By (26), we have

$$i\mathcal{R}_1^{p+1}(\theta; ; a, b; u)\mathcal{G}(\theta; u) + \mathcal{R}_1^p(\theta; ; a, b; u)\mathcal{H}(\theta; u) \\ = \sum_{m=0}^{\infty} \mathcal{A}_m(; a, b; u) \lambda_{p+1+m} \frac{(i\theta)^m}{m!} \tag{31}$$

for $u \in (1, 1+\delta]$. Furthermore, it follows from the above consideration that the left-hand side of (31) is holomorphic for $\theta \in \mathcal{D}(\pi/2)$ and continuous for $(\theta, u) \in \mathcal{D}(\pi/2) \times [1, 1+\delta]$. Hence, by the same method as in the proof of (8), we obtain, for $\gamma \in \mathbf{R}$ with $0 < \gamma < \pi/2$,

$$\frac{|\mathcal{A}_m(; a, b; u)|}{m!} \le \frac{\mathfrak{M}_1(\gamma)}{\gamma^m} \quad (m \in \mathbf{N}_0,\ u \in (1, 1+\delta]) , \tag{32}$$

where

$$\mathfrak{M}_1(\gamma) = \max_{\substack{(\theta,u,p)\in \\ C_\gamma\times[1,1+\delta]\times\{0,1\}}} |i\mathcal{R}_1^{p+1}(\theta;\,;a,b;u)\mathcal{G}(\theta;u) + \mathcal{R}_1^{p}(\theta;\,;a,b;u)\mathcal{H}(\theta;u)|\,.$$

Thus we have the assertion in the case when $r = 1$.

Next we assume that the case of r holds. Then, for any $\gamma \in \mathbf{R}$ with $0 < \gamma < \pi/2$, there exists a constant $\mathfrak{M}_r(\gamma)(> 0)$ independent of u such that

$$\frac{|\mathcal{A}_m(k_1,\ldots,k_{r-1};a,b;u)|}{m!} \le \frac{\mathfrak{M}_r(\gamma)}{\gamma^m} \quad (m \in \mathbf{N}_0,\ u \in (1, 1+\delta])\,.$$

In particular when $a = k_r \in \mathbf{N}$ and $b = 0$, it follows from (24) that

$$\frac{2|\mathfrak{L}_{r+1}(k_1,\ldots,k_{r-1},k_r,-m;\psi;u)|}{\pi^{r+1}\,m!} \le \frac{\mathfrak{M}_r(\gamma)}{\gamma^m} \quad (m \in \mathbf{N}_0,\ u \in (1, 1+\delta])\,.$$

Hence the right-hand side of (21) in the case of $r+1$ is uniformly convergent in the wider sense with respect to $(\theta,u) \in (-\pi/2,\pi/2)\times[1,1+\delta]$. Therefore, for $u \in [1, 1+\delta]$, $\mathcal{R}_{r+1}^{p}(\theta;k_1,\ldots,k_r;a,b;u)$ is defined and holomorphic for all $\theta \in \mathcal{D}(\pi/2)$ and continuous for all $(\theta,u) \in \mathcal{D}(\pi/2)\times[1,1+\delta]$. Using (26) in the case of $r+1$ and the same method as above, we have the assertion in the case of $r+1$. By induction, we obtain the proof.

3. Proof of Theorem 1 in the case of depth 2

In this section, we prove Theorem 1 in the case when $r = 2$, namely prove that $L(k,l;\psi) \in \Lambda_1$ and $L(k,l;\psi^2) \in \Lambda_1$ for $k,l \in \mathbf{N}$ with $k+l \equiv 1 \pmod 2$, where $l \ge 2$ in the case of ψ^2.

We formally define $\mathcal{E}_j^1(u) = 2\rho(-j;u)$ for any $j \in \mathbf{Z}$. Note that $\mathcal{E}_j^1(u) = \mathcal{E}_j(u)$ for $j \in \mathbf{N}_0$. From (21) with $b = 0$, we have

$$\mathcal{R}_1^{p}(\theta;\,;a,0;u) = \frac{1}{2\pi}\sum_{N=0}^{\infty}\mathcal{E}_{N-a}^1(u)\lambda_{p+1+N}\frac{(i\theta)^N}{N!}\,, \tag{33}$$

because $\mathfrak{L}_1(s;\psi;u) = \rho(s;u)$. For $k \in \mathbf{N}$, $p \in \mathbf{N}_0$, $u \in [1,1+\delta]$ and $\theta \in [-\pi/2,\pi/2]$, let

$$\mathcal{I}_1^{p}(\theta;k;u) = \mathcal{R}_1^{p}(\theta;\,;k,0;u) - \frac{1}{2\pi}\sum_{j=0}^{k-1}\mathcal{E}_{j-k}^1(u)\lambda_{p+1+j}\frac{(i\theta)^j}{j!}\,, \tag{34}$$

$$\mathcal{J}_1^{p}(\theta;k;u) = \mathcal{R}_1^{p+1}(\theta;\,;k,0;u)\,. \tag{35}$$

If $u \in (1, 1+\delta]$ then

$$\mathcal{I}_1^{p}(\theta;k;u) = \frac{1}{2\pi}\sum_{m=0}^{\infty}\mathcal{E}_m^1(u)\lambda_{m+p+1+k}\frac{(i\theta)^{m+k}}{(m+k)!}\,, \tag{36}$$

$$\mathcal{J}_1^p(\theta;k;u) = \frac{1}{2\pi}\sum_{n=0}^{\infty} \mathcal{E}_{n-k}^1(u)\lambda_{n+p}\frac{(i\theta)^n}{n!}\,. \tag{37}$$

From Proposition 1, we obtain the following (see [6] (2.8)).

LEMMA 5. *Let $k \in \mathbf{N}$ and $\theta \in (-\pi/2, \pi/2)$. Then $\mathcal{R}_1^p(\theta;\,;a,0;u)$, $\mathcal{I}_1(\theta;k;u)$ and $\mathcal{J}_1(\theta;k;u)$ can be defined and holomorphic for $\theta \in \mathcal{D}(\pi/2)$ when $u \in [1, 1+\delta]$, and continuous for $(\theta, u) \in \mathcal{D}(\pi/2) \times [1, 1+\delta]$. Furthermore $\lim_{u\to 1+0} \mathcal{I}_1(\theta;k;u) = 0$.*

For $n \in \mathbf{Z}$ and $u \in (1, 1+\delta]$, we define

$$\mathcal{E}_n^2(k;u) = 2\mathfrak{L}_2(k,-n;\psi;u) - \sum_{j=0}^{k-1}\binom{n}{j}\mathcal{E}_{j-k}^1(u)\lambda_{k+1+j}\rho(j-n;u)\,. \tag{38}$$

In particular when $n \le -1$, we define $\mathcal{E}_n^2(k;1)$ by (38) with $u = 1$. We can prove the following assertions by the same method as in [7].

LEMMA 6. *For $k \in \mathbf{N}$, $p \in \mathbf{N}_0$ and $u \in (1, 1+\delta]$,*

$$i\mathcal{J}_1^p(\theta;k;u)\mathcal{G}(\theta;u) + \mathcal{I}_1^p(\theta;k;u)\mathcal{H}(\theta;u) = \frac{1}{\pi^2}\sum_{N=0}^{\infty} \mathcal{E}_n^2(k;u)\lambda_{k+1+N}\frac{(i\theta)^N}{N!}\,.$$

PROOF. Applying (23) with $(a,b,p,r) = (k,0,k,1)$, we have

$$\begin{aligned} &i\mathcal{R}_1^{p+1}(\theta;\,;k,0;u)\mathcal{G}(\theta;u) + \mathcal{R}_1^p(\theta;\,;k,0;u)\mathcal{H}(\theta;u) \\ &\quad = \frac{2}{\pi^2}\sum_{N=0}^{\infty} \mathfrak{L}_2(k,-N;\psi;u)\lambda_{p+1+N}\frac{(i\theta)^N}{N!}\,. \end{aligned} \tag{39}$$

From (6), (9) and using the binomial theorem, we have

$$\begin{aligned} &\frac{1}{\pi}\sum_{j=0}^{k-1}\rho(k-j;u)\lambda_{k+1+j}\frac{(i\theta)^j}{j!}\mathcal{H}(\theta;u)\,. \\ &\quad = \frac{2}{\pi^2}\sum_{N=0}^{\infty}\left\{\sum_{j=0}^{k-1}\binom{N}{j}\rho(k-j;u)\lambda_{k+1+j}\rho(j-N;u)\right\}\lambda_{k+1+N}\frac{(i\theta)^N}{N!}\,, \end{aligned}$$

because $\lambda_{p+r}\lambda_{q+r} = \lambda_{p+r}\lambda_{p+q}$. By (38), we have the assertion.

LEMMA 7. *For $k \in \mathbf{N}$ and $\gamma \in \mathbf{R}$ with $0 < \gamma < \pi/2$, there exists a constant $M_2(\gamma) > 0$ independent of u such that*

$$\frac{|\mathcal{E}_n^2(k;u)|}{n!} \le \frac{M_2(\gamma)}{\gamma^n} \quad (n \in \mathbf{N}_0,\ u \in (1, 1+\delta])\,, \tag{40}$$

in particular

$$\liminf_{n\to\infty}\left\{\frac{|\mathcal{E}_n^2(k;u)|}{n!}\right\}^{-1/n} \ge \frac{\pi}{2} \quad (u \in (1, 1+\delta])\,. \tag{41}$$

Furthermore

$$\lim_{u\to 1+0} \mathcal{E}_n^2(k;u)\lambda_{k+1+n} = 0 \quad (n \in \mathbf{N}_0)\,. \tag{42}$$

PROOF. By Proposition 1 with $r=2$ and $b=0$, we obtain

$$\frac{|\mathfrak{L}_2(k,-n;\psi;u)|}{n!} \le \frac{\tilde{\mathfrak{M}}_2(\gamma)}{\gamma^n} \quad (n \in \mathbf{N}_0,\ u \in (1,1+\delta])$$

for a certain $\tilde{\mathfrak{M}}_2(\gamma)(>0)$ independent of u. Combining this and (8), it follows from (38) that (40), namely (41) holds. Hence the equation of Lemma 6 holds for $u=1$, and tends to 0 as $u \to 1+0$ by (11) and Lemma 5. Thus we obtain (42).

PROPOSITION 2. *For $k,l \in \mathbf{N}$, $\mu \in \{0,1\}$, $u \in (1,1+\delta]$ and $\theta \in (-\pi/2,\pi/2)$,*

$$\begin{aligned}&\mathcal{R}_2^{k+l+\mu}(\theta;k;l,0;u) - \frac{1}{2\pi}\sum_{j=0}^{k-1}\mathcal{E}_{j-k}^1(u)(-1)^j\lambda_{k+1+j}\mathcal{R}_1^{k+l+\mu}(\theta;;l,j;u)\\ &\quad = \frac{1}{2\pi^2}\sum_{m=-l}^{\infty}\mathcal{E}_m^2(k;u)\lambda_{k+1+m+\mu}\frac{(i\theta)^{m+l}}{(m+l)!}\,.\end{aligned} \tag{43}$$

PROOF. By (22) with $(a,b,c,p,r) = (0,0,l,k+\mu,2)$ and $(0,j,l,k+\mu,1)$, we obtain (43) when $\theta \in (-\pi/2,\pi/2)$.

By (40), the right-hand side of (43) is uniformly convergent with respect to $u \in (1,1+\delta]$. Hence we let $u \to 1+0$ in (43). Then it follows from (42) that

$$\begin{aligned}&\mathcal{R}_2^{k+l}(\theta;k;l,0;1) - \frac{1}{2\pi}\sum_{j=0}^{k-1}\mathcal{E}_{j-k}^1(1)(-1)^j\lambda_{k+1+j}\mathcal{R}_1^{k+l}(\theta;;l,j;1)\\ &\quad = \frac{1}{2\pi^2}\sum_{m=-l}^{-1}\mathcal{E}_m^2(k;1)\lambda_{k+1+m}\frac{(i\theta)^{m+l}}{(m+l)!}\\ &\quad = \frac{1}{2\pi^2}\sum_{\nu=0}^{l-1}\mathcal{E}_{\nu-l}^2(k;1)\lambda_{k+1+l+\nu}\frac{(i\theta)^{\nu}}{\nu!}\,.\end{aligned} \tag{44}$$

In particular when $l \ge 2$ and $\mu = 0$, both sides of (44) are continuous for $\theta \in [-\pi/2,\pi/2]$. Hence (44) holds for $\theta \in [-\pi/2,\pi/2]$.

Now we assume that $k+l \equiv 0 \pmod 2$ in (44). Substituting (16) into (44) with $\theta = \pi/2$ and using (15) and (18), we have

$$\begin{aligned}&-\frac{1}{2\pi}\sum_{j=0}^{k-1}\mathcal{E}_{j-k}^1(1)(-1)^j\lambda_{k+1+j}\mathcal{R}_1^0(\pi/2;;l,j;1)\\ &\quad = \frac{1}{2\pi^2}\sum_{j=0}^{[(l-2)/2]}\mathcal{E}_{2j+1-l}^2(k;1)\frac{(i\pi/2)^{2j+1}}{(2j+1)!}\,.\end{aligned} \tag{45}$$

Putting $l-2=2m+\eta$ with $m\in\mathbf{N}_0$ and $\eta\in\{0,1\}$ such that $\eta\equiv k \pmod 2$, and multiplying $4\pi/i$ on both sides of (45), we have

$$\sum_{j=0}^{m}\mathcal{E}^2_{2j-2m-\eta-1}(k;1)\frac{(i\pi/2)^{2j}}{(2j+1)!}=-\frac{2}{\pi}\sum_{j=0}^{k-1}\mathcal{E}^1_{j-k}(1)(-1)^j\lambda_{k+1+j}\times\frac{\pi}{i}\mathcal{R}^0_1(\pi/2;\,;2m+2+\eta,j;1)\in\Lambda_1\,. \tag{46}$$

Indeed, it is known that

$$\rho(2j+1;1)=\frac{(-1)^j\pi^{2j+1}}{(2j)!2^{2j+2}}\mathcal{E}^1_{2j}(1)\quad(j\in\mathbf{N}_0) \tag{47}$$

(see, for example, [3] § 1). Hence, if $0\le j<k$ then

$$\frac{1}{\pi}\mathcal{E}^1_{j-k}(1)\lambda_{k+1+j}=\frac{2}{\pi}\rho(k-j)\lambda_{k+1+j}\in\mathbf{Q}[\pi^2]$$

and $(\pi/i)\mathcal{R}^0_1(\pi/2;\,;m+2,j;1)\in\Lambda_1$ from (18). Hence (46) holds.

We recall the following lemma (see [8] Lemma 4.1).

LEMMA 8. *Let $\xi\in\{0,1\}$. Suppose $\{\mathcal{P}_m\}$ and $\{\mathcal{Q}_m\}$ are sequences which satisfy the relation*

$$\sum_{j=0}^{m}\mathcal{P}_{m-j}\frac{(i\pi/2)^{2j}}{(2j+\xi)!}=\mathcal{Q}_m\quad(m\in\mathbf{N}_0)\,.$$

Then the relation

$$\mathcal{P}_m=\sum_{\nu=0}^{m}\mathfrak{B}_{m-\nu,\xi}\frac{(i\pi/2)^{2m-2\nu}}{(2m-2\nu)!}\mathcal{Q}_\nu$$

holds for any $m\in\mathbf{N}_0$, where $\{\mathfrak{B}_{n,q}\}_{n\ge 0}$ are the rational numbers defined by

$$\frac{2t^\xi}{e^t+(-1)^\xi e^{-t}}=\sum_{n=0}^{\infty}\mathfrak{B}_{n,\xi}\frac{t^{2n}}{(2n)!}\quad(\xi\in\{0,1\})\,.$$

Applying Lemma 8 with $\xi=1$, $\mathcal{P}_m=\mathcal{E}^2_{-2m-\eta-1}(k;1)$ and using (46), we can easily check that $\mathcal{E}^2_{-2m-\eta-1}(k;1)\in\Lambda_1$, namely

$$\mathcal{E}^2_{-N}(k;1)\lambda_{k+1+N}\in\Lambda_1 \tag{48}$$

for $N\in\mathbf{N}$, because $\pi\in\Lambda_1$. By (38), we see that $\mathfrak{L}_2(k,N;\psi;1)\lambda_{k+1+N}\in\Lambda_1$. Namely $L(k,N;\psi)\in\Lambda_1$ for $k,N\in\mathbf{N}$ with $k+N\equiv 1 \pmod 2$.

Next we assume $k+l \equiv 1 \pmod 2$ in (44). Similarly, substituting (16) with $u=1$ into (44) and using (15) and (18), we have

$$\begin{aligned} &-\frac{1}{\pi^2}\mathfrak{L}_2(k,l;\psi^2;1)-\frac{1}{2\pi}\sum_{j=0}^{k-1}\mathcal{E}_{j-k}(1)(-1)^j\lambda_{k+1+j}\sum_{\sigma=0}^{[(j-1)/2]}\binom{l-2+j-\sigma}{j-\sigma-1} \\ &\quad\times(-1)^{\sigma+1}\mathfrak{L}_1(l+j-2\sigma-1;\psi^2;1)\frac{(\pi/2)^{2\sigma+1}}{(2\sigma+1)!} \\ &=\frac{1}{2\pi^2}\sum_{j=0}^{[(l-1)/2]}\mathcal{E}^2_{2j-l}(k;1)\frac{(i\pi/2)^{2j}}{(2j)!}\,. \end{aligned} \tag{49}$$

Combining (48) and (49), we have $L(k,l;\psi^2)=\mathfrak{L}_2(k,l;\psi^2;1)\in\Lambda_1$ for $k,l\in\mathbf{N}$ with $k+l\equiv 1 \pmod 2$. Thus we obtain the assertion of Theorem 1 in the case when $r=2$.

EXAMPLE. Putting $(k,l)=(1,3)$ in (46) and using (18), we have

$$\mathcal{E}^2_{-2}(1;1)=-L(3;\psi^2)=-\frac{7}{8}\zeta(3)\,,$$

because $L(s;\psi^2)=\left(1-2^{-s}\right)\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. Putting $(k,n)=(1,-2)$ in (38) and $(k,l)=(1,2)$ in (49), we have

$$\begin{aligned} L(1,2;\psi)&=L(1;\psi)L(2;\psi)-\frac{1}{2}L(3;\psi^2)=\frac{\pi}{4}L(2;\psi)-\frac{7}{16}\zeta(3)\,,\\ L(1,2;\psi^2)&=\frac{1}{2}L(3;\psi^2)=\frac{7}{16}\zeta(3)\,. \end{aligned}$$

By the same method as above, we obtain, for example,

$$\begin{aligned} L(2,3;\psi)&=-3L(1;\psi)L(4;\psi)+\frac{\pi}{6}L(1;\psi)L(3;\psi^2)+2L(5;\psi^2)\\ &=-\frac{3}{4}\pi L(4;\psi)+\frac{7}{192}\pi^2\zeta(3)+\frac{31}{16}\zeta(5)\,. \end{aligned}$$

4. Proof of Theorem 1 in the case of an arbitrary depth

In this section, we aim to complete the proof of Theorem 1 by the same method as in Section 4 of [7].

For $r\in\mathbf{N}$ and $u\in[1,1+\delta]$, we denote by $\Lambda_r(u)$ the $\mathbf{Q}$-algebra generated by

$$\bigcup_{m=1}^{r}\ \bigcup_{\chi\in\{\psi,\psi^2\}}\{\mathfrak{L}_m(j_1,\ldots,j_m;\chi;u)\,\big|\,(j_1,\ldots,j_m)\in\mathbf{N}^m,\ j_m>1\ (\text{if }\chi=\psi^2)\}\,.$$

Note that $\Lambda_r(1) = \Lambda_r$. Furthermore, for $p \in \mathbf{N}_0$, we denote by $\mathcal{V}_r(p; u)$ the $\Lambda_r(u)$-module generated by

$$\bigcup_{j=1}^{r} \{\mathcal{R}_j^p(\theta; k_1, \ldots, k_{j-1}; a, b; u) \mid (k_1, \ldots, k_{j-1}) \in \mathbf{N}^{j-1}, \quad a \in \mathbf{N},\ b \in \mathbf{N}_0 \text{ with } a + \lambda_{b+j+p} \geq 2\}.$$

By (15), we have $\mathcal{V}_r(p; u) = \mathcal{V}_r(p+2; u)$. It follows from (19) that if $g(\theta; u) \in \mathcal{V}_r(\mu; u)$ for $\mu \in \{0, 1\}$ then $(i\pi)^{\mu-1} g(\pi/2; 1) \in \pi^{-r-1+\mu}\Lambda_r$. We define the $\Lambda_r(u)$-linear operator $\tilde{\Delta}^{(l)} : \mathcal{V}_r(p; u) \to \mathcal{V}_{r+1}(p+l+1; u)$ for $l, r \in \mathbf{N}$, $p \in \mathbf{N}_0$ and $u \in [1, 1+\delta]$ by

$$\begin{aligned} &\tilde{\Delta}^{(l)}(\mathcal{R}_j^p(\theta; k_1, \ldots, k_{j-1}; a, b; u)) \\ &\quad = \sum_{\nu=0}^{b} \binom{a-1+b-\nu}{b-\nu} \mathcal{R}_{j+1}^{p+l+1}(\theta; k_1, \ldots, k_{j-1}, a+b-\nu; l, \nu; u), \end{aligned} \tag{50}$$

where $j \in \mathbf{N}$ with $1 \leq j \leq r$. We further define

$$\begin{aligned} \tilde{\Gamma}_{2,\mu}(\theta; k, l; u) = {} & \tilde{\Delta}^{(l)}(\mathcal{R}_1^{k-1+\mu}(\theta; ; k, 0; u)) \\ & - \frac{1}{2\pi} \sum_{j=0}^{k-1} \mathcal{E}_{j-k}^1(u)(-1)^j \lambda_{k+1+j} \mathcal{R}_1^{k+l+\mu}(\theta; ; l, j; u). \end{aligned}$$

for $k, l \in \mathbf{N}$, $\mu \in \{0, 1\}$, $\theta \in [-\pi/2, \pi/2]$ and $u \in [1, 1+\delta]$. Then we have

$$\begin{aligned} &\tilde{\Gamma}_{2,\mu}(\theta; k, l; u) - \mathcal{R}_2^{k+l+\mu}(\theta; k; l, 0; u) \\ &\quad = -\frac{1}{2\pi} \sum_{j=0}^{k-1} \mathcal{E}_{j-k}^1(u)(-1)^j \lambda_{k+1+j} \mathcal{R}_1^{k+l+\mu}(\theta; ; l, j; u) \in \mathcal{V}_1(k+l+\mu; u). \end{aligned} \tag{51}$$

From Proposition 2 and (50), we have

$$\tilde{\Gamma}_{2,\mu}(\theta; k, l; u) = \frac{1}{2\pi^2} \sum_{m=-l}^{\infty} \mathcal{E}_m^2(k; u) \lambda_{k+1+m+\mu} \frac{(i\theta)^{m+l}}{(m+l)!}. \tag{52}$$

These results can be generalized as follows.

PROPOSITION 3. *For $r \in \mathbf{N}$ with $r \geq 2$, $(k_1, \ldots, k_r) \in \mathbf{N}^r$, $\mu \in \{0, 1\}$, $u \in [1, 1+\delta]$ and $\theta \in [-\pi/2, \pi/2]$, there exist*

$$\tilde{\Gamma}_{r,\mu}(\theta; k_1, \ldots, k_r; u) \in \mathcal{V}_r\left(\sum_{j=1}^{r}(k_j+1) + \mu; u\right)$$

and $\{\mathcal{E}_m^r(k_1,\ldots,k_{r-1};u)\}_{m\in\mathbf{Z}}$ *such that the following conditions hold*:

$$\tilde{\Gamma}_{r,\mu}(\theta;k_1,\ldots,k_r;1)-\mathcal{R}_r^{\sum_{j=1}^r(k_j+1)+\mu} \times(\theta;k_1,\ldots,k_{r-1};k_r,0;1)\in\mathcal{V}_{r-1}\left(\sum_{j=1}^r(k_j+1)+\mu\right); \tag{53}$$

$$\tilde{\Gamma}_{r,\mu}(\theta;k_1,\ldots,k_r;u)=\frac{1}{2\pi^r}\sum_{m=-k_r}^{\infty}\mathcal{E}_m^r(k_1,\ldots,k_{r-1};u)\lambda_{\sum_{j=1}^{r-1}(k_j+1)+\mu+m}\frac{(i\theta)^{m+k_r}}{(m+k_r)!}; \tag{54}$$

$$\lim_{u\to1+0}\mathcal{E}_m^r(k_1,\ldots,k_{r-1};u)\lambda_{\sum_{j=1}^{r-1}(k_j+1)+r+m}=0\quad(m\in\mathbf{N}_0); \tag{55}$$

$$\mathcal{E}_{-N}^r(k_1,\ldots,k_{r-1};1)\lambda_{\sum_{j=1}^{r-1}(k_j+1)+r+N}\in\Lambda_{r-1}\quad(N\in\mathbf{N}). \tag{56}$$

Furthermore, for any $\gamma\in\mathbf{R}$ *with* $0<\gamma<\pi/2$, *there exists a constant* $M_r(\gamma)>0$ *such that*

$$\frac{|\mathcal{E}_m^r(k_1,\ldots,k_{r-1};u)|}{m!}\le\frac{M_r(\gamma)}{\gamma^m}\quad(m\in\mathbf{N}_0). \tag{57}$$

PROOF. We prove this proposition by induction on $r(\ge 2)$. The case of $r=2$ is what we mentioned above. Indeed, it follows from (40), (42), (48), (51) and (52) that we obtain (53)–(57) in the case when $r=2$.

Now we assume that we define $\tilde{\Gamma}_{r,\mu}(\theta;k_1,\ldots,k_r;u)\in\mathcal{V}_r(\sum_{j=1}^r(k_j+1)+\mu;u)$ and $\{\mathcal{E}_m^r(k_1,\ldots,k_{r-1};u)\}_{m\in\mathbf{Z}}$ satisfying (53)–(57), and prove the assertion in the case of $r+1$. Suppose $u>1$ and let $p=\sum_{j=1}^r(k_j+1)$. By the assumption, we can write $\tilde{\Gamma}_{r,0}(\theta;k_1,\ldots,k_r;u)\in\mathcal{V}_r(p;u)$ as the following finite sum:

$$\tilde{\Gamma}_{r,0}(\theta;k_1,\ldots,k_r;u)=\sum_{\sigma}\mathcal{C}_\sigma(u)\,\mathcal{R}_{d_\sigma}^{p}(\theta;l_{\sigma,1},\ldots,l_{\sigma,d_\sigma-1};a_\sigma,b_\sigma;u),$$

where $\mathcal{C}_\sigma(u)\in\Lambda_r(u)$ and $d_\sigma\le r$ for any σ. By Lemma 2 and (54), we see that

$$\tilde{\Gamma}_{r,1}(\theta;k_1,\ldots,k_r;u)=\sum_{\sigma}\mathcal{C}_\sigma(u)\,\mathcal{R}_{d_\sigma}^{p+1}(\theta;l_{\sigma,1},\ldots,l_{\sigma,d_\sigma-1};a_\sigma,b_\sigma;u)$$

holds. By (53), we can assume that $\mathcal{C}_1(u)=1$ and

$$\begin{cases}(d_1;l_{1,1},\ldots,l_{1,d_1-1};a_1,b_1)=(r;k_1,\ldots,k_{r-1};k_r,0) & (\sigma=1);\\ d_\sigma\le r-1 & (\sigma\ne1).\end{cases} \tag{58}$$

Let $\mu_0 \in \{0, 1\}$ with $\mu_0 \equiv r \pmod 2$, and put

$$\begin{aligned} &\mathcal{I}_r(\theta; k_1, \ldots, k_r; u) \\ &\quad = \tilde{\Gamma}_{r,\mu_0}(\theta; k_1, \ldots, k_r; u) \\ &\quad\quad - \frac{1}{2\pi^r} \sum_{m=-k_r}^{-1} \mathcal{E}^r_m(k_1, \ldots, k_{r-1}; u)\lambda_{p-(k_r+1)+\mu_0+m} \frac{(i\theta)^{m+k_r}}{(m+k_r)!} \\ &\quad = \tilde{\Gamma}_{r,\mu_0}(\theta; k_1, \ldots, k_n; u) - \frac{1}{2\pi^r} \sum_{j=0}^{k_r-1} \mathcal{E}^r_{j-k_r}(k_1, \ldots, k_{r-1}; u)\lambda_{p+1+\mu_0+j} \frac{(i\theta)^j}{j!}, \end{aligned} \tag{59}$$

and

$$\mathcal{J}_r(\theta; k_1, \ldots, k_r; u) = \tilde{\Gamma}_{r,1-\mu_0}(\theta; k_1, \ldots, k_r; u). \tag{60}$$

By (54) and (55), we have

$$\lim_{u\to 1} \mathcal{I}_r(\theta; k_1, \ldots, k_r; u) = 0 \quad \left(\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right). \tag{61}$$

In the same way as in the proof of Lemma 6, it follows from (9), (26) and the binomial theorem that

$$\begin{aligned} &i\mathcal{J}_r(\theta; k_1, \ldots, k_r; u)\mathcal{G}(\theta; u) + \mathcal{I}_r(\theta; k_1, \ldots, k_r; u)\mathcal{H}(\theta; u) \\ &\quad = \sum_{m=0}^{\infty} \Bigg\{ \sum_{\sigma} \mathcal{C}_\sigma(u)\mathcal{A}_m(l_{\sigma,1}, \ldots, l_{\sigma,j_\sigma-1}; a_\sigma, b_\sigma; u) \\ &\quad\quad - \frac{i}{\pi^{r+1}} \sum_{j=0}^{k_r-1} \mathcal{E}^r_{j-k_r}(k_1, \ldots, k_{r-1}; u)\binom{m}{j}\rho(j-m; u)\lambda_{p+1+\mu_0+j} \Bigg\} \\ &\quad\quad \times \lambda_{p+1+\mu_0+m} \frac{(i\theta)^m}{m!}, \end{aligned} \tag{62}$$

since $\lambda_{p+1+\mu_0+m}\lambda_{j+m} = \lambda_{p+1+\mu_0+j}\lambda_{p+1+\mu_0+m}$. Hence we define

$$\begin{aligned} \mathcal{E}^{r+1}_m(k_1, \ldots, k_r; u) &= \pi^{r+1} \sum_{\sigma} \mathcal{C}_\sigma(u)\mathcal{A}_m(l_{\sigma,1}, \ldots, l_{\sigma,d_\sigma-1}; a_\sigma, b_\sigma; u) \\ &\quad - i \sum_{j=0}^{k_r-1} \mathcal{E}^r_{j-k_r}(k_1, \ldots, k_{r-1}; u)\binom{m}{j}\rho(j-m; u)\lambda_{p+1+r+j} \end{aligned} \tag{63}$$

for $m \in \mathbf{Z}$. Then (62) can be written as

$$\begin{aligned} &i\mathcal{J}_r(\theta; k_1, \ldots, k_r; u)\mathcal{G}(\theta; u) + \mathcal{I}_r(\theta; k_1, \ldots, k_r; u)\mathcal{H}(\theta; u) \\ &\quad = \frac{1}{\pi^{r+1}} \sum_{m=0}^{\infty} \mathcal{E}^{r+1}_m(k_1, \ldots, k_{r-1}, k_r; u)\lambda_{p+1+r+m} \frac{(i\theta)^m}{m!}, \end{aligned} \tag{64}$$

because $\mu_0 \equiv r \pmod 2$. By (24) and (63), we can define

$$\mathcal{E}_m^{r+1}(k_1,\ldots,k_r;1) = \lim_{u\to 1+0} \mathcal{E}_m^{r+1}(k_1,\ldots,k_r;u) \tag{65}$$

for $m \in \mathbf{Z}$ with $m \le -1$. Let $\gamma \in \mathbf{R}$ with $0 < \gamma < \pi/2$. Combining (8), (28), (57) and (62), there exists a constant $M_{r+1}(\gamma)(>0)$ independent of u such that

$$\frac{|\mathcal{E}_m^{r+1}(k_1,\ldots,k_r;u)|}{m!} \le \frac{M_{r+1}(\gamma)}{\gamma^m} \quad (m \in \mathbf{N}_0)\,, \tag{66}$$

which means that (57) in the case of $r+1$ holds. Hence the left-hand side of (64) is uniformly convergent in the wider sense with respect to $(\theta,u) \in (-\pi/2,\pi/2)\times[1,1+\delta]$. Therefore we can let $u \to 1+0$ on both sides of (64), namely (64) holds for $u \in [1,1+\delta]$. Combining (11), (61) and (64), we have

$$\lim_{u\to 1+0} \mathcal{E}_m^{r+1}(k_1,\ldots,k_r;u)\lambda_{\sum_{j=1}^r(k_j+1)+(r+1)+m} = 0 \;\; (m \in \mathbf{N}_0)\,, \tag{67}$$

because $p = \sum_{j=1}^r (k_j+1)$. For $k_{r+1} \in \mathbf{N}$ and $\mu \in \{0,1\}$, we define

$$\begin{aligned}\tilde{\Gamma}_{r+1,\mu}(\theta;k_1,\ldots,k_{r+1};u) &= \tilde{\Delta}^{(k_{r+1})}(\tilde{\Gamma}_{r,\mu}(\theta;k_1,\ldots,k_r;u))\\ &-\frac{1}{2\pi^r}\sum_{j=0}^{k_r-1}\mathcal{E}_{j-k_r}^r(k_1,\ldots,k_{r-1};u)(-1)^j\lambda_{p+j+1}\mathcal{R}_1^{p+k_{r+1}+1+\mu}(\theta;\,;k_{r+1},j;u)\,,\end{aligned} \tag{68}$$

where $p = \sum_{j=1}^r (k_j+1)$. This means that (55) in the case of $r+1$ holds. Furthermore, by (27) and (50), we have

$$\begin{aligned}&\tilde{\Delta}^{(k_{r+1})}(\tilde{\Gamma}_{r,\mu}(\theta;k_1,\ldots,k_r;u))\\ &\quad=\sum_\sigma \mathcal{C}_\sigma(u)\tilde{\Delta}^{(k_{r+1})}(\mathcal{R}_{d_\sigma}^{p+\mu}(\theta;l_{\sigma,1},\ldots,l_{\sigma,d_\sigma-1};a_\sigma,b_\sigma;u))\\ &\quad=\sum_\sigma \mathcal{C}_\sigma(u)\sum_{\nu_\sigma=0}^{b_\sigma}\binom{a_\sigma-1+b_\sigma-\nu_\sigma}{b_\sigma-\nu_\sigma}\\ &\qquad\times\mathcal{R}_{d_\sigma+1}^{p+\mu+k_{r+1}+1}(\theta;l_{\sigma,1},\ldots,l_{\sigma,d_\sigma-1},a_\sigma+b_\sigma-\nu_\sigma;k_{r+1},\nu_\sigma;u)\\ &\quad=\frac{1}{2}\sum_\sigma \mathcal{C}_\sigma(u)\sum_{m=-k_{r+1}}^{\infty}A_m(l_{\sigma,1},\ldots,l_{\sigma,d_\sigma-1};a_\sigma,b_\sigma;u)\lambda_{p+\mu+m}\frac{(i\theta)^{m+k_{r+1}}}{(m+k_{r+1})!}\,.\end{aligned} \tag{69}$$

By (69), we see that (68) states

$$\begin{aligned}&\tilde{\Gamma}_{r+1,\mu}(\theta;k_1,\ldots,k_{r+1};u)\\ &\quad=\frac{1}{2\pi^{r+1}}\sum_{m=-k_{r+1}}^{\infty}\mathcal{E}_m^{r+1}(k_1,\ldots,k_r;u)\lambda_{p+\mu+m}\frac{(i\theta)^{m+k_{r+1}}}{(m+k_{r+1})!}\end{aligned} \tag{70}$$

for $\mu \in \{0,1\}$, which means that (54) in the case of $r+1$ holds. From the assumption (58), we have

$$\tilde{\Delta}^{(k_{r+1})}(\tilde{\Gamma}_{r,\mu}(\theta;k_1,\ldots,k_r;u)) - \mathcal{R}_{r+1}^{q+\mu}(\theta;k_1,\ldots,k_r;k_{r+1},0;u) \in \mathcal{V}_r(q+\mu;u)\,, \tag{71}$$

where $q=\sum_{j=1}^{r+1}(k_j+1)$. Hence, from (56) and (68), we have

$$\tilde{\Gamma}_{r+1,\mu}(\theta;k_1,\ldots,k_{r+1};u) - \mathcal{R}_{r+1}^{q+\mu}(\theta;k_1,\ldots,k_r;k_{r+1},0;u) \in \mathcal{V}_r(q+\mu;u)\,, \tag{72}$$

which means that (53) in the case of $r+1$ holds. Note that

$$\tilde{\Delta}^{(k_{r+1})}\left(\mathcal{R}_r^{\sum_{j=1}^r(k_j+1)+\mu}(\theta;k_1,\ldots,k_{r-1};k_r,0;u)\right) = \mathcal{R}_{r+1}^{q+\mu}(\theta;k_1,\ldots,k_r;k_{r+1},0;u)\,.$$

Now we fix $(k_1,\ldots,k_r)\in\mathbf{N}^r$. Let $k_{r+1}\in\mathbf{N}$ with $\sum_{j=1}^{r+1}k_j \equiv r+1 \pmod 2$ and $\mu\in\{0,1\}$ with $\mu\equiv r+1 \pmod 2$, namely $q\equiv 0 \pmod 2$. Let

$$\begin{aligned} h(\theta;k_1,\ldots,k_{r+1};u) = {} & \tilde{\Gamma}_{r+1,\mu}(\theta;k_1,\ldots,k_{r+1};u) \\ & - \mathcal{R}_{r+1}^{\mu}(\theta;k_1,\ldots,k_r;k_{r+1},0;u)\,. \end{aligned} \tag{73}$$

Since $q\equiv 0 \pmod 2$, it follows from (72) that $h(\theta;k_1,\ldots,k_{r+1};u)\in\mathcal{V}_r(\mu;u)=\mathcal{V}_r(r+1;u)$. By combining (70) and (73), we have

$$\begin{aligned} & \mathcal{R}_{r+1}^{\mu}(\theta;k_1,\ldots,k_r;k_{r+1},0;u) + h(\theta;k_1,\ldots,k_{r+1};u) \\ & = \frac{1}{2\pi^{r+1}} \sum_{m=-k_{r+1}}^{\infty} \mathcal{E}_m^{r+1}(k_1,\ldots,k_r;u)\lambda_{p+\mu+m}\frac{(i\theta)^{m+k_{r+1}}}{(m+k_{r+1})!}\,, \end{aligned} \tag{74}$$

where $p=\sum_{j=1}^r(k_j+1)$. Assume $k_{r+1}\geq 2$. Then, by (66), we can let $u\to 1+0$ on both sides of (74). Furthermore, by (18) and (67) and the assumptions $q\equiv 0$ and $\mu\equiv r+1 \pmod 2$, we can let $\theta\to\pi/2$, and obtain

$$\begin{aligned} & h(\pi/2;k_1,\ldots,k_{r+1};1) \\ & = \frac{1}{2\pi^{r+1}} \sum_{m=-k_{r+1}}^{-1} \mathcal{E}_m^{r+1}(k_1,\ldots,k_r;1)\lambda_{\sum_{j=1}^r(k_j+1)+r+1+m}\frac{(i\pi/2)^{m+k_{r+1}}}{(m+k_{r+1})!} \\ & = \frac{1}{2\pi^{r+1}} \sum_{\nu=0}^{k_{r+1}-1} \mathcal{E}_{\nu-k_{r+1}}^{r+1}(k_1,\ldots,k_r;1)\lambda_{r+\nu}\frac{(i\pi/2)^{\nu}}{\nu!}\,. \end{aligned} \tag{75}$$

Let $\xi\in\{0,1\}$ with $\xi\equiv r \pmod 2$. As well as (46), we put $k_{r+1}-1-\xi=2m+\eta$ with $\eta\equiv k_{r+1}+1+\xi \pmod 2$. Since $q=\sum_{j=1}^{r+1}(k_j+1)\equiv 0$ and $\xi\equiv r \pmod 2$, we have

$$\eta \equiv \sum_{j=1}^r (k_j+1)+r \equiv \sum_{j=1}^r k_j \pmod 2\,. \tag{76}$$

Putting $\nu = 2j + \xi$, (75) states that

$$h(\pi/2; k_1, \dots, k_r, 2m+1+\xi+\eta; 1) = \frac{1}{2\pi^{r+1}} \sum_{j=0}^{m} \mathcal{E}^{r+1}_{2j-2m-1-\eta}(k_1, \dots, k_r; 1) \frac{(i\pi/2)^{2j+\xi}}{(2j+\xi)!} \quad (m \in \mathbf{N}_0)\,. \tag{77}$$

Since $h(\theta; k_1, \dots, k_r, 2m+1+\xi+\eta; u) \in \mathcal{V}_r(r+1; u) = \mathcal{V}_r(1-\xi; u)$, it follows from (19) that

$$(i\pi)^{-\xi} h(\pi/2; k_1, \dots, k_r, 2m+1+\xi+\eta; 1) \in \frac{1}{\pi^{r+1-\xi}} \Lambda_r \subset \frac{1}{\pi^{r+1}} \Lambda_r\,.$$

Applying Lemma 8 with

$$\mathcal{P}_m = \frac{1}{2\pi^{r+1}} \mathcal{E}^{r+1}_{-2m-1-\eta}(k_1, \dots, k_r; 1)\,,$$

$$\mathcal{Q}_m = (i\pi)^{-\xi} h(\pi/2; k_1, \dots, k_r, 2m+1+\xi+\eta; 1) \in \frac{1}{\pi^{r+1}} \Lambda_r$$

for $m \in \mathbf{N}_0$, we have

$$\frac{1}{2\pi^{r+1}} \mathcal{E}^{r+1}_{-2m-1-\eta}(k_1, \dots, k_r; 1) \in \frac{1}{\pi^{r+1}} \Lambda_r \quad (m \in \mathbf{Z} \text{ with } m \le -1)\,. \tag{78}$$

Using (76), we obtain

$$\mathcal{E}^{r+1}_{-N}(k_1, \dots, k_r; 1) \lambda_{\sum_{j=1}^{r}(k_j+1)+r+1+N} \in \Lambda_r \quad (N \in \mathbf{N})\,. \tag{79}$$

Hence it follows from (66), (67), (70), (72) and (79) that we obtain the proof of Proposition 3 by induction.

Finally we give the proof of Theorem 1 in §1 as follows.

Suppose $p = \sum_{j=1}^{r} (k_j + 1) = \sum_{j=1}^{r} k_j + r \equiv 1 \pmod 2$, namely $\sum_{j=1}^{r} k_j$ and r are of different parity. Then the condition (53) gives

$$\tilde{\Gamma}_{r,\mu}(\theta; k_1, \dots, k_r; u) - \mathcal{R}^{1+\mu}_r(\theta; k_1, \dots, k_{r-1}; k_r, 0; u) \in \mathcal{V}_{r-1}(1-\mu; u)$$

for $\mu \in \{0, 1\}$. Choose $\mu \in \{0, 1\}$ with $\mu \equiv r \pmod 2$. Then, by (19), we have

$$(i\pi)^{-\mu} \left(\tilde{\Gamma}_{r,\mu}(\pi/2; k_1, \dots, k_r; 1) - \frac{i^\mu}{\pi^r} L(k_1, \dots, k_r; \psi^2) \right) \in \frac{1}{\pi^r} \Lambda_{r-1}\,,$$

namely

$$(i\pi)^{-\mu} \pi^r \tilde{\Gamma}_{r,\mu}(\pi/2; k_1, \dots, k_r; 1) - \pi^{-\mu} L(k_1, \dots, k_r; \psi^2) \in \Lambda_{r-1}\,. \tag{80}$$

On the other hand, by (57) and the condition $k_r \geq 2$, we can let $\theta = \pi/2$ and $u \to 1+0$ in both sides of (54). Then by (55), we have

$$\begin{aligned}
&\tilde{\Gamma}_{r,\mu}(\pi/2; k_1, \ldots, k_r; 1) \\
&\quad = \frac{1}{2\pi^r} \sum_{m=-k_r}^{-1} \mathcal{E}_m^r(k_1, \ldots, k_{r-1}; 1) \lambda_{\sum_{j=1}^{r-1}(k_j+1)+\mu+m} \frac{(i\pi/2)^{m+k_r}}{(m+k_r)!} \\
&\quad = \frac{1}{2\pi^r} \sum_{N=0}^{k_r-1} \mathcal{E}_{N-k_r}(k_1, \ldots, k_{r-1}; 1) \lambda_{N+\mu} \frac{(i\pi/2)^N}{N!} \\
&\quad = \frac{1}{2\pi^r} \sum_{\nu=0}^{[(k_r-1-\mu)/2]} \mathcal{E}_{2\nu+\mu-k_r}(k_1, \ldots, k_{r-1}; 1) \frac{(i\pi/2)^{2\nu+\mu}}{(2\nu+\mu)!} \,.
\end{aligned} \tag{81}$$

By the assumptions $\mu \equiv r$ and $\sum_{j=1}^{r} (k_j+1) \equiv 1 \pmod 2$, it follows from (56) that

$$\mathcal{E}_{2\nu+\mu-k_r}(k_1, \ldots, k_{r-1}; 1) \in \Lambda_{r-1} \,.$$

Hence, from (81) and the fact that $\pi \in \Lambda_{r-1}$, we have

$$(i\pi)^{-\mu} \pi^r \tilde{\Gamma}_{r,\mu}(\pi/2; k_1, \ldots, k_r; 1) \in \Lambda_{r-1} \,. \tag{82}$$

By combining (80) and (81), we have

$$L(k_1, \ldots, k_r; \psi^2) \in \pi^\mu \Lambda_{r-1} \subset \Lambda_{r-1} \,.$$

Hence we obtain the former assertion of Theorem 1.

Next we prove the latter assertion by induction on $r \geq 2$. The case of $r = 2$ has already been proved in Section 3. Hence we assume that the case of r holds, and prove the case of $r+1$. Choose $(k_1, \ldots, k_{r+1}) \in \mathbf{N}^{r+1}$ with $\sum_{j=1}^{r+1} k_j$ is odd, namely

$$\sum_{j=1}^{r+1} \left(k_j + 1\right) \equiv r \pmod 2 \,. \tag{83}$$

By (58) and (63) with $u = 1$ and $m = -k_{r+1}$, we have

$$\begin{aligned}
\mathcal{E}^{r+1}_{-k_{r+1}}(k_1, \ldots, k_r; 1) &= \pi^{r+1} \mathcal{A}_{-k_{r+1}}(k_1, \ldots, k_{r-1}; k_r, 0; 1) \\
&\quad + \pi^{r+1} \sum_{\sigma \neq 1} \mathcal{C}_\sigma(u) \mathcal{A}_{-k_{r+1}}(l_{\sigma,1}, \ldots, l_{\sigma,d_\sigma-1}; a_\sigma, b_\sigma; 1) \\
&\quad - i \sum_{j=0}^{k_r-1} \mathcal{E}^r_{j-k_r}(k_1, \ldots, k_{r-1}; 1) \lambda_{\sum_{j=1}^{r}(k_j+1)+1+r+j} \binom{-k_{r+1}}{j} \rho(j+k_{r+1}; 1) \,.
\end{aligned} \tag{84}$$

It follows from (24) that the first term on the right-hand side of (84) coincides with $2L(k_1,\ldots,k_r,k_{r+1};\psi)$, the second term belongs to Λ_r because $d_\sigma < r$ $(\sigma \neq 1)$. Furthermore, from (56), we have

$$\begin{aligned}&\mathcal{E}^{r}_{j-k_r}(k_1,\ldots,k_{r-1};1)\lambda_{\sum_{j=1}^{r}(k_j+1)+1+r+j}\\&\qquad=\mathcal{E}^{r}_{j-k_r}(k_1,\ldots,k_{r-1};1)\lambda_{\sum_{j=1}^{r-1}(k_j+1)+r+(k_r-j)}\in\Lambda_{r-1}\subset\Lambda_r\,.\end{aligned}$$

Hence the third term on the right-hand side of (84) belongs to Λ_r. On the other hand, from (56) and (83), we have

$$\mathcal{E}^{r+1}_{-k_{r+1}}(k_1,\ldots,k_r;1)=\mathcal{E}^{r+1}_{-k_{r+1}}(k_1,\ldots,k_r;1)\lambda_{\sum_{j=1}^{r}(k_j+1)+(r+1)+k_{r+1}}\in\Lambda_r\,.$$

Combining these results, we obtain $L(k_1,\ldots,k_{r+1};\psi)\in\Lambda_r$. Hence we see that the assertion in the case of $r+1$ holds. Thus, by induction, we obtain the latter assertion of Theorem 1. This completes the proof of Theorem 1.

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