TOKYO J. MATH.
VOL. 29, NO. 2, 2006

Essential Cycles in Graph Divides as a Link Representation

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(Communicated by K. Taniyama)

Abstract. Graph divide links were introduced as an extension of the class of A'Campo's divide links. We show that some of such links cannot be represented as the links of any graph divides including less circles or less cycles.

1. Introduction

In [1], A'Campo constructed the links of divides as an extension of the class of algebraic links. A *divide* is a generic relative immersion of a disjoint union of intervals (and circles) in a 2-dimensional disk. We review the links of divides in Section 2. In [2], he showed that any divide link is ambient isotopic to a transverse **C**-link, where a *transverse* **C**-*link* is the link represented as the transversal intersection of a complex plane curve and the unit sphere in the 2-dimensional complex vector space $\mathbf{C}^2$ [17]. An *algebraic link* is the link of a singularity of a complex plane curve, which is a transverse **C**-link. We note that there exist transverse **C**-links which are not algebraic [16].

Ishikawa [10] and the author [12] independently showed that the links of divides are quasipositive, by using the visualization algorithm due to Hirasawa [8]. A *quasipositive braid* is the product of conjugate braids with positive braids, and a *quasipositive link* is an oriented link which has a closed quasipositive braid diagram. A *positive braid* is the product of canonical generators of the braid group, that is, a braid which has a diagram without negative crossings. It is well known that any algebraic link admits a representation as a closed positive braid. In [16], Rudolph showed that quasipositive links are transverse **C**-links. In [3], Boileau and Orevkov proved that transverse **C**-links are quasipositive, by the theory of pseudoholomorphic curves.

In [7], Gibson and Ishikawa constructed links of *free divides*, non-relative immersions of intervals in a 2-dimensional disk. In [13], as an extension of the class of the links of divides and free divides, the author constructed the links of *graph divides*, generic immersions of finite graphs. We review the definitions of a graph divide and its links in Section 2. She also

Received April 19, 2005; revised May 10, 2006
The author was partially supported by Grant-in-Aid for Young Scientists (B) (No. 15740044), The Ministry of Education, Culture, Sports, Science and Technology.

showed that any graph divide link is quasipositive, and that there exist quasipositive links which cannot be graph divide links.

A'Campo [2], Gibson and Ishikawa [7] determined the gordian numbers of the links of (free) divides given as immersed intervals. In [13] the author determined the four-dimensional clasp numbers of the links of *tree divides*, graph divides given as immersed trees. In [5] Gibson determined the gordian numbers of such links under some conditions. In [14] the author determined the gordian numbers and the four-dimensional clasp numbers of the links of divides given as certain immersed circles, and showed that her formula does not hold if a divide consists of a limaçon or concentric circles. In their arguments, the immersed image of circles or cycles may be obstructions for the decision of such invariants.

If the set of immersed intervals and trees could represent all divide links, then their gordian numbers or four-dimensional clasp numbers would be determined. In Section 3, we give an example of immersed circles of divides which can be replaced with immersed intervals as link representations. Though in general, we cannot replace circles with trees in graph divides as a link representation. We show it in Section 4. Furthermore we show in Section 5 that there exists a graph divide link which cannot be represented by any graph divide given as a sum of immersed trees and circles.

2. Review of the divide link theory

In this section, we review the links of divides defined by A'Campo [1], the links of oriented divides defined by Gibson and Ishikawa [6], and the links of graph divides [13].

Let D be the unit disk in the real plane $\mathbf{R}^2$, that is $D = \{x = (x_1, x_2) \in \mathbf{R}^2 \mid ||x||^2 = x_1^2 + x_2^2 \leq 1\}$. A *divide* P is a generic relative immersion in the unit disk $(D, \partial D)$ of a finite number of 1-manifolds, i.e., copies of the unit interval $(I, \partial I)$ and the unit circle [1, 2, 8, 10]. We also call the image of such an immersion a *divide*. We call a divide P a *circle divide* (resp. an *interval divide*) if P is an immersion of only circles (resp. intervals).

Let $T_x X$ be the tangent space at a point x of a manifold X, and TX be the tangent bundle over the manifold X. We identify the 3-sphere S^3 with the set

$$ST\mathbf{R}^2 = \{(x, u) \in T\mathbf{R}^2 \mid x \in \mathbf{R}^2, u \in T_x\mathbf{R}^2, ||x||^2 + ||u||^2 = 1\}.$$

We orient the 3-sphere and the link $L(P)$ as follows. We identify the tangent bundle $T\mathbf{R}^2 = \mathbf{R}^4$ with the 2-dimensional complex vector space $\mathbf{C}^2$ by the map

$$((x_1, x_2), (u_1, u_2)) \mapsto (x_1 + \sqrt{-1}u_1, x_2 + \sqrt{-1}u_2).$$

The tangent bundle $T\mathbf{R}^2$ is oriented by the complex orientation of $\mathbf{C}^2$, and the 3-sphere is naturally oriented by the complex orientation of the 4-ball

$$\{(x, u) \in T\mathbf{R}^2 \mid x \in \mathbf{R}^2, u \in T_x\mathbf{R}^2, ||x||^2 + ||u||^2 \leq 1\}.$$

The *link of a divide* P is the set given by

$$L(P) = \{(x, u) \in ST\mathbf{R}^2 \mid x \in P, u \in T_x P\}.$$

Let $[a, b]$ be a small interval with $a < b$. Let $\phi : [a, b] \to D$ be an embedding whose image lies on P. We orient a part of the link $L(P)$ as the image of the map $t \mapsto (\phi(t), \frac{\sqrt{1 - ||\phi(t)||^2}}{||\dot{\phi}(t)||}\dot{\phi}(t))$, where $\dot{\phi}(t)$ is the differential of $\phi(t)$. We can extend this orientation to $L(P)$. A *divide link* is the oriented link ambient isotopic to the link of some divide.

In [6], Gibson and Ishikawa constructed links associated with oriented divides. An *oriented divide* is the image of a generic immersion of finite number of copies of the unit circle in the unit disk, with a specific orientation assigned to each immersed circle. The *link of an oriented divide* Q is the set of $L_{ori}(Q)$ given by

$$L_{ori}(Q) = \{(x, u) \in ST\mathbf{R}^2 \mid x \in Q, u \in \vec{T}_x Q\},$$

where $\vec{T}_x Q$ is the set of tangent vectors in the same direction as the assigned orientation of Q. The link $L_{ori}(Q)$ naturally inherits its orientation from Q.

In [4], Gibson showed the following theorem.

THEOREM 2.1 ([4]). *Any link can be represented as the link of some oriented divide.* □

We note that a regular isotopy of a given divide or oriented divide in the space of generic immersions does not change the isotopy type of its link.

There exist some transformations which do not change the isotopy type of the links of divides or oriented divides. The transformations illustrated on the top of Figure 1 are *inverse self-tangency moves*. The bottom transformation on oriented divides is a *triangle move*. If the bottom one is on divides (or graph divides), we call it a *divide triangle move*.

LEMMA 2.2 ([7]). *The isotopy type of the link of an oriented divide does not change under inverse self-tangency moves and triangle moves. Furthermore, the isotopy type of the link of a divide or a graph divide does not change under divide triangle moves.*

REMARK 2.3. In [4], Gibson showed that there exist several other transformations on oriented divides which do not change the link isotopy type.

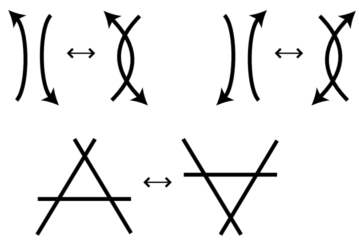

FIGURE 1. Inverse self-tangency moves and a (divide) triangle move.

In [13], a link is associated to any immersed graphs and circles in the unit disk as follows.

Let G be a disjoint union of copies of the unit circle and finite graphs without point components. A *graph divide* $P = (G, \varphi)$ is a generic immersion $\varphi : G \to D$ or its image, with the following properties. Each graph might have loops and multiple edges. The singularities are only transversal double points of two arcs in edges and circles. We suppose that the point of $P \cap \partial D$ is the image of a vertex of degree 1. We regard the unit interval as a finite graph. The image of vertices of degree 1 might not lie in the boundary of the unit disk. We call a point of such image a *free endpoint* of P and denote by E_P the set of all free endpoints of P. We denote by T_P the set of all vertices except free endpoints and points in ∂D. We denote $V_P = E_P \cup T_P$. If T_P is empty and φ is a non-relative immersion, P is called a *free divide* [7].

A *branch* of P is the image of each component of G. We shall call the image of an interval component an *interval branch*, the image of a circle component a *circle branch*, and the image of a tree component a *tree branch*. In [7], Gibson and Ishikawa considered free divides with only interval branches, though a free divide means a sum of interval and circle branches in this article. We note that a divide is also a free divide as they commented in [7] and hence it is a graph divide.

In [13], the definition of the links of divides was extended as follows. We need 'signs' to vertices of a graph divide, because the link is not associated to a graph divide alone. Let x be a vertex of G. We also denote the image $\varphi(x)$ by x. If x lies in ∂D, x does not need a sign. If x is a point of V_P, we give x a sign $\varepsilon_x = +$ or $\varepsilon_x = -$.

For a given graph divide $P = (G, \varphi)$ and given signs of vertices, we construct an oriented divide $d(P; \{\varepsilon_x\}_{x \in V_P})$ as follows. For each branch B of P, except near the points of V_P, we draw the boundary of 'very small' neighborhood of B in the disk D, assigned with the clockwise orientation, as illustrated in Figure 2 (a), (b), and (c), where interrupted curves represent ∂D. In particular, we draw a 'sharp' around each double point of P as (c), and draw a 'hairpin curve' around each point of $P \cap \partial D$ as (b). We suppose that such hairpin curves lie in the interior of D. Around $x \in E_P$ with $\varepsilon_x = -$, we draw a 'hairpin curve', as illustrated in Figure 2 (e). Around $x \in E_P$ with $\varepsilon_x = +$, we draw a 'small kink', as illustrated in Figure 2 (d). Around $x \in T_P$ with $\varepsilon_x = -$, we draw oriented curves such that each curve approaches x along an edge and turns to its neighbor edge on the left, as illustrated in Figure 2 (g). Around $x \in T_P$ with $\varepsilon_x = +$, we draw oriented curves such that each curve approaches x along an edge and turns to its neighbor edge on the right, as illustrated in Figure 2 (f). We denote the obtained curves by $d(P; \{\varepsilon_x\}_{x \in V_P})$ and call it the *doubling* of the graph divide P with signs $\{\varepsilon_x\}_{x \in V_P}$.

DEFINITION 2.4. The *link of a graph divide* P with signs $\{\varepsilon_x\}_{x \in V_P}$ is the set given by

$$L(P; \{\varepsilon_x\}_{x \in V_P}) = L_{ori}(d(P; \{\varepsilon_x\}_{x \in V_P})) .$$

We note that the link of a given graph divide depends on signs of vertices. For fixed signs of vertices ($\{\varepsilon_x\}_{x \in V_P}$), a regular isotopy of P in the space of generic immersions does not change the isotopy type of the oriented divide $d(P; \{\varepsilon_x\}_{x \in V_P})$, therefore it does not change

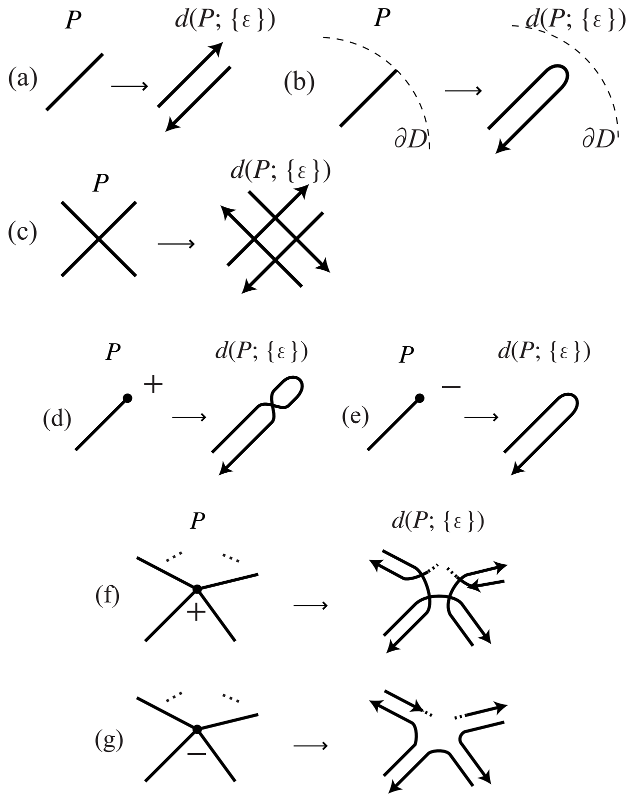

FIGURE 2. The doubling of graph divides.

the isotopy type of the link $L(P; \{\varepsilon_x\}_{x\in V_P})$. A *graph divide link* is the oriented link ambient isotopic to the link of some graph divide with some signs of vertices.

EXAMPLE 2.5 ([13], cf. [5]). For a graph divide P with signed vertices illustrated in Figure 3, the doubling of P is illustrated as the right of P. Then the link of P is the knot illustrated at the bottom of Figure 3. It is known that this knot is not fibered if n is a positive integer. Then it is not a divide link since divide links are all fibered.

Indeed, the knot of P with signed vertices illustrated in Figure 3 cannot be represented by any free divide for $n \geq 2$[1] ([13], cf. [5]). In [7], Gibson and Ishikawa checked that the knot of the free divide with only one double point must be the trefoil or the mirror image of the knot 5_2 in the table due to Rolfsen [15]. In [13], the author showed that the number of the double points of a tree divide is equal to the 4-dimensional clasp number of its link.

[1] The author would like to apologize for incorrectly writing $n \geq 0$ in the paragraph after Remark 6.6 in [13].

EXAMPLE 2.6 ([13]). For a graph divide P with signed vertices illustrated in Figure 4, the doubling of P is illustrated as the right of P. Then the link of P is the knot illustrated at the bottom of Figure 4. This knot is the mirror image of 8_{21} in the table of Rolfsen [15]. It is well known that the knot 8_{21} is fibered. We note that there is no (free) divide which represent this knot as checked in [13].

By the argument in [13], for any graph divide $P = (G, \varphi)$, there exists a graph divide $P' = (G', \varphi')$, where G' is a sum of uni-trivalent graphs and copies of circles, such that $L(P'; \{\varepsilon_x\}_{x \in V_{P'}})$ is ambient isotopic to $L(P; \{\varepsilon_x\}_{x \in V_P})$.

The above definition of a graph divide gives a natural extension of the class of divide links. If P is a classical divide, this definition is same as the visualization algorithm due to Hirasawa [8]. If P has no circle branches and $E_P = V_P$, the link $L(P; \{\varepsilon_x\}_{x \in V_P})$ is ambient isotopic to the *link of a free divide* originally constructed by Gibson and Ishikawa [7].

Furthermore, the class of links of tree divides which Gibson defined in [5] is the subclass of links of graph divides [13]. In this article, we call P a *tree divide* if G is a union of trees and intervals.

3. A family of circle branches which can be changed to interval branches

Figure 5 gives an example of a circle divide which can be changed to an interval divide or tree divide as the link representation. An embedded circle in the disk is the divide which represents the Hopf link. The sum of a vertical line and a horizontal line also gives the divide

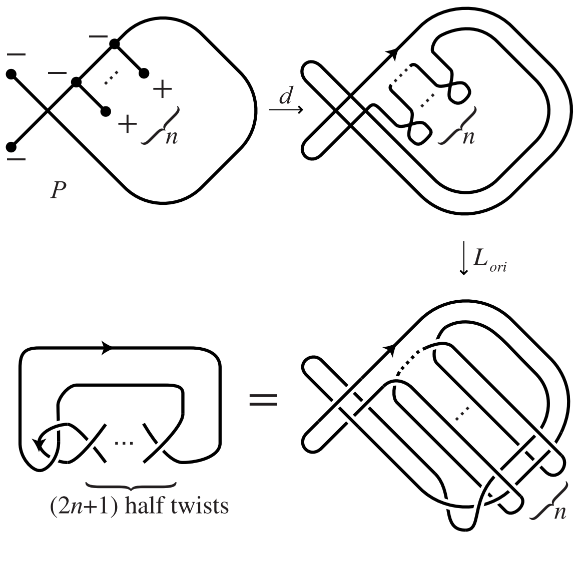

FIGURE 3. An example of a (non-fibered) graph divide knot.

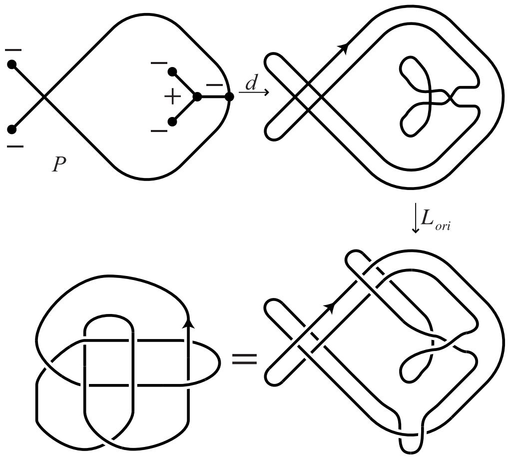

FIGURE 4. An example of a graph divide knot.

representing the same link. In this section, we observe more circle branches which can be changed to interval branches as representations of links.

Let B be an intersection of a given graph divide P and a small square $[a_1, b_1] \times [a_2, b_2]$ in the unit disk D. Let f_B be the projection $(x_1, x_2) \mapsto x_2$ restricted to B. We suppose that $f_B^{-1}(y)$ is the set of n points for any $y \in [a_2, b_2]$ except $\{y_1, \dots, y_k\}$, and that each $f_B^{-1}(y_j)$ is the set of $n-2$ regular points and the double point. We suppose that f_B has neither maximal points nor minimal points. We shall call such B an *n-braid part* of P. The n-braid part of P may be regarded as the projection image of an n-braid to the plane.

Let P be a graph divide which has a braid part B surrounded with a circle branch C as illustrated on the left of Figure 6. Let P' and P'' be graph divides obtained from P by changing C as illustrated in Figure 6. We suppose that the sign ε_x of each $x \in V_P$ is preserved. The divide P' is obtained from P by divide triangle moves. The oriented divide $d(P'; \{\varepsilon_x\}_{x \in V_{P'}})$ is obtained from $d(P''; \{\varepsilon_x\}_{x \in V_{P''}})$ by inverse self-tangency moves and triangle moves. By Lemma 2.2, the links $L(P'; \{\varepsilon_x\}_{x \in V_{P'}})$ and $L(P''; \{\varepsilon_x\}_{x \in V_{P''}})$ are ambient

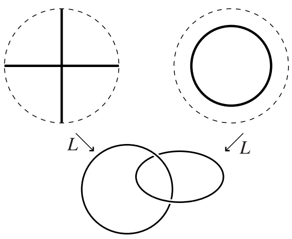

FIGURE 5. Divides representing the Hopf link.

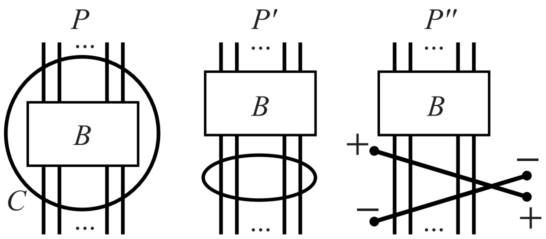

FIGURE 6. Changing circle branches to interval branches.

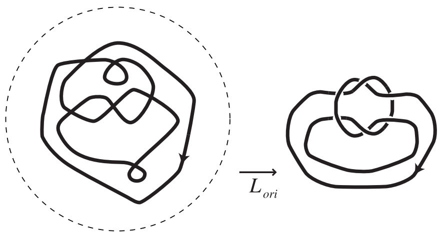

FIGURE 7. An oriented divide which represents the square knot.

isotopic to $L(P; \{\varepsilon_x\}_{x \in V_P})$. Then the links $L(P'; \{\varepsilon_x\}_{x \in V_{P'}})$ and $L(P''; \{\varepsilon_x\}_{x \in V_{P''}})$ are ambient isotopic to $L(P; \{\varepsilon_x\}_{x \in V_P})$.

4. Circle divide links without tree divide representations

In this section, we prove the following theorem.

THEOREM 4.1. *There exist circle divides whose links cannot be represented as the link of any tree divides.*

In [13], the author showed there exist links which cannot be graph divide links. Before the proof of Theorem 4.1, we present such examples.

LEMMA 4.2 ([13]). *If a graph divide knot K is slice, then K is trivial.*

EXAMPLE 4.3. We denote the mirror image of a link L by $L!$. The knot 3_1 in the knot table by Rolfsen [15] is the left hand trefoil. The square knot $3_1\sharp(3_1!)$, the connected sum of the left hand and right hand trefoils, is slice. By Lemma 4.2, the knot $3_1\sharp(3_1!)$ is not a graph divide link. However, by Theorem 2.1, there exists an oriented divide Q with $L_{ori}(Q) = 3_1\sharp(3_1!)$. Figure 7 gives an example of such Q.

EXAMPLE 4.4 ([13]). The mirror image of the knot 8_{20} in the Rolfsen's table [15] is quasipositive. It is not a graph divide link because it is slice. By Theorem 2.1, there exists an oriented divide which represents $8_{20}!$. Figure 8 gives such an example.

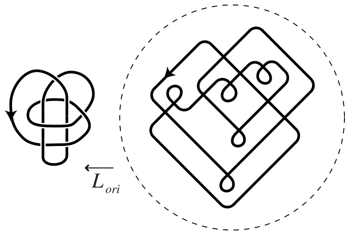

FIGURE 8. The oriented divide which represents the knot $8_{20}!$.

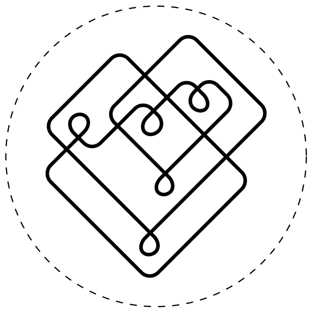

FIGURE 9. An example of an essential circle branch.

REMARK 4.5. The *slice euler characteristic* of a link L is the maximal number of euler characteristics for compact oriented 2-manifolds in the 4-ball with boundary L. We denote it by $\chi_s(L)$. In [11], Ishikawa showed that the equality $-\chi_s(L) = TB(L)$ holds for any graph divide link L, where $TB(L)$ is the maximal Thurston-Bennequin number of L. He also noted that this equality does not hold for the knot $8_{20}!$.

Theorem 4.1 is obtained from the following proposition.

PROPOSITION 4.6. *Let $P = (S^1, \varphi)$ be a divide given by an immersed circle. We give P an orientation o and denote this oriented divide by (P, o). If the knot $L_{ori}(P, o)$ is not a graph divide link, then the link $L(P)$ cannot be represented as the link of any tree divides.*

PROOF. We assume that a tree divide $P' = (T, \varphi')$ with signs $\{\varepsilon\}$ represents the link $L(P)$. The link $L(P)$ has two components since P is an immersed circle. Then T is the disjoint sum of two connected trees T_1 and T_2. Each component of $L(P)$ is ambient isotopic to one of $L((T_i, \varphi'|_{T_i}); \{\varepsilon\}|_{T_i})$. The non-trivial knot $L_{ori}(P, o)$ is a component of $L(P)$. It contradicts the supposition. □

PROOF OF THEOREM 4.1. Let K be a non-trivial knot. We suppose that K is not a graph divide link. By Theorem 2.1, there exists an oriented divide Q_K with $L_{ori}(Q_K) = K$. Let P_K be a circle divide obtained from Q_K by ignoring the orientation. We apply Proposition

4.6 to P_K. Then the 2-component link $L(P_K)$ cannot be represented as the link of any tree divide. □

For example, the circle divide shown in Figure 9 has an essential circle branch, that is, we cannot expect a graph divide representation of its link without circle branches.

5. Essential cycles of graph divides

In this section, we prove the following theorem.

THEOREM 5.1. *There exist graph divide links which cannot be represented as the links of graph divides with only tree branches and circle branches.*

In the proof, the following result is used.

LEMMA 5.2 ([6]). *Let o_1 and o_2 be different orientations of an immersed circle P. Let (P, o_i) be an oriented divide obtained from P and the orientation o_i. Then the link $L_{ori}(P, o_2)$ is the same link as $L_{ori}(P, o_1)$ but with the opposite orientation.*

PROOF OF THEOREM 5.1. Let K be a non-trivial knot. We suppose that K is not a graph divide link. Let P_K be the circle divide constructed in the proof of Theorem 4.1. We add an immersed edge α to P_K as illustrated in Figure 10. Figure 11 is the case $K = 8_{20}!$. By Lemma 5.2, for any sign ε of the vertex of degree 3, the link $L(P_K \cup \alpha; \{\varepsilon\})$ is a 2-component link such that one component is the connected sum of the knots K and $3_1!$, and the other is K with the opposite orientation.

We assume that there exists a graph divide P' such that each branch of P' is a circle branch or a tree branch and that the link $L(P'; \{\varepsilon'_x\}_{x \in V_{P'}})$ is ambient isotopic to $L(P_K \cup \alpha; \{\varepsilon\})$ for some signs $\{\varepsilon'_x\}_{x \in V_{P'}}$. This link has 2 components, then P' is either an immersion of a single circle or an immersion of two trees. By almost same argument as that in the proof of Proposition 4.6, P' cannot be a tree divide. Hence P' should be a circle divide (S^1, φ').

By Lemma 5.2, the components of the link $L(P')$ are the same knot but with opposite orientations to each other. Then K is ambient isotopic to the connected sum of K and $3_1!$. It contradicts the unique factorization theorem and hence the link $L(P_K \cup \alpha; \{\varepsilon\})$ is not the link of any circle divide. Therefore, the graph divide P cannot be changed to a sum of circle branches and tree branches. □

6. Strongly quasipositivity

A *strongly quasipositive braid* is the product of positive bands. The *positive band* is the braid $\sigma_{ij} = \sigma_i \cdots \sigma_{j-1}\sigma_j {\sigma_{j-1}}^{-1} \cdots {\sigma_i}^{-1}$, where σ_k's are the canonical generators of the braid group and i is less than j. A *strongly quasipositive link* is the closure of a strongly quasipositive braid. For any strongly quasipositive link, the slice euler characteristic is equal to the euler characteristic. Hirasawa recently showed that any divide link is strongly quasipositive

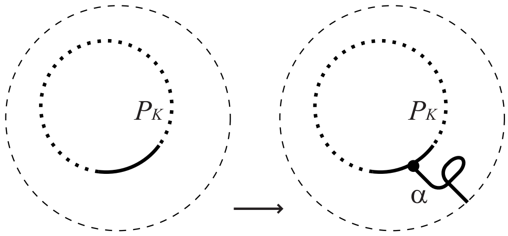

FIGURE 10. Adding α to the circle divide P_K.

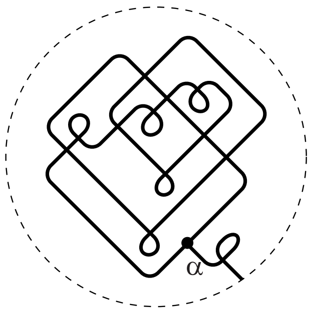

FIGURE 11. An example of an essential cycle in a graph divide.

[9]. The knot of Example 2.6 is fibered, but not strongly quasipositive, because the slice euler characteristic is -1 but the euler characteristic is -3, so the slice euler characteristic is not equal to the euler characteristic.

The result due to Rudolph on plumbing of Seifert surface of strongly quasipositive links [18] implies the following proposition.

PROPOSITION 6.1 (cf. [18]). *Let L_1 and L_2 be strongly quasipositive links. A connected sum of L_1 and L_2 is strongly quasipositive if and only if L_1 and L_2 are strongly quasipositive.*

By this proposition, the link $L(P_K \cup \alpha; \{\varepsilon\})$, which is constructed in the proof of Theorem 5.1, is strongly quasipositive, since this link is a connected sum of divide links. Instead of α, we attach the graph divide in Figure 4 to the circle divide P_K as illustrated in Figure 12. The link of this graph divide is $L(P_K)\sharp(8_{21}!)$. The knot $8_{21}!$ is not strongly quasipositive as commented in [13]. Then the link $L(P_K)\sharp(8_{21}!)$ is not strongly quasipositive by Proposition 6.1. By same argument as that for $P_K \cup \alpha$, the link $L(P_K)\sharp(8_{21}!)$ is neither a tree divide link nor a circle divide link.

Figure 13 describes the inclusion relations for graph divide links and quasipositive links shown by A'Campo [1, 2], Boileau, Orevkov [3], Gibson, Ishikawa [7, 10], Hirasawa [9], Rudolph [16], and the author [12, 13]. Let L_1 be a divide link without tree divide representation, whose existence is shown in Theorem 4.1. Let L_2 be the link of the graph divide

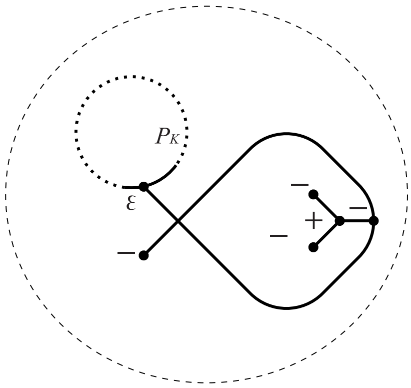

FIGURE 12. A graph divide whose link is not strongly quasipositive.

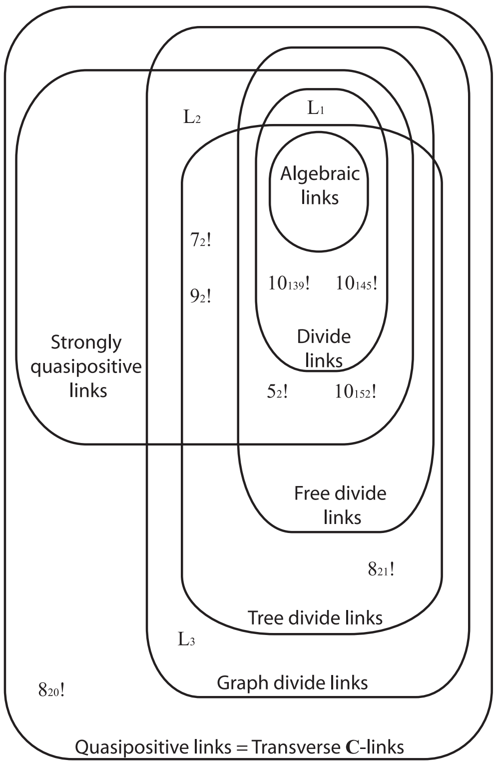

FIGURE 13. The inclusion relations for graph divide links.

without circle and tree divide representation, constructed in Section 5. Let L_3 be the link of the graph divide which has no circle and tree divide representation, and which is not strongly quasipositive. Then the links L_1, L_2, and L_3 lie as described in Figure 13.

ACKNOWLEDGMENT. The author would like to thank Yuichi Yamada for his suggestion for the existence of circle branches which can be changed to tree branches. She also would like to thank Takuji Nakamura for his comment on strongly quasipositivity of graph divide links.

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