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A Calculus Scheme for Clifford Distributions

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Abstract. The aim of the paper is to construct the fundamental solution of an arbitrary complex power of the Dirac operator, these powers being defined as convolution operators with a kernel expressed in terms of specific distributions in Euclidean space. The desired fundamental solution is found, at least formally, in terms of the same families of distributions as those arising in the kernel of the corresponding operator. Clearly, in order to prove these results in a rigorous way, we first have to investigate the definition and properties of both the convolution and the product of arbitrary elements of the families of distributions under consideration, leading to a very attractive pattern of mutual relations between them.

1. Introduction

The Dirac operator $\underline{\partial}$ is at the heart of Clifford analysis, a direct and elegant generalization to higher dimension of the theory of holomorphic functions in the complex plane. Its fundamental solution is the Cauchy convolution kernel yielding so-called monogenic functions.

In Section 2, we recall the basic notions and concepts of Clifford analysis which will be used in the course of this paper. For a detailed study of the Clifford function theory, we refer the reader to [2, 8].

Complex powers of the Dirac operator $\underline{\partial}$ were already considered in [8]. They are convolution operators, the kernels of which are given by certain combinations of T_λ^*- and U_λ^*-distributions (see [3, 4, 5]). In Section 3, the definitions and the most relevant properties of these distributions in $\mathbf{R}^m$ are given.

The eventual aim of this paper is to construct a fundamental solution for an arbitrary complex power of $\underline{\partial}$. Except for some particular cases depending on the dimension m, the desired solution can be found in a formal way in terms of specific T_λ^*- and U_λ^*-distributions, related to the ones arising in the kernel of the considered operator. Hence, since the action of the operator on its proposed fundamental solution clearly involves the convolution of arbitrary

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members of the T_λ^*- and the U_λ^*-families, this induces the need for a thorough study of the convolvability of these distributions. This problem is addressed in Section 4.

In Section 5, we pass to frequency space, for a similar study of the product of the distributions under consideration.

The study of both convolution and product of arbitrary T_λ^*- and the U_λ^*-distributions reveals a whole scheme of mutual relations both inside and in between the considered families, caused by the action of some well-known operators (Dirac, Hilbert, Laplace, ...); moreover, a mirrored pattern arises in frequency space. This dance of the Clifford distributions is addressed in Section 6.

In Section 7, the fundamental solution of $\underline{\partial}^\mu$ is given for each $\mu \in \mathbf{C}$. Special attention is paid to the exceptional case where the dimension m is even and $\mu = m + n$, $n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\} = \{0, 1, 2, \dots\}$, giving rise to logarithmic terms in the solution.

Finally, in a last section, the historical background and connections of the T_λ^*- and the U_λ^*-distributions and their convolution is briefly sketched.

2. Clifford analysis

In this section we briefly recall the basic notions and results of Clifford analysis, a function theory which may be regarded as a generalization to a higher dimensional setting of the theory of holomorphic functions in the complex plane. For more details, we refer the reader to [2, 8].

Let $\mathbf{R}^{0,m}$ be the real vector space $\mathbf{R}^m$, endowed with a non-degenerate quadratic form of signature $(0, m)$, let $(e_1, \dots, e_m)$ be an orthonormal basis for $\mathbf{R}^{0,m}$, and let $\mathbf{R}_{0,m}$ be the universal real Clifford algebra constructed over $\mathbf{R}^{0,m}$. The non-commutative multiplication in $\mathbf{R}_{0,m}$ is governed by the rules

$$e_i^2 = -1\,, \quad i = 1, 2, \dots, m \quad \text{and} \quad e_i e_j + e_j e_i = 0\,, \quad 1 \leq i \neq j \leq m\,.$$

A basis for $\mathbf{R}_{0,m}$ is obtained by considering, for any set $A = \{i_1, \dots, i_h\} \subset \{1, \dots, m\}$ with $1 \leq i_1 < i_2 < \cdots < i_h \leq m$, the element $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$. For the empty set ϕ, we put $e_\phi = 1$, the latter being the identity element. Then any $a \in \mathbf{R}_{0,m}$ may be written as

$$a = \sum_A a_A\, e_A\,, \quad a_A \in \mathbf{R}\,,$$

or still as $a = \sum_{k=0}^m [a]_k$, where $[a]_k = \sum_{|A|=k} a_A\, e_A$ is a so-called k-vector ($k = 0, 1, \dots, m$). Denoting the space of k-vectors by $\mathbf{R}_{0,m}^k$, we have that $\mathbf{R}_{0,m} = \bigoplus_{k=0}^m \mathbf{R}_{0,m}^k$, which leads to the identification of $\mathbf{R}$ and $\mathbf{R}^{0,m}$ with respectively $\mathbf{R}_{0,m}^0$ and $\mathbf{R}_{0,m}^1$. We will also identify an element $\underline{x} = (x_1, \dots, x_m) \in \mathbf{R}^m$ with the one-vector (or vector for short)

$$\underline{x} = \sum_{j=1}^m x_j\, e_j\,.$$

For any two vectors $\underline{x}$ and $\underline{y}$ we have $\underline{x}\,\underline{y} = \underline{x} \bullet \underline{y} + \underline{x} \wedge \underline{y}$ where

$$\underline{x} \bullet \underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^{m} x_j y_j = \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\underline{x})$$

is a scalar and

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_i e_j (x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\underline{x})$$

is a 2-vector, also called bivector. In particular $\underline{x}^2 = \underline{x} \bullet \underline{x} = -|\underline{x}|^2$.

Conjugation in $\mathbf{R}_{0,m}$ is defined as the anti-involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, m$. In particular for a vector $\underline{x}$ we have $\overline{\underline{x}} = -\underline{x}$.

The Dirac operator in $\mathbf{R}^m$ is the first order vector valued differential operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j\, \partial_{x_j}\ ,$$

its fundamental solution being given by

$$E_m(\underline{x}) = \frac{1}{a_m}\ \frac{\overline{\underline{x}}}{|\underline{x}|^m}\ ,$$

where $a_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$ stands for the area of the unit sphere S^{m-1} in $\mathbf{R}^m$.

We consider functions $f : \mathbf{R}^m \to \mathbf{R}_{0,m}$; such a function is said to be left (respectively right) monogenic in the open region Ω of $\mathbf{R}^m$ iff f is continuously differentiable in Ω and moreover satisfies the equation $\underline{\partial}\, f = 0$ (respectively $f\,\underline{\partial} = 0$). As $\overline{\underline{\partial}\, f} = \overline{f}\ \overline{\underline{\partial}} = -\overline{f}\underline{\partial}$, a function f is left monogenic in Ω iff $\overline{f}$ is right monogenic in Ω.

As moreover the Dirac operator factorizes the Laplace operator, i.e. $-\underline{\partial}^2 = \underline{\partial}\,\overline{\underline{\partial}} = \overline{\underline{\partial}}\,\underline{\partial} = \Delta$, where $\Delta = \sum_{j=1}^{m} \partial_{x_j}^2$, a (left or right) monogenic function in Ω is harmonic and hence also a C_∞ function in Ω.

Introducing spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$, gives rise to the Clifford-vector valued locally integrable function $\underline{\omega}$, which is to be seen as the higher dimensional analogue of the *signum*-distribution on the real line; we will encounter $\underline{\omega}$ as one of the distributions discussed below.

3. Some specific classical and Clifford distributions

The distributions T_λ and U_λ ($\lambda \in \mathbf{C}$) and their generalizations have been extensively studied in [3, 4]. The first family $\mathcal{T} = \{T_\lambda : \lambda \in \mathbf{C}\}$ of distributions considered is very classical. It consists of the radial distributions $T_\lambda = Fp\ r^\lambda = Fp\ (x_1^2 + \cdots + x_m^2)^{\lambda/2}$, and it contains a.o. the respective fundamental solutions of natural powers of the Laplace operator. As convolution operators they give rise to the traditional Riesz potentials. The second family $\mathcal{U} = \{U_\lambda : \lambda \in \mathbf{C}\}$ of distributions arises in a natural way by the action of the Dirac operator on $\mathcal{T}$. The U_λ-distributions are Clifford-vector valued and it is shown, in Section 5, that they also arise as products of T_λ-distributions with the distribution $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$, mentioned above. In Section 6, it is shown moreover that the two families $\mathcal{T}$ and $\mathcal{U}$ are linked by the Hilbert transform as well. Typical examples in the $\mathcal{U}$-family are the fundamental solutions of the Dirac operator and of its odd natural powers.

The normalized distributions T_λ^* and U_λ^* arise when their singularities are removed by dividing them by an appropriate Gamma-function. In this section we summarize the definitions and properties of these normalized versions. For a brief sketch of the historical background of these distributions we refer the reader to Section 8.

The normalized distributions T_λ^* are defined by

$$\begin{cases} T_\lambda^* = \pi^{\frac{\lambda+m}{2}} \dfrac{T_\lambda}{\Gamma\left(\frac{\lambda+m}{2}\right)}\,, & \lambda \neq -m-2l\,, \\ T_{-m-2l}^* = \dfrac{\pi^{\frac{m}{2}-l}}{2^{2l}\Gamma\left(\frac{m}{2}+l\right)}(-\Delta)^l \delta(\underline{x})\,, & l \in \mathbf{N}_0\,, \end{cases}$$

where the action of $T_\lambda = Fp\ r^\lambda$ on a test function $\phi \in \mathcal{S}$ is given by

$$\langle T_\lambda, \phi \rangle = a_m \langle Fp\ r_+^\mu, \Sigma^{(0)}[\phi] \rangle$$

with $\mu = \lambda + m - 1$. In the above expressions $Fp\ r_+^\mu$ is the classical "finite part" distribution on the real r-axis and $\Sigma^{(0)}$ is the scalar valued generalized spherical mean, defined on scalar valued test functions $\phi(\underline{x})$ by

$$\Sigma^{(0)}[\phi] = \frac{1}{a_m} \int_{S^{m-1}} \phi(\underline{x})\, dS(\underline{\omega})\,.$$

We call *Riesz potential of the first kind* $\mathcal{P}_T^\gamma$, $\gamma \in \mathbf{C}$, the scalar valued convolution operator given by

$$\mathcal{P}_T^\gamma[f] = T_{\gamma-m}^* * f\,, \quad f \in \mathcal{S}\,.$$

For $\gamma \neq -2l, l \in \mathbf{N}_0$, we have more explicitly:

$$\mathcal{P}_T^\gamma[f] = \frac{\pi^{\frac{\gamma}{2}}}{\Gamma\left(\frac{\gamma}{2}\right)}\ Fp \int_{\mathbf{R}^m} |\underline{x} - \underline{u}|^{\gamma-m} f(\underline{u})\, dV(\underline{u})\,, \tag{3.1}$$

while for $\gamma = -2l, l \in \mathbf{N}_0$ we have

$$\mathcal{P}_T^{-2l}[f] = \frac{\pi^{\frac{m}{2}-l}}{2^{2l}\,\Gamma\left(\frac{m}{2}+l\right)}\,(-\Delta)^l f = \frac{\pi^{\frac{m}{2}-l}}{2^{2l}\,\Gamma\left(\frac{m}{2}+l\right)}\,\underline{\partial}^{2l} f\,.$$

So

$$\mathcal{P}_T^{\gamma}[f] = \frac{2^{\gamma}\pi^{\frac{\gamma+m}{2}}}{\Gamma\left(\frac{m-\gamma}{2}\right)}\,I^{\gamma}[f]\,,\quad \gamma \neq m+2k\,,\;\; k \in \mathbf{N}_0\,,$$

where $I^{\gamma}[f]$ is the traditional Riesz potential.
Note that $\mathcal{P}_T^{\gamma}[f]$ is an entire function of γ, whereas $I^{\gamma}[f]$ shows simple poles at $\gamma = m + 2k, k \in \mathbf{N}_0$.

The Clifford-vector valued distributions U_{λ}^{*} are typical Clifford analysis objects. They originate from the action of the Dirac operator on the T_{λ}^{*}-distributions (see below). They are defined by

$$\begin{cases} U_{\lambda}^{*} = \pi^{\frac{\lambda+m+1}{2}}\,\dfrac{U_{\lambda}}{\Gamma\left(\frac{\lambda+m+1}{2}\right)}\,, & \lambda \neq -m-2l-1\,,\\[2ex] U_{-m-2l-1}^{*} = -\dfrac{\pi^{\frac{m}{2}-l}}{2^{2l+1}\,\Gamma\left(\frac{m}{2}+l+1\right)}\,\underline{\partial}^{2l+1}\delta(\underline{x})\,, & l \in \mathbf{N}_0\,, \end{cases}$$

where the action of U_{λ} on a test function $\phi \in \mathcal{S}$ is given by

$$\langle U_{\lambda}, \phi\rangle = a_m \langle Fp\; r_{+}^{\mu}, \Sigma^{(1)}[\phi]\rangle\,,$$

with $\mu = \lambda + m - 1$. The Clifford-vector valued generalized spherical mean $\Sigma^{(1)}$ is defined on scalar valued test functions $\phi(\underline{x})$ by

$$\Sigma^{(1)}[\phi] = \frac{1}{a_m}\int_{S^{m-1}} \underline{\omega}\,\phi(\underline{x})\,dS(\underline{\omega})\,.$$

We call *Riesz potential of the second kind* $\mathcal{P}_U^{\gamma}$, $\gamma \in \mathbf{C}$, the Clifford-vector valued convolution operator

$$\mathcal{P}_U^{\gamma}[f] = U_{\gamma-m}^{*} * f\,,\quad f \in \mathcal{S}\,.$$

For $\gamma \neq -2l-1, l = 0, 1, 2, \ldots$, we have more explicitly:

$$\begin{aligned} \mathcal{P}_U^{\gamma}[f] &= \pi^{\frac{\gamma+1}{2}}\,\frac{U_{\gamma-m}}{\Gamma\left(\frac{\gamma+1}{2}\right)} * f \\ &= \frac{\pi^{\frac{\gamma+1}{2}}}{\Gamma\left(\frac{\gamma+1}{2}\right)}\,Fp \int_{\mathbf{R}^m} |\underline{x}-\underline{u}|^{\gamma-m}(\underline{\omega}-\underline{\xi})\,f(\underline{u})\,dV(\underline{u})\,, \end{aligned}$$

while for $\gamma = -2l-1$, $l = 0, 1, 2, \ldots$, we have

$$\mathcal{P}_U^{-2l-1}[f] = U^*_{-m-2l-1} * f = -\frac{\pi^{\frac{m}{2}-l}}{2^{2l+1}\,\Gamma\left(\frac{m}{2}+l+1\right)}\,\underline{\partial}^{2l+1} f\,.$$

We also put

$$\mathcal{P}_U^{\gamma}[f] = -\frac{2^{\gamma}\pi^{\frac{\gamma+m+1}{2}}}{\Gamma\left(\frac{m-\gamma+1}{2}\right)}\,J^{\gamma}[f]\,,\quad \gamma \neq m+2k+1,\ k \in \mathbf{N}_0\,.$$

Note that $\mathcal{P}_U^{\gamma}[f]$ is an entire function of γ, whereas $J^{\gamma}[f]$ shows simple poles at $\gamma = m + 2k+1,\ k \in \mathbf{N}_0$.

The normalized distributions T^*_λ and U^*_λ are holomorphic mappings from $\lambda \in \mathbf{C}$ to the space $\mathcal{S}'(\mathbf{R}^m)$ of tempered distributions. As already mentioned they are intertwined by the action of the Dirac operator. They enjoy the following properties: for all $\lambda \in \mathbf{C}$ one has

(i) $\underline{x}\ T^*_\lambda = \frac{\lambda+m}{2\pi}\ U^*_{\lambda+1}$; $\quad \underline{x}\ U^*_\lambda = U^*_\lambda\ \underline{x} = -T^*_{\lambda+1}$

(ii) $\underline{\partial}\ T^*_\lambda = \lambda\ U^*_{\lambda-1}$; $\quad \underline{\partial}\ U^*_\lambda = U^*_\lambda\ \underline{\partial} = -2\pi\ T^*_{\lambda-1}$

(iii) $\Delta T^*_\lambda = 2\pi\lambda T^*_{\lambda-2}$; $\quad \Delta U^*_\lambda = 2\pi(\lambda-1)U^*_{\lambda-2}$

(iv) $\mathcal{F}[T^*_\lambda] = T^*_{-\lambda-m}$; $\quad \mathcal{F}[U^*_\lambda] = -i\ U^*_{-\lambda-m}$.

For property (iv) the following definition of the Fourier transformation has been adopted:

$$\mathcal{F}[f(\underline{x})](\underline{y}) = \int_{\mathbf{R}^m} f(\underline{x})\exp(-2\pi i\,\langle \underline{x}, \underline{y}\rangle)\,d\underline{x}\,.$$

4. Convolution of the distributions T^*_λ and U^*_λ

The convolution and the product of distributions is by no means straightforward, especially when distributions with non-compact support are concerned. Problems in the convolution are caused by bad behavior at infinity, while the product heavily depends on local regularity.

This section is devoted to the problem of the convolvability of the T_λ- and U_λ-distributions.

In the sequel we will use the following subsets of $\mathbf{C} \times \mathbf{C}$:

$$\begin{aligned}
\Phi_m &= \{(\alpha,\beta) \in \mathbf{C}\times\mathbf{C} \mid \mathbf{R}e\,\alpha > -m,\ \mathbf{R}e\,\beta > -m,\ \mathbf{R}e(\alpha+\beta) < -m\}\,;\\
S_{m,1} &= \{(\alpha,\beta) \in \mathbf{C}\times\mathbf{C} \mid \alpha = -m-2k,\ k\in\mathbf{N}_0 \text{ and } \beta \neq 2l,\ l \in \mathbf{N}_0\}\,;\\
S_{m,2} &= \{(\alpha,\beta) \in \mathbf{C}\times\mathbf{C} \mid \alpha \neq 2k,\ k\in\mathbf{N}_0 \text{ and } \beta = -m-2l,\ l \in \mathbf{N}_0\}\,;\\
\Psi_3 &= \{(\alpha,\beta) \in (\mathbf{C}\times\mathbf{C})\setminus(S_{m,1}\cup S_{m,2}) \mid m \geq 3 \text{ and } (\mathbf{R}e\,\alpha \leq -m,\ -2 > \mathbf{R}e\,\beta > -m\\
&\qquad \text{or } \mathbf{R}e\,\beta \leq -m,\ -2 > \mathbf{R}e\,\alpha > -m)\}\,;\\
\Psi_4 &= \{(\alpha,\beta) \in (\mathbf{C}\times\mathbf{C})\setminus(S_{m,1}\cup S_{m,2}) \mid m \geq 4 \text{ and } \mathbf{R}e\,\alpha \leq -m,\ \mathbf{R}e\,\beta \leq -m\}\,.
\end{aligned}$$

Finally, we will also use the set

$$\Omega = \Phi_m \cup S_{m,1} \cup S_{m,2} \cup \Psi_3 \cup \Psi_4 .$$

Starting point is the observation that the Riesz normalizations (see [12]) of the distributions T_λ, viz.

$$R_\alpha = \frac{1}{2^\alpha \pi^{\frac{m}{2}}} \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \, Fp\, r^{\alpha-m} = \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{2^\alpha \pi^{\frac{\alpha+m}{2}}} \, T^*_{\alpha-m} , \quad \alpha \neq -2l,\ \alpha \neq m+2l,\ l \in \mathbf{N}_0$$

$$R_{-2l} = (-\Delta)^l \delta = \frac{2^{2l}\Gamma\left(\frac{m}{2}+l\right)}{\pi^{\frac{m}{2}-l}} \, T^*_{-m-2l},\ l \in \mathbf{N}_0$$

$$R_{m+2l} = \frac{2(-1)^l}{\pi^{\frac{m}{2}} 2^{m+2l} \Gamma\left(\frac{m}{2}+l\right) l!} \, r^{2l} \left(\log \frac{1}{\pi r} + A_{m,l}\right) , \quad l \in \mathbf{N}_0$$

where

$$A_{m,l} = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{l} - C\right) + \frac{1}{2}\frac{\Gamma'\left(\frac{m}{2}+l\right)}{\Gamma\left(\frac{m}{2}+l\right)}$$

and C is Euler's constant, satisfy the convolution formula

$$R_\alpha * R_\beta = R_{\alpha+\beta} , \quad \forall(\alpha,\beta) \in \mathbf{C} \times \mathbf{C}$$

The following result is then obtained immediately.

PROPOSITION 1. *For all* $(\alpha,\beta) \in \mathbf{C}\times\mathbf{C}$ *such that* $\alpha \neq 2j$, $j \in \mathbf{N}_0$, $\beta \neq 2k$, $k \in \mathbf{N}_0$ *and* $\alpha+\beta+m \neq 2l$, $l \in \mathbf{N}_0$ *the convolution* $T^*_\alpha * T^*_\beta$ *is the tempered distribution given by*

$$T^*_\alpha * T^*_\beta = c_m(\alpha,\beta)\ T^*_{\alpha+\beta+m} . \tag{4.2}$$

where

$$c_m(\alpha,\beta) = \pi^{\frac{m}{2}} \ \frac{\Gamma\left(-\frac{\alpha+\beta+m}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)} .$$

Nevertheless let us give some additional comments, since the equality (4.2) should be interpreted with care.

A. If the complex parameters α and β fulfil the conditions of the set Φ_m, viz. $\mathbf{R}e\ \alpha > -m$, $\mathbf{R}e\ \beta > -m$ and $\mathbf{R}e\ (\alpha+\beta) < -m$ then the distributions T^*_α, T^*_β and $T^*_{\alpha+\beta+m}$ are L_1^{loc}-functions, and the convolvability of T^*_α and T^*_β is a classical result (see [10], Prop. 2.36).

B. If $(\alpha,\beta) \in \Psi_3 \cup \Psi_4$, then the distributions T^*_α and T^*_β are not L_1^{loc}-functions anymore; however the convolution of those tempered distributions is still defined.

C. If $(\alpha,\beta) \in S_{m,1} \cup S_{m,2}$ then either T^*_α or T^*_β reduces to a natural power of the Laplace operator, from which a direct calculation leads to the desired result.

D. Finally, if $(\alpha, \beta) \notin \Omega$ then $T^*_\alpha * T^*_\beta$ does not exist as a genuine convolution anymore. However for each $\beta \in \mathbf{C} \setminus \{2j | j \in \mathbf{N}_0\}$, the expression $c_m(\alpha, \beta)\ T^*_{\alpha+\beta+m}$ is a holomorphic mapping of α, for all $\alpha \in \mathbf{C} \setminus \{2k | k \in \mathbf{N}_0\}$ such that $\alpha \neq -\beta - m + 2l$, $l \in \mathbf{N}_0$. Hence, we may define the expression $T^*_\alpha * T^*_\beta$ at the left-hand side of (4.2) in this α-region by analytic continuation. Moreover, this reasoning clearly allows for the rôles of α and β to be interchanged. This leads to a sound interpretation of the above result (4.2) in this case, involving a "$*$"-operation which, although not being a genuine convolution, still satisfies the basic properties of a convolution, as will be shown below in Corollary 1.

Similarly as above, we now give sense to the distributions $T^*_\alpha * U^*_\beta$, $U^*_\alpha * T^*_\beta$ and $U^*_\alpha * U^*_\beta$. In Proposition 2, we trace out the largest subset of $\mathbf{C} \times \mathbf{C}$ for which each of these distributions is a convolution in the classical sense. Next, for all admissible couples (α, β) not belonging to that subset, we provide a "natural" definition.

PROPOSITION 2. *For $(\alpha + 1, \beta) \in \Omega$, the distributions $U^*_\alpha * T^*_\beta$ and $T^*_\beta * U^*_\alpha$ exist as a convolution in $\mathcal{S}'$, their action on a test function $\phi \in \mathcal{S}$ being given by*

$$\langle U^*_\alpha * T^*_\beta, \phi \rangle = \langle T^*_\beta * U^*_\alpha, \phi \rangle \ = \ c_m(\alpha - 1, \beta)\ \langle U^*_{\alpha+\beta+m}, \phi \rangle\ .$$

For $(\alpha + 1, \beta + 1) \in \Omega$, the distribution $U^*_\alpha * U^*_\beta$ exists as a convolution in $\mathcal{S}'$, its action on a test function $\phi \in \mathcal{S}$ being given by

$$\langle U^*_\alpha * U^*_\beta, \phi \rangle = \frac{-2\pi}{\alpha + \beta + m}\ c_m(\alpha - 1, \beta - 1)\ \langle T^*_{\alpha+\beta+m}, \phi \rangle.$$

PROOF. We only treat the case of $U^*_\alpha * T^*_\beta$, the other cases being similar.

Take $(\alpha + 1, \beta) \in \Omega$, then $T^*_{\alpha+1} * T^*_\beta$ is a genuine convolution and hence:

$$\underline{\partial}(T^*_{\alpha+1} * T^*_\beta) = (\underline{\partial} T^*_{\alpha+1}) * T^*_\beta\ . \tag{4.3}$$

On account of (4.2) and the properties mentioned in Section 3, this leads to

$$(\alpha + \beta + m + 1)\ c_m(\alpha + 1, \beta)\ U^*_{\alpha+\beta+m} = (\alpha + 1)\ U^*_\alpha * T^*_\beta\ ,$$

or

$$U^*_\alpha * T^*_\beta = c_m(\alpha - 1, \beta)\ U^*_{\alpha+\beta+m}\ .$$

□

Note that, according to Proposition 1, one has

$$T^*_{\alpha+1} * T^*_\beta = c_m(\alpha + 1, \beta)\ T^*_{\alpha+\beta+m+1}\ . \tag{4.4}$$

So the action of the Dirac operator $\underline{\partial}$ on both sides of (4.4) produces

$$\underline{\partial}(T^*_{\alpha+1} * T^*_\beta) = (\alpha + 1)\ c_m(\alpha - 1, \beta)\ U^*_{\alpha+\beta+m}\ , \tag{4.5}$$

which inspires the following definition.

DEFINITION 1. (i) For $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that $\alpha \neq 2j+1,\ \beta \neq 2k,\ \alpha+\beta \neq -m+2l+1,\ j,k,l \in \mathbf{N}_0$ one puts

$$U_\alpha^* * T_\beta^* = T_\beta^* * U_\alpha^* \ = c_m(\alpha-1, \beta)\ U_{\alpha+\beta+m}^* \,. \tag{4.6}$$

(ii) For $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that $\alpha \neq 2j+1,\ \beta \neq 2k+1,\ \alpha+\beta \neq -m+2l,\ j,k,l \in \mathbf{N}_0$ one puts

$$U_\alpha^* * U_\beta^* = \pi^{\frac{m}{2}+1} \frac{\Gamma\left(-\frac{\alpha+\beta+m}{2}\right)}{\Gamma\left(\frac{-\alpha+1}{2}\right)\Gamma\left(\frac{-\beta+1}{2}\right)}\ T_{\alpha+\beta+m}^* \,. \tag{4.7}$$

REMARK 1. In terms of the Riesz potentials of the first and the second kind, the obtained convolution formulae read:

(i) for $\lambda \neq m+2j$, $\mu \neq m+2k$ and $\lambda+\mu \neq m+2l,\ j,k,l \in \mathbf{N}_0$, one has

$$I^\lambda[I^\mu[f]] = I^{\lambda+\mu}[f]\,;$$

(ii) for $\lambda \neq m+2j+1$, $\mu \neq m+2k$ and $\lambda+\mu \neq m+2l+1,\ j,k,l \in \mathbf{N}_0$, one has

$$J^\lambda[I^\mu[f]] = I^\mu[J^\lambda[f]] = J^{\lambda+\mu}[f]\,;$$

(iii) for $\lambda \neq m+2j+1$, $\mu \neq m+2k+1$ and $\lambda+\mu \neq m+2l,\ j,k,l \in \mathbf{N}_0$, one has

$$J^\lambda[J^\mu[f]] = I^{\lambda+\mu}[f]\,.$$

Now we extend the basic convolution properties to all newly defined convolutions between members of the families of distributions $\mathcal{T} = \{T_\lambda^* : \lambda \in \mathbf{C}\}$ and $\mathcal{U} = \{U_\lambda^* : \lambda \in \mathbf{C}\}$.

COROLLARY 1. *Let X_1 and X_2 be two distributions of $\mathcal{T} \cup \mathcal{U}$. Then, as long as the distributions involved are defined, the following properties hold*:

(i) [*Commutativity*] $X_1 * X_2 = X_2 * X_1$;

(ii) [*Derivation*] $\underline{\partial}^l(X_1 * X_2) = (\underline{\partial}^l X_1) * X_2 = X_1 * (\underline{\partial}^l X_2)$
$(X_1 * X_2)\underline{\partial}^l = (X_1\underline{\partial}^l) * X_2 = X_1 * (X_2\underline{\partial}^l), \quad \forall l \in \mathbf{N}.$

PROOF. (i) The commutativity property can readily be checked.

(ii) In all cases where the "$*$"-operator denotes a genuine convolution, the proof is trivial. In the other cases, restricting α and β to admissible values, we have e.g.

$$\underline{\partial}(T_\alpha^* * T_\beta^*) = (\alpha+\beta+m)\ c_m(\alpha, \beta)\ U_{\alpha+\beta+m-1}^* \,.$$

The right-hand side equals

$$\frac{\alpha+\beta+m}{\alpha}\ \frac{c_m(\alpha,\beta)}{c_m(\alpha-2,\beta)}\ (\underline{\partial}T_\alpha^*) * T_\beta^* = (\underline{\partial}T_\alpha^*) * T_\beta^* \,,$$

which also can be written as

$$\frac{\alpha+\beta+m}{\beta}\ \frac{c_m(\alpha,\beta)}{c_m(\alpha,\beta-2)}\ T_\alpha^* * (\underline{\partial}T_\beta^*) = T_\alpha^* * (\underline{\partial}T_\beta^*) \,,$$

from which we may conclude that

$$\underline{\partial}(T^*_\alpha * T^*_\beta) = (\underline{\partial}T^*_\alpha) * T^*_\beta \ = \ T^*_\alpha * (\underline{\partial}T^*_\beta)\,.$$

The other cases are treated similarly. □

5. Products of the distributions T^*_λ and U^*_λ

In general, the product of arbitrary distributions is not defined. However, if the convolution of two distributions f and g exists, one can always give meaning to the product of their Fourier transforms, since for $f, g \in \mathcal{S}'$ and $\phi \in \mathcal{S}$:

$$\langle \mathcal{F}(f) \cdot \mathcal{F}(g), \phi \rangle \ = \ \langle \mathcal{F}(f * g), \phi \rangle \ = \ \langle f * g, \mathcal{F}^{-1}\phi \rangle\,.$$

Hence, in view of the fact that the Fourier transform maps the set $\mathcal{T}$ (resp. $\mathcal{U}$) onto the set $\mathcal{T}$ (resp. $-i\,\mathcal{U}$), it makes sense to consider products of distributions T^*_λ and U^*_λ. In this section we will give a rigorous definition for the distributions $T^*_\alpha \cdot T^*_\beta$, $T^*_\alpha \cdot U^*_\beta$, $U^*_\alpha \cdot T^*_\beta$ and $U^*_\alpha \cdot U^*_\beta$, for all allowed values of $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$. Note that we use the dot notation in order to emphasize that we are dealing with a product.

PROPOSITION 3. *For all $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$, such that $(-\alpha - m, -\beta - m) \in \Omega$, one has*

(i) *$T^*_\alpha \cdot T^*_\beta$, $T^*_\alpha \cdot U^*_\beta$, $U^*_\alpha \cdot T^*_\beta$ and $U^*_\alpha \cdot U^*_\beta$ are well-defined as products of distributions in $\mathcal{S}'$;*

(ii) *the action of these distributions on a test function $\phi \in \mathcal{S}$ is given by*

$$\langle T^*_\alpha \cdot T^*_\beta, \phi \rangle = c_m(-\alpha - m, -\beta - m)\ \langle T^*_{\alpha+\beta}, \phi \rangle$$
$$\langle T^*_\alpha \cdot U^*_\beta, \phi \rangle = c_m(-\alpha - m, -\beta - m - 1)\ \langle U^*_{\alpha+\beta}, \phi \rangle$$
$$\langle U^*_\alpha \cdot T^*_\beta, \phi \rangle = c_m(-\alpha - m - 1, -\beta - m)\ \langle U^*_{\alpha+\beta}, \phi \rangle$$
$$\langle U^*_\alpha \cdot U^*_\beta, \phi \rangle = \frac{-2\pi}{\alpha + \beta + m}\ c_m(-\alpha - m - 1, -\beta - m - 1)\ \langle T^*_{\alpha+\beta}, \phi \rangle\,.$$

PROOF. We only treat the product $T^*_\alpha \cdot T^*_\beta$, the other cases being similar.

Take $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that $(-\alpha - m, -\beta - m) \in \Omega$, i.e. let (α, β) be a couple of complex parameters for which the distribution $T^*_{-\alpha-m} * T^*_{-\beta-m}$ is a genuine convolution (see Proposition 1). We then have

$$\begin{aligned}\mathcal{F}[T^*_{-\alpha-m}] \cdot \mathcal{F}[T^*_{-\beta-m}] &= \mathcal{F}[T^*_{-\alpha-m} * T^*_{-\beta-m}] \\ &= c_m(-\alpha - m, -\beta - m)\ \mathcal{F}[T^*_{-\alpha-\beta-m}]\,,\end{aligned}$$

which, by means of the properties listed in Section 3, reduces to

$$T^*_\alpha \cdot T^*_\beta = c_m(-\alpha - m, -\beta - m)\ T^*_{\alpha+\beta}\,. \tag{5.8}$$

The desired formula follows if we let both sides act on a test function ϕ. □

For all $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that $(-\alpha - m, -\beta - m) \notin \Omega$ and for which $T^*_{-\alpha-m} * T^*_{-\beta-m}$ is defined (see Proposition 1), we still have

$$\mathcal{F}[T^*_{-\alpha-m} * T^*_{-\beta-m}] = c_m(-\alpha - m, -\beta - m)\ T^*_{\alpha+\beta}\ ,$$

but, as the distribution $T^*_{-\alpha-m} * T^*_{-\beta-m}$ is no longer defined as a genuine convolution, we cannot rewrite the left-hand side as the product of two Fourier transforms. However, the above formula inspires a definition for the products of distributions in $\mathcal{T} \cup \mathcal{U}$ in these cases as well.

DEFINITION 2. (i) For $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that $\alpha \neq -m - 2j,\ \beta \neq -m - 2k,\ \alpha + \beta \neq -m - 2l,\ \ j, k, l \in \mathbf{N}_0$ one puts

$$T^*_\alpha \cdot T^*_\beta = c_m(-\alpha - m, -\beta - m) T^*_{\alpha+\beta}\ . \tag{5.9}$$

(ii) For $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that $\alpha \neq -m - 2j,\ \beta \neq -m - 2k - 1,\ \alpha + \beta \neq -m - 2l - 1,\ \ j, k, l \in \mathbf{N}_0$ one puts

$$T^*_\alpha \cdot U^*_\beta = c_m(-\alpha - m, -\beta - m - 1) U^*_{\alpha+\beta}\ . \tag{5.10}$$

(iii) For $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that $\alpha \neq -m - 2j - 1,\ \beta \neq -m - 2k,\ \alpha + \beta \neq -m - 2l - 1,\ \ j, k, l \in \mathbf{N}_0$ one puts

$$U^*_\alpha \cdot T^*_\beta = c_m(-\alpha - m - 1, -\beta - m) U^*_{\alpha+\beta}\ . \tag{5.11}$$

(v) For $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$ such that $\alpha \neq -m - 2j - 1,\ \beta \neq -m - 2k - 1,\ \alpha + \beta \neq -m - 2l,\ \ j, k, l \in \mathbf{N}_0$ one puts

$$U^*_\alpha \cdot U^*_\beta = -\pi^{\frac{m}{2}+1} \frac{\Gamma\left(\frac{\alpha+\beta+m}{2}\right)}{\Gamma\left(\frac{\alpha+m+1}{2}\right)\Gamma\left(\frac{\beta+m+1}{2}\right)}\ T^*_{\alpha+\beta}\ . \tag{5.12}$$

By the above definition we have now given meaning to the distribution $T^*_\alpha \cdot T^*_\beta$ for all $(\alpha, \beta) \in \mathbf{C} \times \mathbf{C}$. However, one should always keep in mind that only for the couples (α, β) mentioned in Proposition 3, $T^*_\alpha \cdot T^*_\beta$ can be interpreted as a real product of distributions.

REMARK 2. On account of Corollary 1(i), the "$\cdot$"-operator acting on $(\mathcal{T} \cup \mathcal{U}) \times (\mathcal{T} \cup \mathcal{U})$ is commutative. In particular, we have, for allowed values of α and β, that $U^*_\beta \cdot T^*_\alpha = T^*_\alpha \cdot U^*_\beta$.

6. Mutual relations between the T^*_λ- and U^*_λ-distributions

From the above multiplication rules for the T^*_λ- and U^*_λ-distributions it follows that, returning to their non-normalized versions, in particular for $\lambda \neq -m - n, n \in \mathbf{N}_0$, one has:

$$T_\lambda U_0 = U_0 T_\lambda = U_\lambda\ ,$$

or

$$T_\lambda \underline{\omega} = U_\lambda\ .$$

This means that, for $\lambda \neq -m-n$, $n \in \mathbf{N}_0$, the U_λ-distributions may be seen as the products of the corresponding T_λ-distributions with the locally integrable Clifford-vector valued function $U_0 = \underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$. This last distribution may be regarded as the higher dimensional analogue of the $\mathrm{sgn}(x) = \frac{x}{|x|}$ distribution on the real line. This becomes even more clear in frequency space where, as is the case for $\mathrm{sgn}(x)$, its spectrum

$$\mathcal{F}[U_0] = -i\,\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}}\,U_{-m} = \frac{2i}{a_{m+1}}\;Pv\frac{\overline{\omega}}{r^m}$$

is, up to the imaginary unit, the convolution kernel J^0 of the higher dimensional Hilbert transform in the Clifford analysis setting (see [6, 7, 9]).
For this Hilbert transform the T^*_λ- and U^*_λ-distributions are paired, since the convolution formulae give, for $\lambda \notin \mathbf{N}_0$:

$$\mathcal{H}[T^*_\lambda] = J^0[T^*_\lambda] = -\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1-\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}U^*_\lambda$$

and

$$\mathcal{H}[U^*_\lambda] = J^0[U^*_\lambda] = -\sqrt{\pi}\frac{\Gamma\left(-\frac{\lambda}{2}\right)}{\Gamma\left(\frac{1-\lambda}{2}\right)}T^*_\lambda\,.$$

As long as $\mathbf{R}e\,\lambda \leq 2$ these are genuine Hilbert transforms, which are then analytically continuated to $\mathbf{C} \setminus \mathbf{N}_0$.

So, in retrospect, we have two convolution operators linking the T^*_λ- and U^*_λ-distributions, viz. the Dirac operator $\underline{\partial}\delta$ and the Hilbert operator $\mathcal{H}$, and two convolution operators acting inside each of the two families $\mathcal{T}$ and $\mathcal{U}$, viz. the Laplace operator $\Delta\delta$ and the Hilbert-Dirac operator $\mathcal{H}\underline{\partial} = \underline{\partial}\mathcal{H}$ (see also [5]):

(i) $\underline{\partial}\delta * T^*_\lambda = \lambda U^*_{\lambda-1}$ and $\underline{\partial}\delta * U^*_\lambda = U^*_\lambda * \underline{\partial}\delta = -2\pi T^*_{\lambda-1}$

(ii) $\mathcal{H}[T^*_\lambda] = -\frac{1}{\sqrt{\pi}}\dfrac{\Gamma\left(\frac{1-\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}U^*_\lambda$ and $\mathcal{H}[U^*_\lambda] = -\sqrt{\pi}\dfrac{\Gamma\left(-\frac{\lambda}{2}\right)}{\Gamma\left(\frac{1-\lambda}{2}\right)}T^*_\lambda$

(iii) $\Delta\delta * T^*_\lambda = 2\pi\lambda T^*_{\lambda-2}$ and $\Delta\delta * U^*_\lambda = 2\pi(\lambda-1)U^*_{\lambda-2}$

(iv) $\mathcal{H}\underline{\partial}[T^*_\lambda] = 2\sqrt{\pi}\dfrac{\Gamma\left(\frac{1-\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}T^*_{\lambda-1}$ and $\mathcal{H}\underline{\partial}[U^*_\lambda] = 2\sqrt{\pi}\dfrac{\Gamma\left(1-\frac{\lambda}{2}\right)}{\Gamma\left(\frac{1-\lambda}{2}\right)}U^*_{\lambda-1}$

By the Fourier transform these four convolution operators are turned into multiplication operators, two of them acting between the $\mathcal{T}$- and the $\mathcal{U}$-families, viz. $i\underline{x}$ and $i\underline{\omega}$, the other two acting inside each family, viz. $r^2 = (i\underline{x})^2$ and $r = |i\underline{x}| = (i\underline{x})(i\underline{\omega})$:

(i) $i\underline{x}T^*_{-\lambda-m} = -i\frac{\lambda}{2\pi}U^*_{-\lambda-m+1}$ and $i\underline{x}U^*_{-\lambda-m} = -iT^*_{-\lambda-m+1}$

(ii) $i\underline{\omega}T^*_{-\lambda-m} = -i\frac{\lambda}{2\pi}U^*_{-\lambda-m}$ and $i\underline{\omega}U^*_{-\lambda-m} = -iT^*_{-\lambda-m}$

(iii) $r^2T^*_{-\lambda-m} = -\frac{\lambda}{2\pi}T^*_{-\lambda-m+2}$ and $r^2U^*_{-\lambda-m} = \frac{1-\lambda}{2\pi}U^*_{-\lambda-m+2}$

(iv) $rT^*_{-\lambda-m} = -\frac{\lambda}{2\pi}T^*_{-\lambda-m+1}$ and $rU^*_{-\lambda-m} = \frac{1-\lambda}{2\pi}U^*_{-\lambda-m+1}$

Note that all those operators themselves belong to the $\mathcal{T}$- or the $\mathcal{U}$-families:

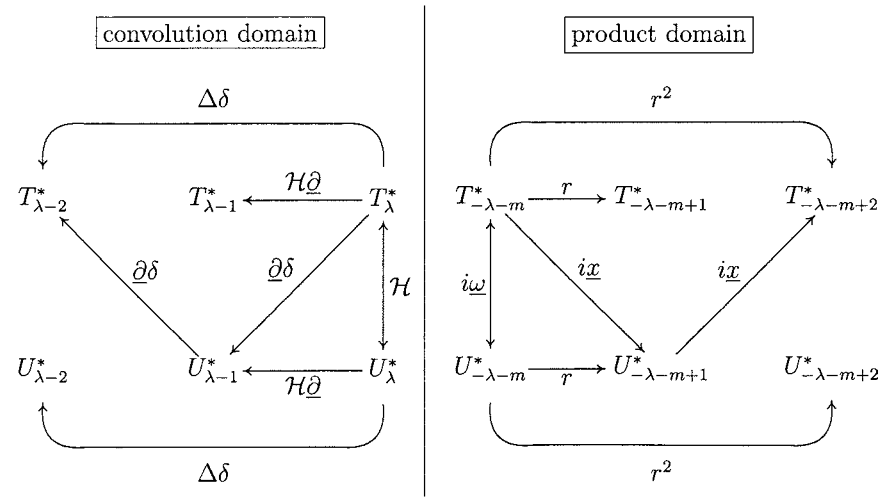
(i) $\underline{\partial}\delta = -2\frac{\Gamma\left(\frac{m}{2}+1\right)}{\pi^{\frac{m}{2}}}U^*_{-m-1}$ and $i\underline{x} = i\frac{\Gamma\left(\frac{m}{2}+1\right)}{\pi^{\frac{m}{2}+1}}U^*_1$

(ii) $\mathcal{H} = -\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}}U^*_{-m}$ and $i\underline{\omega} = i\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}}U^*_0$

(iii) $\Delta\delta = 4\frac{\Gamma\left(\frac{m}{2}+1\right)}{\pi^{\frac{m}{2}-1}}T^*_{-m-2}$ and $r^2 = \frac{\Gamma\left(\frac{m}{2}+1\right)}{\pi^{\frac{m}{2}+1}}T^*_2$

(iv) $\mathcal{H}\underline{\partial} = 2\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m-1}{2}}}T^*_{-m-1}$ and $r = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}}T^*_1$

which nicely illustrates the strongly unifying character of the T^*_λ- and the U^*_λ-distributions. This dance of both families of distributions is schematically shown in the picture below.



7. The fundamental solution of the operator $\underline{\partial}^\mu$

In the previous sections, we have collected all preliminary results needed to construct, in a rigorous way, the fundamental solution $E_\mu(\underline{x})$ of the operator $\underline{\partial}^\mu$. This operator is a convolution operator, acting on tempered distributions as follows (see [8, 5]):

$$\underline{\partial}^{\mu} f = \left[\frac{1+e^{i\pi\mu}}{2} \frac{2^{\mu}\Gamma\left(\frac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}}\, T^{*}_{-m-\mu} - \frac{1-e^{i\pi\mu}}{2} \frac{2^{\mu}\Gamma\left(\frac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}}\, U^{*}_{-m-\mu} \right] * f$$

$$= \frac{2^{\mu}}{\pi^{\frac{m}{2}}}\, Fp \frac{1}{|\underline{x}|^{\mu+m}} \left[\frac{1+e^{i\pi\mu}}{2} \frac{\Gamma\left(\frac{m+\mu}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)} - \frac{1-e^{i\pi\mu}}{2} \frac{\Gamma\left(\frac{m+\mu+1}{2}\right)}{\Gamma\left(-\frac{\mu-1}{2}\right)} \underline{\omega} \right] * f\ .$$

It is well-defined for all $\mu \in \mathbf{C}$, except for $\mu = -m-k$, $k \in \mathbf{N}_0$. Hence for the moment we exclude the above values of μ for which the operator $\underline{\partial}^{\mu}$ is not yet defined.

As can be seen from the structure of this section, we need to distinguish between several cases for the parameter $\mu \in \mathbf{C}$, depending on the dimension m.

7.1. The case $\mu = 1, 2, \ldots, m-1$. For these values of μ we already know that (see e.g. [3]):

$$\underline{\partial}^{2l}\left(\frac{1}{a_m} \frac{1}{2l-2} \cdots \frac{1}{4} \frac{1}{2} \frac{1}{m-2} \frac{1}{m-4} \cdots \frac{1}{m-2l}\, T_{-m+2l} \right) = \delta(\underline{x})\ ,$$

$$\underline{\partial}^{2l+1}\left(-\frac{1}{a_m} \frac{1}{2l} \cdots \frac{1}{4} \frac{1}{2} \frac{1}{m-2} \frac{1}{m-4} \cdots \frac{1}{m-2l}\, U_{-m+2l+1} \right) = \delta(\underline{x})\ .$$

Taking into account the respective definitions of T^{*}_{λ} and U^{*}_{λ}, this leads to the fundamental solutions:

$$E_{2l}(\underline{x}) = \frac{\Gamma\left(\frac{m}{2}-l\right)}{2^{2l}\pi^{\frac{m}{2}+l}}\, T^{*}_{-m+2l}\ , \tag{7.13}$$

$$E_{2l+1}(\underline{x}) = -\frac{\Gamma\left(\frac{m}{2}-l\right)}{2^{2l+1}\pi^{\frac{m}{2}+l+1}}\, U^{*}_{-m+2l+1}\ . \tag{7.14}$$

7.2. The case $\mu = m+k$, $k \in \mathbf{N}_0$. For *odd* dimension m, it is sufficient to notice that

$$T^{*}_{-2m-2k-1} * T^{*}_{2k+1} = \frac{\pi^{m}}{\Gamma\left(m+k+\frac{1}{2}\right)\Gamma\left(-k-\frac{1}{2}\right)}\ \delta(\underline{x})$$

and

$$U^{*}_{-2m-2k} * U^{*}_{2k} = \frac{\pi^{m+1}}{\Gamma\left(m+k+\frac{1}{2}\right)\Gamma\left(-k+\frac{1}{2}\right)}\ \delta(\underline{x})\ .$$

Hence, in this case, in view of the definitions of $\underline{\partial}^{m+2k}$ and $\underline{\partial}^{m+2k+1}$ in terms of U^{*}_{-2m-2k} and $T^{*}_{-2m-2k-1}$, we arrive at the fundamental solutions:

$$E_{m+2k}(\underline{x}) = -\frac{\Gamma\left(\frac{1}{2}-k\right)}{2^{m+2k}\pi^{\frac{1}{2}+m+k}}\ U^{*}_{2k}\ , \tag{7.15}$$

$$E_{m+2k+1}(\underline{x}) = \frac{\Gamma\left(-\frac{1}{2}-k\right)}{2^{m+2k+1}\pi^{\frac{1}{2}+m+k}}\ T^*_{2k+1}\ . \tag{7.16}$$

In case of an *even* dimension m, note that (7.13) and (7.14) are ill-defined whenever $\mu = m + k$ ($k \in \mathbf{N}_0$). Thus, one has to use different techniques in order to find a fundamental solution. As

$$\underline{\partial} \ln r = \frac{2}{a_m}\ U^*_{-1}\ ,$$

it can be directly seen that, for even m, the fundamental solution of $\underline{\partial}^m$ is given by

$$E_m(\underline{x}) = -\frac{1}{2^{m-1}\Gamma\left(\frac{m}{2}\right)}\ \ln r\ .$$

This fact inspired us to propose, for the remaining natural powers of $\underline{\partial}$, a fundamental solution containing a logarithmic term, which has eventually lead to the following proposition.

PROPOSITION 4. *Let the dimension m be even. Then, for all $k \in \mathbf{N}$, the fundamental solution $E_{m+k}(\underline{x})$ of $\underline{\partial}^{m+k}$ is given by*

$$E_{m+2k-1}(\underline{x}) = (p_{2k-1} \ln r + q_{2k-1}) U^*_{2k-1}\ , \tag{7.17}$$

$$E_{m+2k}(\underline{x}) = (p_{2k} \ln r + q_{2k}) T^*_{2k}\ , \tag{7.18}$$

with

$$\begin{aligned} p_{2k} &= \left(\frac{-1}{4\pi}\right)^k \frac{p_0}{k!} \\ q_{2k} &= -\left(\frac{-1}{4\pi}\right)^k \frac{p_0}{k!} \sum_{j=1}^{k} \left[\frac{1}{m+2k-2j} + \frac{1}{2k-2j+2}\right] \\ p_{2k+1} &= 2\left(\frac{-1}{4\pi}\right)^{k+1} \frac{p_0}{k!} \\ q_{2k+1} &= -2\left(\frac{-1}{4\pi}\right)^{k+1} \frac{p_0}{k!} \left\{ \sum_{j=1}^{k} \left[\frac{1}{m+2k-2j} + \frac{1}{2k-2j+2}\right] + \frac{1}{m+2k} \right\}, \end{aligned} \tag{7.19}$$

and

$$p_0 = -\frac{1}{2^{m-1}\Gamma\left(\frac{m}{2}\right)}\ .$$

PROOF. First, assuming that the desired fundamental solutions take the proposed forms (7.17) and (7.18), we establish recurrence relations between the coefficients (p_{2k}, q_{2k}) and (p_{2k-1}, q_{2k-1}). To this end, note that

$$\underline{\partial}^{m+2k} E_{m+2k}(\underline{x})\ =\ \delta(\underline{x})\ \Leftrightarrow\ \underline{\partial}(p_{2k} \ln r + q_{2k})\ T^*_{2k}\ =\ E_{m+2k-1}(\underline{x})\ ,$$

which, on account of (7.17) and the product rules from Proposition 3, leads to

$$\begin{cases} p_{2k} = \dfrac{1}{2k}\; p_{2k-1} \\ q_{2k} = \dfrac{1}{2k}\left(q_{2k-1} - \dfrac{1}{2k} p_{2k-1} \right) . \end{cases} \tag{7.20}$$

Similarly, we have

$$\underline{\partial}^{m+2k+1} E_{m+2k+1}(\underline{x}) \;=\; \delta(\underline{x}) \;\Leftrightarrow\; \underline{\partial}(p_{2k+1} \ln r + q_{2k+1}) U^*_{2k+1} \;=\; E_{m+2k}(\underline{x}) \;,$$

yielding

$$\begin{cases} p_{2k+1} = -\dfrac{1}{2\pi}\; p_{2k} \\ q_{2k+1} = -\dfrac{1}{2\pi}\left(q_{2k} - \dfrac{1}{m+2k} p_{2k} \right) . \end{cases} \tag{7.21}$$

From (7.20) and (7.21), a straightforward calculation leads to the desired closed expressions (7.19) for all coefficients. □

7.3. The case $\mu \in \mathbf{C} \setminus \mathbf{N}$, $\mu \neq -m-k$, $k \in \mathbf{N}_0$. It remains to construct fundamental solutions $E_\mu(\underline{x})$ for all non-natural powers of $\underline{\partial}$. Seen the definition of $\underline{\partial}^\mu$ as a convolution operator in terms of $T^*_{-m-\mu}$ and $U^*_{-m-\mu}$ and the fact that

$$T^*_{-m-\mu} * T^*_{-m+\mu} = \frac{\pi^m}{\Gamma\left(\frac{m+\mu}{2}\right)\Gamma\left(\frac{m-\mu}{2}\right)}\; \delta(\underline{x})$$

and

$$U^*_{-m-\mu} * U^*_{-m+\mu} = \frac{\pi^{m+1}}{\Gamma\left(\frac{m+\mu+1}{2}\right)\Gamma\left(\frac{m-\mu+1}{2}\right)}\; \delta(\underline{x}) \;,$$

it seems rather natural to define $E_\mu(\underline{x})$ as a linear combination of $T^*_{-m+\mu}$ and $U^*_{-m+\mu}$. Hence we put

$$E_\mu(\underline{x}) = C_T(m,\mu)\; T^*_{-m+\mu} + C_U(m,\mu)\; U^*_{-m+\mu} \;,$$

where the complex coefficients $C_T(m,\mu)$ and $C_U(m,\mu)$ still need to be determined in order to fulfil $\underline{\partial}^\mu E_\mu(\underline{x}) = \delta(\underline{x})$.

The action of $\underline{\partial}^\mu$ on the distribution $E_\mu(\underline{x})$ produces four terms, two of which are "mixed", in the sense that they contain a T^* as well as a U^*; they are given by

$$\begin{aligned} &\frac{1+e^{i\pi\mu}}{2} \frac{2^\mu \Gamma\left(\frac{m+\mu}{2}\right)}{\pi^{\frac{m-\mu}{2}}}\; C_U(m,\mu)\; T^*_{-m-\mu} * U^*_{-m+\mu} \\ &\quad = \frac{1+e^{i\pi\mu}}{2} \frac{2^\mu \Gamma\left(\frac{m+1}{2}\right)}{\pi^{-\frac{\mu}{2}} \Gamma\left(\frac{m-\mu+1}{2}\right)}\; C_U(m,\mu)\; U^*_{-m} \end{aligned}$$

and

$$-\frac{1-e^{i\pi\mu}}{2}\ \frac{2^{\mu}\Gamma\left(\frac{m+\mu+1}{2}\right)}{\pi^{\frac{m-\mu+1}{2}}}\ C_T(m,\mu)\ U^*_{-m-\mu} * T^*_{-m+\mu}$$
$$= -\frac{1-e^{i\pi\mu}}{2}\ \frac{2^{\mu}\Gamma\left(\frac{m+1}{2}\right)}{\pi^{-\frac{\mu-1}{2}}\Gamma\left(\frac{m-\mu}{2}\right)}\ C_T(m,\mu)\ U^*_{-m}\ .$$

Clearly, we need these terms to cancel each other, leading to the condition

$$\frac{C_T(m,\mu)}{C_U(m,\mu)} = \pi^{\frac{1}{2}}\frac{1+e^{i\pi\mu}}{1-e^{i\pi\mu}}\ \frac{\Gamma\left(\frac{m-\mu}{2}\right)}{\Gamma\left(\frac{m-\mu+1}{2}\right)}\ . \tag{7.22}$$

One can easily verify that with

$$C_T(m,\mu) = \frac{e^{-i\pi\mu}+1}{2}\ \frac{\Gamma\left(\frac{m-\mu}{2}\right)}{2^{\mu}\pi^{\frac{m+\mu}{2}}}$$

and

$$C_U(m,\mu) = \frac{e^{-i\pi\mu}-1}{2}\ \frac{\Gamma\left(\frac{m-\mu+1}{2}\right)}{2^{\mu}\pi^{\frac{m+\mu+1}{2}}}$$

not only condition (7.22) is satisfied, but moreover

$$\underline{\partial}^{\mu}[C_T(\mu,m)T^*_{-m+\mu} + C_U(\mu,m)U^*_{-m+\mu}] = \delta(\underline{x})\ .$$

This means that for arbitrary $\mu \in \mathbf{C} \setminus \mathbf{N}$, the fundamental solution of $\underline{\partial}^{\mu}$ is given by

$$E_{\mu} = \frac{1+e^{-i\pi\mu}}{2}\ \frac{\Gamma\left(\frac{m-\mu}{2}\right)}{2^{\mu}\pi^{\frac{m+\mu}{2}}}\ T^*_{-m+\mu} - \frac{1-e^{-i\pi\mu}}{2}\ \frac{\Gamma\left(\frac{m-\mu+1}{2}\right)}{2^{\mu}\pi^{\frac{m+\mu+1}{2}}}\ U^*_{-m+\mu}\ .$$

Note that the case of the natural powers of the Dirac operator, already mentioned in [3], is included in this formula as well (see (7.13)–(7.14) and even (7.15)–(7.16)). Only the fundamental solutions $E_{m+k}(\underline{x})$ (with $k \in \mathbf{N}_0$ and m even) escape from this unifying structure.

7.4. The case $\mu = -m-k$, $k \in \mathbf{N}_0$. We are now able to define $\underline{\partial}^{-m-k}$ as the convolution operator

$$\underline{\partial}^{-m-k} f = E_{m+k} * f$$

where E_{m+k} has been constructed in subsection 7.2.

We put

$$E_{-m-k} = \underline{\partial}^{m+k}\delta$$

and observe that indeed

$$\underline{\partial}^{-m-k} E_{-m-k} = E_{m+k} * \underline{\partial}^{m+k}\delta = \delta\ .$$

Note that, depending on the parity of the dimension m and the natural number k, E_{-m-k} is indeed a distribution in the $\mathcal{T}$- or the $\mathcal{U}$-family.

8. Historical comment

The radial distributions $T_\lambda = Fp\, r^\lambda$, $r = |\underline{x}|$, $\lambda \in \mathbf{C}$, are of course well known. In [12] Riesz introduced their normalizations R_α (see Section 4) and the corresponding Riesz potentials

$$I^\gamma[f] = R_\gamma * f\,, \quad f \in \mathcal{S}\,.$$

Note that, if ignoring the additional definition of R_{m+2l}, R_α shows simple poles at $\alpha = m+2l$, $l \in \mathbf{N}_0$ and that $\mathcal{F}[R_{-2l}]$ and $\mathcal{F}[R_{m+2l}]$, $l \in N_0$ are no Riesz kernels anymore. In our approach T_λ^* is an entire function of $\lambda \in \mathbf{C}$ and the Fourier transform is a bijection in the family $\{T_\lambda^* : \lambda \in \mathbf{C}\}$.

In [11] Horváth introduced the vectorial kernels

$$\vec{N}_\alpha = -\vec{\nabla} R_{\alpha+1}\,,$$

which, for $\alpha \neq -2l-1$ and $\alpha \neq m+2l+1$, $l \in \mathbf{N}_0$, are given by

$$\vec{N}_\alpha = \frac{1}{2^\alpha \pi^{\frac{m}{2}}}\, \frac{\Gamma\left(\frac{m-\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}\, \frac{\vec{x}}{r^{m-\alpha+1}}\,.$$

These kernels satisfy the convolution formula

$$\vec{N}_\alpha * \vec{N}_\beta = -R_{\alpha+\beta}\,,$$

where the convolution of the two vector valued distributions has to be taken in the sense of a scalar product.

If the Euclidean vector $\vec{x}$ is identified with the Clifford-vector $\underline{x}$, then the Horváth kernels $\vec{N}_\alpha$ correspond to the Clifford distributions

$$\vec{N}_\alpha \approx \frac{\Gamma\left(\frac{m-\alpha+1}{2}\right)}{2^\alpha \pi^{\frac{\alpha+m+1}{2}}}\, U^*_{\alpha-m}\,.$$

Again note that $\vec{N}_\alpha$ shows simple poles at $\alpha = m+2l+1$, $l \in \mathbf{N}_0$, whereas $U^*_{\alpha-m}$ is an entire function of $\lambda \in \mathbf{C}$. The convolution of $\vec{N}_\alpha$ with an appropriate function, say a rapidly decreasing one, then gives rise to the vectorial counterpart of our Clifford-vector valued Riesz potentials of the second kind. Moreover, after identification, up to a minus sign, of the scalar product of two vectors with the inner product of two Clifford-vectors, the above convolution formula for the Horváth kernels $\vec{N}_\alpha$, finds its equivalent in our formula (4.7) for the convolution of U_λ-distributions where the Clifford geometric multiplication is involved.

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