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Partial Sums of Multiple Zeta Value Series

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1. Introduction

For $\vec{s} = (s_1, \ldots, s_d) \in \mathbf{N}^d$ with $s_d > 1$, the Multiplze Zeta Value (MZV) series $\zeta_d(\vec{s})$ is defined as

$$\zeta_d(\vec{s}) = \zeta_d(s_1, \ldots, s_d) = \sum_{1 \le k_1 < \cdots < k_d} k_1^{-s_1} \cdots k_d^{-s_d} .$$

We call d its depth and $wt(\vec{s}) := \sum_{i=1}^{d} s_i$ its weight.

Now we consider partial sums of these MZV series. More precisely, for $\vec{s} \in \mathbf{N}^d$ and a non-negative integer n, the nth partial sum of MZV series $H_d(\vec{s}; n)$ is defined by

$$H_d(\vec{s}\,; n) = H_d(s_1, \ldots, s_d\,; n) = \sum_{1 \le k_1 < \cdots < k_d \le n} k_1^{-s_1} \cdots k_d^{-s_d} , \tag{1.1}$$

where $H_d(\vec{s}\,; r) = 0$ for $r = 0, \ldots, d-1$. Then the following theorem was obtained by Wolstenholme ([3], p. 89):

THEOREM 1.1. *For any prime number* $p \ge 5$, $H_1(1; p-1) \equiv 0 \pmod{p^2}$.

In [6] Zhao studied the p-divisibility of $H_d(\vec{s}\,; p-1)$ for general $\vec{s}$, which turned out to be closely related to the Bernoulli numbers B_t defined by $\frac{x}{e^x-1} = \sum_{t=0}^{\infty} \frac{B_t}{t!} x^t$. When (p, t) is an irregular pair, i.e., $p \,|\, B_t$ for even t with $2 \le t \le p-3$, $H_d(\vec{s}\,; p-1)$ is divisible by higher power of p than usually expected. As another generalization of Theorem 1.1, Bayat considered the p-divisibility of $H_1^*(s\,; p^a - 1)$ for a positive integer a in [1], where $H_1^*(s\,; n) = \sum_{\substack{1 \le k \le n \\ p \nmid k}} k^{-s}$ for any $n \in \mathbf{N}$ and we have $H_1^*(s\,; p-1) = H_1(s\,; p-1)$.

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In this paper, we study the p-divisibility of the series $H_d^*(\vec{s}\,;p^a-1)$ for $\vec{s}=(s_1,\dots,s_d)\in\mathbf{N}^d$, where

$$H_d^*(\vec{s}\,;n)=\sum_{\substack{1\le k_1<\cdots<k_d\le n\\ p\nmid k_1,\dots,p\nmid k_d}} k_1^{-s_1}\cdots k_d^{-s_d}\quad\text{for}\quad n\in\mathbf{N}\,.$$

This series was originally introduced by Zhao in [7], and although H_d^* depends on p, we do not put p here by following Zhao's notation. We generalize Zhao's results in [6] to $a\ge 2$, where he treated the case $a=1$, and these contain Bayat's when $d=1$ (Lemma 2.2 in Sec. 2). Here we will see that these sums are related to irregular pairs of higher order introduced by Keller, which is very interesting. The definition of such a prime is the following:

DEFINITION 1.2 ([5], p. 3). Let $\hat{B}(l)=B_l/l$. A pair (p,l) is called *an irregular pair of order a* if $p^a\,|\,\hat{B}(l)$ with $2\le l\le p^{a-1}(p-1)$ and even l.

Here is the outline of the paper: We start with MZV series of depth 1 in Section 2, and deal with homogeneous MZV series, i.e., $\vec{s}=(\{s\}^d)$ the set formed by repeating s d times in Section 3; In Section 4 we treat with the non-homogeneous series of depth 2, and relate the result to the p-divisibility set $J(s_1,s_2\,|\,p)$, the set of $n\in\mathbf{Z}_{\ge 0}$ such that $p\,|\,H_2^*(s_1,s_2\,;n)$ (see Corollary 4.5).

2. Zeta-value series

Let a be a positive integer throughout this paper. The results in this section are generalizations of those in [6] to $a\ge 2$.

We need the following Clausen-von Staudt Theorem (see [4], p. 233):

LEMMA 2.1. *For a positive integer m, $B_{2m}+\sum_{p-1|2m}\frac{1}{p}$ is an integer.*

Now we begin with the following lemma.

LEMMA 2.2 (cf. [6, Lemma 2.2]). *Let p be an odd prime and s be a positive integer. Then*

$$H_1^*(s\,;p^a-1)\equiv\begin{cases}0 \pmod{p^{\wp(s+1)a}} & \text{if } s \text{ is odd and } p-1\nmid s+1\,,\\ & \quad\text{or } s \text{ is even and } p-1\nmid s\,,\\ -p^{a-1}\pmod{p^a} & \text{if } p-1\,|\,s\,,\\ -p^{2a-1}(m+1)/2 \pmod{p^{2a}} & \text{if } s+1=m(p-1)\,,\end{cases}$$

where $\wp(i)$ denotes the parity of i which is 1 if i is odd and 2 otherwise.

PROOF. Suppose s is odd, and choose n such that $3 \leq t := np^{2a}(p-1) - s \leq p^{2a-1}(p-1) + 2$. Then t is odd. By Fermat's Little Theorem we have

$$H_1^*(s\,;\,p^a - 1) = \sum_{\substack{k=1 \\ k \not\equiv 0 \pmod p}}^{p^a-1} \frac{1}{k^s} \equiv \sum_{k=1}^{p^a-1} k^t - \sum_{k=1}^{p^{a-1}-1} (pk)^t \pmod{p^{2a}}\,.$$

Now we use the formula of power-sum

$$\sum_{k=1}^{p^a-1} k^t = \frac{1}{t+1}(B_{t+1}(p^a) - B_{t+1}) \quad \text{for } t \geq 1\,, \tag{2.1}$$

where $B_{t+1}(z)$ are the Bernoulli polynomials defined by $\frac{xe^{zx}}{e^x-1} = \sum_{t=0}^{\infty} \frac{B_t(z)}{t!} x^t$ and so

$$B_{t+1}(p^a) = \sum_{k=0}^{t+1} \binom{t+1}{k} B_{t+1-k}\, p^{ka}\,.$$

Thus,

$$H_1^*(s\,;\,p^a - 1) \equiv p^a B_t + \frac{t}{2}\, p^{2a} B_{t-1} \pmod{p^{2a}}\,.$$

When $p-1 \nmid s+1$, this lemma follows from the fact that B_j is 0 for odd j greater than 1 and that B_{t-1} is p-integral by Lemma 2.1.

If $s+1 = m(p-1)$, then we can put $1 \leq s \leq p^{2a-1}(p-1) - 1$ and $2 \leq m \leq p^{2a-1}$. Taking $n = m$, we get $t = mp^{2a}(p-1) - m(p-1) + 1$. From $pB_{t-1} \equiv -1 \pmod p$ by Lemma 2.1 again, we obtain

$$H_1^*(s;\, p^a - 1) \equiv \frac{t}{2}\, p^{2a} B_{t-1} \equiv -\frac{p^{2a-1}}{2}\,(m+1) \pmod{p^{2a}}\,.$$

Similarly, we can prove this lemma for an even s. □

REMARK. $H_1^*(s\,;\,p^a - 1)$ is the particular case of $H_1^*(s\,;\,pn)$ for $n = p^{a-1}$ (cf. [7, Lemma 3.2]). For odd $s < p-1$, Lemma 2.2 is a refinement of what Bayat obtained in [1, Theorem 4].

The proof of Lemma 2.2 is effective in later discussion. In fact, a close look at the proof gives us the following refinement:

THEOREM 2.3 (cf. [6, Theorem 2.8]). *Suppose n is a positive integer and p is an odd prime such that $p \geq 2n+3$. Then we have the congruences*:

$$\begin{aligned} \frac{-2}{2n-1} \cdot H_1^*(2n-1;\, p^a - 1) &\equiv p^a \cdot H_1^*(2n;\, p^a - 1) \pmod{p^{3a}} \\ &\equiv p^{2a} \cdot \frac{2n}{2n + p^{a-1}} B_{p^{a-1}(p-1)-2n} \pmod{p^{3a}}\,. \end{aligned} \tag{2.2}$$

In particular, the following are equivalent:

(a) $(p, p^{a-1}(p-1)-2n)$ *is an irregular pair of order* a.
(b) $H_1^*(2n\,; p^a-1) \equiv 0 \pmod{p^{2a}}$.
(c) $H_1^*(2n-1\,; p^a-1) \equiv 0 \pmod{p^{3a}}$.
(d) $H_2^*(n, n\,; p^a-1) \equiv 0 \pmod{p^{2a}}$.

PROOF. Take m such that $a+1 \le t := mp^{3a-1}(p-1)-2n \le a+p^{3a-1}(p-1)$. Then even $t \equiv p^{a-1}(p-1)-2n \pmod{p^{a-1}(p-1)}$ and $p-1 \nmid t$, so B_t/t is a p-integer ([4], p. 238). From (2.1) we have

$$H_1^*(2n-1\,; p^a-1) \equiv \sum_{\substack{k=1 \\ k\not\equiv 0 \pmod p}}^{p^a-1} k^{t+1} \equiv \frac{t+1}{2} p^{2a} B_t \pmod{p^{3a}}$$

and

$$H_1^*(2n\,; p^a-1) \equiv \sum_{\substack{k=1 \\ k\not\equiv 0 \pmod p}}^{p^a-1} k^{t} \equiv p^a B_t \pmod{p^{2a}},$$

and by the Kummer Congruences ([4], p. 239) we obtain

$$B_t \equiv \frac{2nB_{p^{a-1}(p-1)-2n}}{2n+p^{a-1}} \pmod{p^a}.$$

Hence (2.2) and the equivalences of (a) to (c) follow. For the equivalence of (b) and (d), we use the shuffle relation

$$H_1^*(n; p^a-1)^2 = 2H_2^*(n, n; p^a-1) + H_1^*(2n; p^a-1), \tag{2.3}$$

(see Section 3), and apply Lemma 2.2. □

COROLLARY 2.4 (cf. [6, Proposition 2.9]). *Let* n *and* p *be as in Theorem* 2.3. *Then there exists* n' *which satisfies* $2n \equiv 2n' \pmod{p-1}$ *and* $2 \le 2n' \le p-3$. *If* $(p, p-2n'-1)$ *is an irregular pair, then the following conditions are equivalent*:

(a) $(p, p^{a-1}(p-1)-2n)$ *is an irregular pair of order* $2a$.
(b) $H_1^*(2n\,; p^a-1) \equiv 0 \pmod{p^{3a}}$.
(c) $H_1^*(2n-1\,; p^a-1) \equiv 0 \pmod{p^{4a}}$.

When n *is odd, they are all equivalent to*
(d) $H_2^*(n, n\,; p^a-1) \equiv 0 \pmod{p^{3a}}$.

PROOF. Take m such that $2a+1 \le t := mp^{4a-1}(p-1)-2n \le 2a+p^{4a-1}(p-1)$. Then t is even and $t \equiv p^{2a-1}(p-1)-2n \pmod{p^{2a-1}(p-1)}$.

Suppose $p-1 \mid t-2$. Then $p-1 | 2n'+2$, namely $p-2n'-1=2$. Since $(p,2)$ is never an irregular pair, $p-1 \nmid t-2$ and B_{t-2} is p-integral. The equivalences of (a) to (d) follow as in the proof of Theorem 2.3. □

3. Homogeneous MZV series

In this section, we consider homogeneous MZV series, i.e., the series with $\vec{s} = (s, s, \ldots, s)$. Now shuffle relations for H_d explained in [6] can be applied to H_d^*: for positive integers $s_0, s_1, \ldots, s_d$ and n, it holds that

$$H_1^*(s_0\,; n)\cdot H_d^*(s_1,\ldots,s_d\,; n)$$
$$= \sum_{\vec{s}\in \mathrm{Shfl}(s_0,(s_1,\ldots,s_d))} H_{d+1}^*(\vec{s}\,; n) + \sum_{j=1}^{d} H_d^*(s_1,\ldots,s_{j-1}, s_j+s_0, s_{j+1},\ldots,s_d\,; n)\,,$$

where

$$\mathrm{Shfl}(s_0,(s_1,\ldots,s_d)) := \sum_{\substack{\sigma \text{ permutes } \{0,\ldots,d\} \\ \sigma^{-1}(1)<\cdots<\sigma^{-1}(d)}} (s_{\sigma(0)},\ldots,s_{\sigma(d)})\,.$$

Thus, for any $l=1,\ldots,d-1$, we have

$$H_1^*(ls\,; n)\cdot H_{d-l}^*(\{s\}^{d-l}; n) = \sum_{\vec{s}\in \mathrm{Shfl}(\{ls\},\{s\}^{d-l})} H_{d-l+1}^*(\vec{s}\,; n) + \sum_{\vec{s}\in \mathrm{Shfl}(\{(l+1)s\},\{s\}^{d-l-1})} H_{d-l}^*(\vec{s}\,; n)\,.$$

Applying $\sum_{l=1}^{d-1}(-1)^l$ to the both sides, we get the following lemma:

LEMMA 3.1 (cf. [6, Lemma 2.10]). *Let s, d and n be positive integers. Then*

$$d\,H_d^*(\{s\}^d; n) = \sum_{l=1}^{d}(-1)^{l-1} H_1^*(ls\,; n)\cdot H_{d-l}^*(\{s\}^{d-l}; n)\,.$$

Let $P(d)$ be the set of unordered partitions of d. For $\lambda=(\lambda_1,\ldots,\lambda_r)\in P(d)$ we put $H^{*\lambda}(s\,; n) = \prod_{i=1}^{r} H_1^*(\lambda_i s\,; n)$.

LEMMA 3.2 (cf. [6, Lemma 2.11]). *Let s, d and n be positive integers. Then there are integers c_λ such that*

$$d!\,H_d^*(\{s\}^d; n) = \sum_{\lambda\in P(d)} c_\lambda H^{*\lambda}(s; n)\,, \tag{3.1}$$

where $c_{(d)} = (-1)^{d-1}(d-1)!$.

PROOF. The proof goes by induction on d as that of Lemma 2.11 in [6].

For $\lambda = (\lambda_1, \ldots, \lambda_r) \in P(d)$, c_λ is determined recursively by

$$c_\lambda = {\sum_{i=1}^{r}}' \frac{(d-1)!}{(d-\lambda_i)!}(-1)^{\lambda_i - 1} c_{(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_r)} ,$$

where $\widehat{\lambda_i}$ and $\sum'$ mean that we leave out λ_i and that we take non-repeating sub-partitions $(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_r)$ in the sum, respectively. □

The following theorem treats the particular case $n = p^a$ of Proposition 3.3 in [7], and we obtain the stronger result:

THEOREM 3.3 (cf. [6, Lemma 2.13]). *Let s and d be two positive integers, and p be an odd prime satisfying $p \geq d+2$ and $p-1$ divides none of sl and $sl+1$ for $l = 1, 2, \ldots, d$. Then*

$$H_d^*(\{s\}^d;\, p^a - 1) \equiv 0 \pmod{p^{\wp(sd-1)a}} . \tag{3.2}$$

Particularly, if $p \geq sd+3$, then (3.2) *is always true and so $p^a \,|\, H_d^*(\{s\}^d;\, p^a-1)$.*

PROOF. We use (3.1) by substituting $p^a - 1$ for n and apply Lemma 2.2. □

PROPOSITION 3.4 (cf. [6, Proposition 2.14]). *Suppose $(p, p^{a-1}(p-1)-(3s+1))$ is an irregular pair of order a. Then*

$$H_3^*(s,s,s;\, p^a - 1) \equiv \begin{cases} 0 \pmod{p^{3a}} & \text{for } a = 1 , \\ 0 \pmod{p^{3a-2}} & \text{otherwise} . \end{cases}$$

PROOF. By the assumption, s must be odd. From Lemma 3.2 it holds that

$$6H_3^*(\{s\}^3;\, p^a-1) = H_1^*(s;\, p^a-1)^3 - 3H_1^*(s;\, p^a-1)H_1^*(2s;\, p^a-1) + 2H_1^*(3s;\, p^a-1) .$$

We apply Theorem 2.3 and Lemma 2.2 to $H_1^*(3s;\, p^a-1)$ and other terms on the right-hand side, respectively. □

4. Non-homogeneous MZV series of depth 2

For $\vec{s} = (s_1, s_2)$, we obtain the following theorem:

THEOREM 4.1 (cf. [6, Theorem 3.1]). *Let s_1 and s_2 be two positive integers and p be an odd prime. Let $s_1 \equiv m,\ s_2 \equiv n \pmod{p-1}$ and $0 \leq m, n \leq p-2$. Then*

$$H_2^*(s_1, s_2;\, p^a - 1) \equiv p^{a-1} \cdot \omega \pmod{p^a} ,$$

where

$$\omega = \begin{cases} \eta & \text{if } (m,n) = (0,0)\,, \\ 1 & \text{if } (m,n) = (1,0)\,, \\ -1 & \text{if } (m,n) = (0,1)\,, \\ \dfrac{(-1)^n}{m+n}\dbinom{m+n}{m} B_{p-m-n} & \text{if } p \geq m+n \text{ and } m,\ n \geq 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and

$$\eta = \begin{cases} 1 & \text{if } a = 1\,, \\ \dfrac{1}{2} & \text{otherwise}\,. \end{cases}$$

PROOF. Let $s_1 \equiv m_1,\ s_2 \equiv n_1 \pmod{p^{a-1}(p-1)}$ and $0 \leq m_1, n_1 \leq p^{p-1}(p-1)-1$, and let $M = p^{a-1}(p-1) - m_1$ and $N = p^{a-1}(p-1) - n_1$. Then $1 \leq M,\ N \leq p^{a-1}(p-1)$ and

$$H_2^*(s_1, s_2;\, p^a - 1) \equiv \sum_{\substack{1 \leq k_1 < k_2 \leq p^a - 1 \\ p \nmid k_1 and\ p \nmid k_2}} k_1^M k_2^N \pmod{p^a}\,.$$

To proceed the proof for $a \geq 2$, we need the result for $a = 1$ in [6] and the following two lemmas. □

LEMMA 4.2. *For an odd prime p and positive integers M and N,*

$$\sum_{\substack{1 \leq k_1 < k_2 \leq p^a - 1 \\ p \nmid k_1 and\ p \nmid k_2}} k_1^M k_2^N \equiv p^{a-1} \sum_{1 \leq k_1 < k_2 \leq p-1} k_1^M k_2^N \pmod{p^a} \tag{4.1}$$

holds except when $p - 1$ divides both M and N.

PROOF. The left-hand side of (4.1) is written as

$$\begin{aligned} \sum_{\substack{1 \leq k_1 < k_2 \leq p^a - 1 \\ p \nmid k_1 and\ p \nmid k_2}} k_1^M k_2^N &= \sum_{s=0}^{p^{a-1}-1} \sum_{1 \leq k_1 < k_2 \leq p-1} (sp + k_1)^M (sp + k_2)^N \\ &\quad + \sum_{s=0}^{p^{a-1}-1} \sum_{t=s+1}^{p^{a-1}-1} \sum_{k_1=1}^{p-1} \sum_{k_2=1}^{p-1} (sp + k_1)^M (tp + k_2)^N \\ &= \sum_{s=0}^{p^{a-1}-1} \sum_{1 \leq k_1 < k_2 \leq p-1} k_1^M k_2^N \end{aligned}$$

$$+\sum_{s=0}^{p^{a-1}-1}\sum_{l_1=0}^{M}\sum_{\substack{l_2=0\\ l_1+l_2\geq 1}}^{N}\binom{M}{l_1}\binom{N}{l_2}k_1^{M-l_1}k_2^{N-l_2}(ps)^{l_1+l_2}$$

$$+\sum_{s=0}^{p^{a-1}-1}\sum_{t=s+1}^{p^{a-1}-1}\sum_{k_1=1}^{p-1}\sum_{k_2=1}^{p-1}(sp+k_1)^M(tp+k_2)^N . \tag{4.2}$$

By (2.1)

$$\begin{aligned} p^{l_1+l_2}&\sum_{s=0}^{p^{a-1}-1}s^{l_1+l_2}\\ &\equiv p^{l_1+l_2}\left\{B_{l_1+l_2}p^{a-1}+\frac{l_1+l_2-1}{2}B_{l_1+l_2-1}p^{2(a-1)}\right\}\equiv 0 \pmod{p^a} \end{aligned} \tag{4.3}$$

when $l_1+l_2\geq 1$, because p in the denominator of $B_{l_1+l_2+1}$ or $B_{l_1+l_2}$ cancels out with $p^{l_1+l_2}$. Thus, the second term on the right-hand side of (4.2) is congruent to 0 modulo p^a.

As for the third term, we can write it as

$$\begin{aligned} \sum_{s=0}^{p^{a-1}-1}\sum_{t=s+1}^{p^{a-1}-1}\sum_{k_1=1}^{p-1}\sum_{k_2=1}^{p-1}(sp+k_1)^M(tp+k_2)^N &= \sum_{s=0}^{p^{a-1}-1}\sum_{t=s+1}^{p^{a-1}-1}\sum_{k_1=1}^{p-1}\sum_{k_2=1}^{p-1}k_1^Mk_2^N\\ +\sum_{s=0}^{p^{a-1}-1}\sum_{t=s+1}^{p^{a-1}-1}\sum_{k_1=1}^{p-1}\sum_{k_2=1}^{p-1}\sum_{l_1=0}^{M}&\sum_{\substack{l_2=0\\ l_1+l_2\geq 1}}^{N}\binom{M}{l_1}\binom{N}{l_2}k_1^{M-l_1}k_2^{N-l_2}p^{l_1+l_2}s^{l_1}t^{l_2} . \end{aligned}$$

The second term in the above equation is congruent to 0 modulo p^a by the similar calculation as in (4.3), and the first term becomes

$$\sum_{s=0}^{p^{a-1}-1}\sum_{t=s+1}^{p^{a-1}-1}\sum_{k_1=1}^{p-1}\sum_{k_2=1}^{p-1}k_1^Mk_2^N\equiv\frac{p^{a-1}(p^{a-1}-1)}{2}\sum_{k_1=1}^{p-1}k_1^M\sum_{k_2=1}^{p-1}k_2^N\equiv 0 \pmod{p^a}$$

except when $p-1$ divides both M and N by Lemma 2.2. □

For $\vec{s}=(s_1,\dots,s_d)\in\mathbf{N}^d$, we set $\overleftarrow{s}=(s_d,\dots,s_1)$.

LEMMA 4.3 (cf. [6, Lemma 3.2]). *Let p be an odd prime. Then*

$$H_d^*(\vec{s}\,;p^a-1)\equiv(-1)^{\mathrm{wt}(s)}H_d^*(\overleftarrow{s}\,;p^a-1) \pmod{p^a} .$$

PROOF. Substitute p^a-k_i for k_i $(1\leq i\leq d)$ in the definition of $H_d^*(\vec{s};p^a-1)$ in (1.1). □

We go back to the proof of Theorem 4.1. From Lemma 4.2 and the result for $a=1$, Theorem 4.1 follows except when $p-1$ divides both s_1 and s_2.

When $p-1$ divides both s_1 and s_2, from the shuffle relation and Lemma 4.3, we get

$$H_2^*(s_1, s_2\,;\,p^a-1) = \frac{H_1^*(s_1\,;\,p^a-1)H_1^*(s_2\,;\,p^a-1) - H^*(s_1+s_2\ \ p^a-1)}{2},$$

and apply Lemma 2.2. □

The above proof for s_1 and s_2 both divisible by $p-1$ is valid for any positive integers s_1 and s_2 with the same parity:

COROLLARY 4.4 (cf. [6, Corollary 3.4]). *Let p be an odd prime. Suppose s_1 and s_2 are two positive integers satisfying* (i) $s_1 \equiv s_2 \pmod 2$ *and* (ii) $p-1 \nmid s_1+s_2$. *Then*

$$H_2^*(s_1, s_2;\,p^a-1) \equiv 0 \pmod{p^a} \tag{4.4}$$

holds for $a \geq 2$. With one more condition (iii) $p-1 \nmid s_1$ *or* $p-1 \nmid s_2$, (4.4) *holds for $a=1$.*

Now we relate Therem 4.1 to the p-divisibility set $J(s_1, s_2 \mid p)$. Put

$$H(\vec{s}\,;\,n) = \frac{a(\vec{s}\,;\,n)}{b(\vec{s}\,;\,n)}\,, \quad a(\vec{s}\,;\,n),\ b(\vec{s}\,;\,n) \in \mathbf{N}\,, \quad \gcd\,(a(\vec{s}\,;\,n), b(\vec{s}\,;\,n)) = 1\,,$$

and $a(\vec{s}; r) = 0$ and $b(\vec{s}; r) = 1$ for $r = 0, 1, \ldots, d-1$. Then the p-divisibility set of the MZV series $J(\vec{s} \mid p)$ is defined as

$$J(\vec{s} \mid p) = \{n \in \mathbf{Z}_{\geq 0}\ :\ a(\vec{s}\,;\,n) \equiv 0 \pmod p\}\,.$$

In [7] Zhao surmised that $J(\vec{s} \mid p)$ is finite and $J(\vec{s}\,|2) = 0$, which is the generalization of the conjecture for $\vec{s} = (1)$ by Eswarathasan and Levine [2].

When $\vec{s} = (s_1, s_2)$, he obtained that $J(s_1, s_2 \mid p)$ is finite if s_1, $s_2 \leq 20$, $s_2 \geq 2$, and $p = 2, 3, 5$ ([7, Theorem 1.3]), for which we get the following corollary:

COROLLARY 4.5. *Suppose $a \geq 2$. Let s_1 and s_2 be positive integers and m and n be as in Theorem* 4.1. *Then p^a-1 is not an element of $J(s_1, s_2\ \mid p)$ for $(m,n) =$ $(0,0)$, $(1,0)$, $(0,1)$, and all the pairs (m,n) such that $m, n \geq 1$, $m+n$ is odd, and $(p, p-m-n)$ is not an irregular pair.*

PROOF. Since we have

$$H_2(s_1, s_2;\,p^a-1) = \sum_{\substack{1\leq k_1\leq k_2\leq p^a-1\\ p|k_1 \text{ or } p|k_2}} \frac{1}{k_1^{s_1}k_2^{s_2}} + H_2^*(s_1, s_2;\,p^a-1)$$

and $p\,|H_2^*(s_1, s_2;\,p^a-1)$ by Theorem 4.1, our statement is true if $p \nmid \sum_{\substack{1\leq k_1\leq k_2\leq p^a-1\\ p|k_1 \text{ or } p|k_2}} \frac{1}{k_1^{s_1}k_2^{s_2}}$.

From the identity

$$\sum_{\substack{1\leq k_1\leq k_2\leq p^a-1\\ p|k_1 \text{ or } p|k_2}} \frac{1}{k_1^{s_1}k_2^{s_2}} = \sum_{1\leq k_1\leq k_2\leq p^{a-1}-1} \frac{1}{(pk_1)^{s_1}(pk_2)^{s_2}}$$

$$+ \sum_{k_1=1}^{p^{a-1}-1} \sum_{i=k_1}^{p^{a-1}-1} \sum_{k_2=1}^{p-1} \frac{1}{(pk_1)^{s_1}(pi+k_2)^{s_2}}$$

$$+ \sum_{k_2=1}^{p^{a-1}-1} \sum_{j=0}^{k_2-1} \sum_{k_1=1}^{p-1} \frac{1}{(pj+k_1)^{s_1}(pk_2)^{s_2}}, \qquad (4.5)$$

the highest power of p in the denominator comes from the term

$$\frac{1}{p^{(s_1+s_2)(a-1)}} \sum_{1 \le k_1 \le k_2 \le p-1} \frac{1}{k_1^{s_1} k_2^{s_2}},$$

which is a partial sum of the first term in (4.5). For the pairs (m, n) in the assumption, $p \nmid \sum_{1 \le k_1 \le k_2 \le p-1} \frac{1}{k_1^{s_1} k_2^{s_2}}$. Thus $\sum_{\substack{1 \le k_1 \le k_2 \le p^a-1 \\ p|k_1 \text{ or } p|k_2}} \frac{1}{k_1^{s_1} k_2^{s_2}}$ has $p^{(s_1+s_2)(a-1)}$ in the denominator and so $p^a - 1 \notin J(s_1, s_2 \mid p)$. □

REMARK. The above corollary seems true for other pairs (m, n) not mentioned in the assumption by computer calculation.

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