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On $\mathfrak{m}$-Full Powers of Parameter Ideals

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Abstract. Let Q be a parameter ideal in a Noetherian local ring A with the maximal ideal $\mathfrak{m}$. Then A is a regular local ring and $\mathfrak{m}/Q$ is cyclic, if depth A > 0 and Q^n is $\mathfrak{m}$-full for some integer $n \geq 1$. Consequently, A is a regular local ring and all the powers of Q are integrally closed in A once Q^n is integrally closed for some $n \geq 1$.

1. Introduction

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let I be an ideal in A. Then we say that I is $\mathfrak{m}$-full if $\mathfrak{m}I : x = I$ for some $x \in \mathfrak{m}$. The notion of $\mathfrak{m}$-full ideal was introduced by D. Rees and played since integrally closed ideals are $\mathfrak{m}$-full under a certain mild condition ([G2, Theorem (2.4)]), an important role in the analysis of integrally closed ideals (cf. [G2, GH1, GH2, GHK, HUV, MTV]).

The present purpose is to prove the following.

THEOREM 1.1. *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and let Q be a parameter ideal in A. Assume that depth $A > 0$ and Q^n is $\mathfrak{m}$-full for some integer $n \geq 1$. Then the local ring A is regular and $\mathfrak{m}/Q$ is cyclic.*

This theorem provides a new sight of $\mathfrak{m}$-full powers of parameter ideals and gives rise to a sufficiently simple proof of the following result, which has been known if A is excellent and n is sufficiently large ([MTV, Théorème 3]) or if depth $A > 0$ [HUV, Corollary 2.11]).

COROLLARY 1.2. *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and let Q be a parameter ideal in A. Then the following three conditions are equivalent to each other.*

(1) *A is a regular local ring and $\mathfrak{m}/Q$ is cyclic.*
(2) *Q is integrally closed in A.*
(3) *Q^n is integrally closed in A for some $n \geq 1$.*

When this is the case, the ideals Q^ℓ are integrally closed in A for all integers $\ell \geq 1$.

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In Corollary (1.2) our contribution is the implication (3) $\Rightarrow$ (2); the equivalence of conditions (1) and (2) is due to [G2, Theorem (3.1)] as well as the last assertion. Thus, as for the parameter ideals Q in a Noetherian local ring A, the integral closedness of *any* power of Q implies that of *all* the powers of Q and the regularity of A as well.

A global version of Corollary (1.2) is as follows. We suspect that the assumption $\mathrm{Ass}_A A/I = \mathrm{Min}_A A/I$ in condition (2) of Proposition (1.3) is superfluous.

PROPOSITION 1.3. *Let A be a Noetherian ring. Let I be an ideal in A and assume that $\mu_A(I) = \mathrm{ht}_A I$, where $\mu_A(I)$ and $\mathrm{ht}_A I$ denote the number of generators and the height of I, respectively. Then the following conditions are equivalent.*

(1) *I is integrally closed in A.*

(2) *$\mathrm{Ass}_A A/I = \mathrm{Min}_A A/I$ and I^n is integrally closed in A for some integer $n \geq 1$.*
When this is the case, I^ℓ is integrally closed in A for every integer $\ell \geq 1$.

The proof of Theorem (1.1) and Corollary (1.2) shall be given in Section 3. Section 2 is devoted to some preliminaries. In our proof of Theorem (1.1), some results on Ratliff-Rush closures, FLC rings (that is, generalized Cohen-Macaulay local rings), and $\mathfrak{m}$-full ideals will play key roles, which we will briefly summarize in Section 2.

In what follows, otherwise specified, let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let $\mu_A(*)$ and $\ell_A(*)$ denote the number of generators and the length, respectively. For each ideal I in A let $\mathrm{ht}_A(I)$ be the height of I. We denote by $\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{m}}(A)$ the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$. Let $\mathrm{H}^i_{\mathfrak{m}}(*)$ $(i \in \mathbf{Z})$ stand for the i $\underline{th}$ local cohomology functor of A with respect to $\mathfrak{m}$.

2. Preliminaries

Let A be a commutative Noetherian ring and let $\mathcal{F}_A$ denote the set of ideals in A which contain at least one nonzerodivisor in A. For each $I \in \mathcal{F}_A$ let

$$\tilde{I} = \bigcup_{n \geq 0} (I^{n+1} \; I^n)$$

be the Ratliff-Rush closure of I. Then $I \subseteq \tilde{I} \subseteq \bar{I}$ and $\tilde{I} = \tilde{\tilde{I}}$ (cf. [Mc, Lemma 8.2 (vi)]), where $\bar{I}$ denotes the integral closure of I.

PROPOSITION 2.1. (1) (Y. Shimoda) *Let $I \subseteq J$ be ideals in A and assume that $I^n = J^n$ for some integer $n \geq 1$. Then $I^\ell = J^\ell$ for all integers $\ell \geq n$.*

(2) *Let $I \in \mathcal{F}_A$ and assume that $I^n = \bar{I}^n$ for some $n \geq 1$. Then $\bar{I} = \tilde{I}$ and $I^\ell = \bar{I}^\ell$ for all integers $\ell \geq n$.*

PROOF. (1) Since $I^n \subseteq I^{n-1}J \subseteq J^{n-1}J = J^n$, we get $I^n = I^{n-1}J = J^n$. Therefore $I^{n+1} = II^n = I(I^{n-1}J) = I^nJ = J^nJ = J^{n+1}$. Thus $I^\ell = J^\ell$ for all $\ell \geq n$.

(2) We have $I^n = \bar{I}^n$ since $\bar{I}^n \subseteq \overline{I^n} = I^n$. Hence $\bar{I} \subseteq \tilde{I}$ by [Mc, Lemma 8.2 (iv)] so that $\bar{I} = \bar{I}$ by [Mc, Lemma 8.2 (vi)]. The latter equality follows from assertion (1). □

Now let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let $\mathrm{H}^i_{\mathfrak{m}}(*)$ ($i \in \mathbf{Z}$) be the local cohomology functors of A with respect to $\mathfrak{m}$. Then we say that A has FLC (or equivalently, A is a generalized Cohen-Macaulay local ring), if all the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(A)$ ($i \neq d$) are finitely generated.

For each ideal I in A we put

$$\mathcal{R}(I) = A[It] = \bigoplus_{n \geq 0} I^n ,$$

where t denotes an indeterminate. Let

$$\mathrm{G}(I) = \mathcal{R}(I)/I\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$$

be the associated graded ring of I.

Let $G = \mathrm{G}(\mathfrak{m})$ and $M = G_+$ the unique graded maximal ideal in G. Let $\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{m}}(A)$ denote the multiplicity of A. We then have the following.

PROPOSITION 2.2. (1) *Suppose that A has FLC. Then A is a regular local ring, if* $\mathrm{e}(A) = 1$ *and* depth $A > 0$.

(2) *The local ring A has FLC, if the local cohomology modules*

$$H^i_M(G) = \lim_{n \to \infty} \mathrm{Ext}^i_G(G/M^n, G)$$

of G with respect to M are finitely generated for all $i \neq d$.

PROOF. (1) The local ring A is unmixed since A has FLC and depth $A > 0$ (cf. [SV, Appendix, Proposition 16]). Hence A is regular by [N, Theorem 40.6].

(2) See [G1, Proposition (3.1)]). □

The notion of $\mathfrak{m}$-full ideal was introduced by D. Rees, who showed that every integrally closed ideal I is $\mathfrak{m}$-full, provided I is not nilpotent and the residue class field $A/\mathfrak{m}$ of A is infinite [G2, Theorem (2.4)]). The readers may consult [G2] about basic results on $\mathfrak{m}$-full ideals. Here let us note two of them, which we later need to prove Theorem (1.1).

PROPOSITION 2.3. *Let I be an ideal in A and assume that I is $\mathfrak{m}$-full. Then the following assertions hold true.*

(1) *Let J be an ideal in A. Assume $I \subseteq J$ and $\ell_A(J/I) < \infty$. Then $\mu_A(I) \geq \mu_A(J)$.*

(2) *Assume that A/I is an Artinian Gorenstein local ring. Then $\mathfrak{m}/I$ is cyclic.*

PROOF. (1) See [G2, Lemma (2.2) (2)].

(2) Let $x \in \mathfrak{m}$ such that $\mathfrak{m}I : x = I$. Then $I : \mathfrak{m} = (\mathfrak{m}I : x) : \mathfrak{m} = (\mathfrak{m}I : \mathfrak{m}) : x \supseteq I : x$ so that $I : \mathfrak{m} = I : x$. Thus we get the exact sequence

$$0 \to [I : \mathfrak{m}]/I \to A/I \xrightarrow{\hat{x}} A/I \to A/[I + (x)] \to 0 .$$

Hence $\ell_A(A/[I + (x)]) = \ell_A([I : \mathfrak{m}]/I) = 1$ because A/I is Gorenstein. Thus $\mathfrak{m} = I + (x)$, whence $\mathfrak{m}/I$ is cyclic. □

3. Proofs of Theorem 1.1 and Corollary 1.2

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim\ A$. Let Q be a parameter ideal in A.

PROOF OF THEOREM 1.1. Passing to the local ring $A[X]_{\mathfrak{m}A[X]}$ where X is an indeterminate over A, we may assume that the residue class field $k = A/\mathfrak{m}$ of A is infinite. Let $\mathfrak{q}$ be a minimal reduction of $\mathfrak{m}$. Hence $\mathfrak{q}$ is a parameter ideal in A and $\mathfrak{m}^{r+1} = \mathfrak{q}\mathfrak{m}^r$ for some $r \geq 0$ (such an ideal $\mathfrak{q}$ must exist because the field $k = A/\mathfrak{m}$ is infinite), and then $\mathcal{R}(\mathfrak{m})$ is a module-finite extension of $\mathcal{R}(\mathfrak{q})$. Let

$$\varphi : \mathcal{R}(\mathfrak{q})/\mathfrak{m}\mathcal{R}(\mathfrak{q}) \to \mathcal{R}(\mathfrak{m})/\mathfrak{m}\mathcal{R}(\mathfrak{m})$$

be the homomorphism of graded k-algebras induced from the inclusion $\mathcal{R}(\mathfrak{q}) \subseteq \mathcal{R}(\mathfrak{m})$. Hence the homomorphism φ is also finite. We put

$$P = \mathcal{R}(\mathfrak{q})/\mathfrak{m}\mathcal{R}(\mathfrak{q}) \quad \text{and} \quad G = \mathcal{R}(\mathfrak{m})/\mathfrak{m}\mathcal{R}(\mathfrak{m})\,.$$

For each integer $i \geq 0$ let $P_i = \mathfrak{q}^i/\mathfrak{m}\mathfrak{q}^i$ and $G_i = \mathfrak{m}^i/\mathfrak{m}^{i+1}$ denote the homogeneous components of P and G of degree i. Then because $\mathfrak{q}$ is a parameter ideal in A, the ring P is the polynomial ring with d variables over the field k. Therefore φ is a monomorphism since φ is finite and $\dim P = \dim G = d$.

We now look at the $\mathfrak{m}$-full ideal Q^n. Then

$$\mu_A(\mathfrak{m}^n) \leq \mu_A(Q^n) = \binom{d+n-1}{d-1} = \mu_A(\mathfrak{q}^n)$$

by Proposition (2.3) (1), so that the monomorphism φ induces an isomorphism

$$P_n = \mathfrak{q}^n/\mathfrak{m}\mathfrak{q}^n \to G_n = \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

of vector spaces over k. Hence $\mathfrak{m}^n = \mathfrak{q}^n + \mathfrak{m}^{n+1}$ and so $\mathfrak{m}^n = \mathfrak{q}^n$ by Nakayama's lemma. Thus $\mathfrak{m}^\ell = \mathfrak{q}^\ell$ for all integers $\ell \geq n$ by Proposition (2.1) (1). Hence the homomorphism $\varphi : P \to G$ induces an isomorphism between the vector spaces P_ℓ and G_ℓ over k and so

$$\ell_A(\mathfrak{m}^\ell/\mathfrak{m}^{\ell+1}) = \ell_A(\mathfrak{q}^\ell/\mathfrak{m}\mathfrak{q}^\ell) = \binom{d+\ell-1}{d-1}$$

for every $\ell \geq n$. Thus $\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{m}}(A) = 1$ by definition. Let $C = \operatorname{Coker}\varphi$. Then $\dim_k C < \infty$ since $C_\ell = (0)$ if $\ell \geq n$. Therefore because the ring P is the polynomial ring over k, thanks to the exact sequence

$$0 \to P \overset{\varphi}{\to} \to G \to C \to 0$$

of finitely generated graded P-modules, we get $\mathrm{H}^i_M(G) = (0)$ for all $i \neq 0, d$, where $M = G_+$. Hence by Proposition (2.2) (2) the local ring A has FLC, so that the local ring A is regular by Proposition (2.2) (1) because $\mathrm{e}(A) = 1$ and depth $A > 0$.

Since A/Q is an Artinian Gorenstein local ring, to see that $\mathfrak{m}/Q$ is cyclic, by Proposition (2.3) (2) it is enough to show that Q is $\mathfrak{m}$-full. Let $x \in \mathfrak{m}$ such that $\mathfrak{m}Q^n : x = Q^n$ and let $a \in \mathfrak{m}Q : x$. Then $x(aQ^{n-1}) = (xa)Q^{n-1} \subseteq \mathfrak{m}Q^n$ whence $aQ^{n-1} \subseteq \mathfrak{m}Q^n : x = Q^n$. Therefore $a \in Q^n : Q^{n-1} = Q$ because Q is generated by an A-regular sequence. Thus $\mathfrak{m}Q : x = Q$ and so Q is $\mathfrak{m}$-full. $\square$

We are in a position to prove Corollary (1.2). The last assertion and the equivalence of conditions (1) and (2) in Corollary (1.2) are due to [G2, Theorem (3.1)]. Let us include brief proofs of the last assertion and the implication (2) $\Rightarrow$ (1) for the sake of completeness.

PROOF OF COROLLARY 1.2. We may assume that $d = \dim A > 0$. Passing to the local ring $A[X]_{\mathfrak{m}A[X]}$ where X is an indeterminate over A, we may also assume that the residue class field $k = A/\mathfrak{m}$ of A is infinite.

(3) $\Rightarrow$ (2) We will show that A is a regular local ring and Q is integrally closed. Let $W = \mathrm{H}^0_{\mathfrak{m}}(A)$, $B = A/W$, and $\mathfrak{n} = \mathfrak{m}/W$. Then Q^nB is integrally closed in B because $W \subseteq \sqrt{(0)}$ and Q^n is integrally closed in A. Hence by Theorem (1.1) the local ring B is regular because $\operatorname{depth} B > 0$ and Q^nB is $\mathfrak{n}$-full. We must show that $W = (0)$. Since $W \subseteq \sqrt{(0)}$, we have $W \subseteq \overline{Q^n} = Q^n$. Let $\ell > 0$ be an integer and assume that $W \subseteq Q^\ell$. Let $Q = (a_1, a_2, \ldots, a_d)$ and choose $w \in W$. We write $w = \sum_{|\alpha|=\ell} c_\alpha \mathbf{a}^\alpha$ with $c_\alpha \in A$, where $\mathbf{a}^\alpha = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_d^{\alpha_d}$ and $|\alpha| = \sum_{i=1}^d \alpha_i$ for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ with $0 \le \alpha_i \in \mathbf{Z}$. Let $\bar{*}$ denote the image in B. Then

$$\sum_{|\alpha|=\ell} \overline{c_\alpha}\, \overline{a_1}^{\alpha_1} \overline{a_2}^{\alpha_2} \cdots \overline{a_d}^{\alpha_d} = \bar{w} = 0\,.$$

Since the system $\overline{a_1}, \overline{a_2}, \ldots, \overline{a_d}$ of parameters in B forms a regular sequence, we get $\overline{c_\alpha} \in QB$ for every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ with $|\alpha| = \ell$. Thus $c_\alpha \in Q + W$ so that $w \in (Q^{\ell+1} + WQ^\ell) \cap W = (Q^{\ell+1} \cap W) + WQ^\ell$. Consequently, $W = Q^{\ell+1} \cap W \subseteq Q^{\ell+1}$ by Nakayama's lemma. Hence $W \subseteq \bigcap_{\ell>0} Q^\ell = (0)$ and so the local ring A is regular.

Because Q^n is integrally closed, we have $\bar{Q} = \tilde{Q}$ by Proposition (2.1) (2). The ideal Q is generated by a regular sequence, whence

$$\tilde{Q} = \bigcup_{n \ge 0} (Q^{n+1} : Q^n) = Q\,,$$

so that we have $\bar{Q} = \tilde{Q} = Q$.

(2) $\Rightarrow$ (1) Thanks to the above proof, A is a regular local ring. Hence $\mathfrak{m}/Q$ is cyclic by Proposition (2.3) (2) because Q is $\mathfrak{m}$-full and A/Q is an Artinian Gorenstein local ring.

(1) $\Rightarrow$ the last assertion. We may assume that $d \ge 2$. Since $\mathfrak{m}/Q$ is cyclic by our assumption, we may choose a regular system $a_1, a_2, \ldots, a_d$ of parameters of A so that $Q = (a_1, \ldots, a_{d-1}, a_d^q)$ for some $q \ge 1$. Let

$$\mathcal{S} = \mathcal{R}(Q)[t^{-1}] = A[a_1t, \ldots, a_{d-1}t, a_d^q t, t^{-1}]$$

be the extended Rees algebra of Q (here t denotes an indeterminate over A). Let $u = t^{-1}$. Then $\mathrm{G}(Q) = \mathcal{S}/u\mathcal{S}$. We must show that $\mathcal{S}$ is an integrally closed integral domain. We firstly recall that the sequence $a_1, \dots, a_{d-1}, a_d^q$ is regular. Hence the associated graded ring $\mathrm{G}(Q)$ is the polynomial ring with d indeterminates over A/Q, that is

$$\mathrm{G}(Q) = (A/Q)[\overline{a_1 t}, \dots, \overline{a_{d-1} t}, \overline{a_d^q t}]$$

and the elements $\{\overline{a_i t}\}_{1 \le i \le d-1}$ and $\overline{a_d^q t}$ are algebraically independent over A/Q, where $\bar{*}$ denotes the image in $\mathrm{G}(Q)$. In particular, the ring $\mathrm{G}(Q)$ is Cohen-Macaulay. Hence the ring $\mathcal{S}$ is also Cohen-Macaulay because u is a nonzero divisor in $\mathcal{S}$.

Let P be a prime ideal in $\mathcal{S}$ with $\mathrm{ht}_{\mathcal{S}} P = 1$. We will show that the localization $\mathcal{S}_P$ of $\mathcal{S}$ is a discrete valuation ring. We may assume that $u \in P$ (because the ring $\mathcal{S}[u^{-1}] = A[t, t^{-1}]$ is regular). Then $P = (\mathfrak{m}, u)\mathcal{S} = (a_1, a_2, \dots, a_d, u)\mathcal{S}$, since $P/u\mathcal{S}$ is a unique minimal prime ideal in the polynomial ring $\mathrm{G}(Q)$. Hence $P = (a_d, u)\mathcal{S}$ because $a_i = a_i t \cdot u$ for all $1 \le i \le d-1$. Therefore $P\mathcal{S}_P = a_d \mathcal{S}_P$ because $a_d^q t \notin P = (\mathfrak{m}, u)\mathcal{S}$ and $u = a_d^q/(a_d^q t)$. Hence $\mathcal{S}_P$ is a discrete valuation ring with the regular parameter a_d. Thus the Cohen-Macaulay ring $\mathcal{S}$ satisfies Serre's condition (R_1), so that $\mathcal{S}$ is an integrally closed integral domain. Hence Q^n is integrally closed in A for every integer $n \ge 1$, which completes the proof of Corollary (1.2). □

Before closing this paper let us note a brief proof of Proposition (1.3). We suspect the assumption that $\mathrm{Ass}_A A/I = \mathrm{Min}_A A/I$ in condition (2) is superfluous.

PROOF OF PROPOSITION 1.3. (1) ⇒ (2) and the last assertion. This is due to [G2, Theorem (1.1)].

(2) ⇒ (1) Assume that $I \ne \bar{I}$ and choose $P \in \mathrm{Ass}_A \bar{I}/I$. Then $P \in \mathrm{Ass}_A A/I = \mathrm{Min}_A A/I$. Hence the ideal IA_P is a parameter ideal in the local ring A_P because $\mathrm{ht}_A I = \mu_A(I)$. Since

$$(IA_P)^n = I^n A_P = \overline{I^n} A_P = \overline{I^n A_P} = \overline{(IA_P)^n}\,,$$

by Corollary (1.2) the local ring A_P is regular and $IA_P = \overline{IA_P} = \bar{I} A_P$. This is impos sible. □

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