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Shape Derivative of Energy Functional in an Infinite Elastic Strip with a Semi-Infinite Crack

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Abstract. In this paper we study linear elasticity equations in an infinite elastic strip with a semi-infinite crack. We find the derivative of the energy functional as the crack shifts with an angle. Then we obtain the formula given by surface force and the angle.

1. Introduction

Analysis of fracture phenomena has been a major subject of solid mechanics from the mathematical viewpoint since Griffith's work [6]. The important parameter in fracture mechanics "energy release rate" (first in [6]) is the variation of energy with respect to the crack growth in a straight line, which depends only on the coefficients of leading terms of the asymptotic solution of fields near the moving crack tip. There are many pieces of research on the variation of energy by the tangential perturbation of cracks as in [2, 4, 5, 11, 13]. Non-tangential perturbation becomes very interested in shape sensitivity analysis of cracks. Because, in shape sensitivity analysis, results say that shape functional depends only on the normal component of the perturbation (see e.g. Theorem 2.27 (P. 59) in [14] by Sokolowski and Zolésio), however in fracture, the importance is in shape sensitivity in tangential direction. The singularity at the crack tip is the leading cause of the difference. In [8] Khludnev, Ohtsuka and Sokolowski studied the shape derivative under unilateral boundary conditions considered at the crack faces, which implies non-penetration between the crack face. As a general perturbation of the crack is given, they derived the derivative of the energy functional with respect to the perturbation parameter. In this paper we consider linear elasticity equations in an infinite elastic strip with a semi-infinite crack under the free traction condition on the crack. Then, we show here that the derivative of the energy functional changes in arbitrary direction, and derive the formula given by surface force and the angle.

In section 2 we introduce our result for stationary problem. This problem leads to a singular integral equation by the potential theory (see for example, [1], [9]). By proving

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the compactness of the singular integral operator and using the results of [7] and [10], the existence of a unique solution is proved by the Fredholm alternative.

In section 3, using the result of [3], we consider a boundary value problem for shifted crack with an angle from the initial crack. And we can solve the problem by virtue of the method in [7].

In section 4, following [11] and [12], we derive the derivative of energy functional with respect to sliding parameter represented by the surface force and the angle.

2. Preliminaries

By $u = (u_i)_{i=1,2,3}$, $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ and $\sigma = (\sigma_{ij})_{i,j=1,2,3}$ we denote the displacement vector, the strain tensor and the stress tensor, respectively. The linear elasticity equations for a homogeneous isotropic material consist of the constitutive law (Hooke's law)

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}\,, \quad i, j = 1, 2, 3 \tag{1}$$

and the equilibrium conditions without any body forces

$$\frac{\partial}{\partial x_j}\sigma_{ij} = 0\,, \quad i, j = 1, 2, 3\,. \tag{2}$$

Here and in what follows we use the summation convention. λ and μ are *Lamé* constants, δ_{ij} is the Kronecker's delta. The strain-displacement relation is given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\,, \quad u_{i,j} = \partial_j u_i\,, \quad i, j = 1, 2, 3\,. \tag{3}$$

In the state of a plane strain, third component u_3 of the displacement u is zero, while the components u_1 and u_2 are functions of x_1 and x_2 only, hence $\varepsilon_{i3} = 0$, $\sigma_{13} = \sigma_{23} = 0$. Let $\Omega = \{(x_1, x_2) \mid x_1 \in \mathbf{R}, -a < x_2 < a\}$ $(a > 0)$ be a strip in $\mathbf{R}^2$, representing a homogeneous elastic plate. Then (2) gives the system of equations

$$A\,(\partial_x)\,u = 0 \tag{4}$$

for $u = (u_1, u_2)^{\mathrm{T}}$, where $A(\partial_x) = A\big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\big)$,

$$A(\xi_1, \xi_2) = \begin{pmatrix} \mu\xi^2 + (\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2 \\ (\lambda+\mu)\xi_1\xi_2 & \mu\xi^2 + (\lambda+\mu)\xi_2^2 \end{pmatrix}, \quad \xi^2 = \xi_1^2 + \xi_2^2\,.$$

We assume the shearing strain $\mu > 0$ and the modulus of compression $3\lambda + 2\mu \geq 0$, in which case it is easy to see that the operator A is elliptic. Moreover we introduce the boundary stress operator $T(\partial_x) = T\big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\big)$ defined by

$$T(F\xi_1, \xi_2) = \begin{pmatrix} (\lambda+2\mu)\nu_1\xi_1 + \mu\nu_2\xi_2 & \mu\nu_2\xi_1 + \lambda\nu_1\xi_2 \\ \lambda\nu_2\xi_1 + \mu\nu_1\xi_2 & \mu\nu_1\xi_1 + (\lambda+2\mu)\nu_2\xi_2 \end{pmatrix},$$

where $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$ is the unit outward normal to $\partial\Omega$. In the case of $\nu = (0, 1)^{\mathrm{T}}$

$$T(\xi_1, \xi_2) = \begin{pmatrix} \mu\xi_2 & \mu\xi_1 \\ \lambda\xi_1 & (\lambda + 2\mu)\xi_2 \end{pmatrix}.$$

We denote by $\Gamma = \{(x_1, 0) \mid -\infty < x_1 \leq 0\}$ the crack in Ω. On the crack we assume the free traction condition

$$\sigma_{ij}^{+}\nu_j = \sigma_{ij}^{-}\nu_j = 0 \quad \text{on} \quad \Gamma^{\pm}, \tag{5}$$

where $\Gamma^{\pm}$ mean both sides of Γ. Here for every $x \in \Gamma$, $\sigma_{ij}^{\pm}(x) = \lim_{\eta\to+0} \sigma_{ij}(x \pm \eta\nu_x)$ with the normal unit vector ν_x, in this case $\nu_x = (0, 1)$ or $\nu_x = (0, -1)$. The limit values σ_{ij}^{+} and σ_{ij}^{-} may be different in general, therefore σ_{ij} may have a jump on Γ. On $\partial\Omega_+ = \{(x_1, a) \mid x_1 \in \mathbf{R}\}$, $\partial\Omega_- = \{(x_1, -a) \mid x_1 \in \mathbf{R}\}$ $(a > 0)$ the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega_-, \tag{6}$$

$$\sigma_{ij}\nu_j = p_i \quad \text{on} \quad \partial\Omega_+ \tag{7}$$

are imposed, where p_i are given continuous functions on $\partial\Omega_+$.

We introduce the class $\mathcal{K}$ of functions $u(x)$ with the properties (cf. [9]):

1) $u \in C^0(\overline{\Omega \setminus \Gamma}) \cap C^2(\Omega \setminus \Gamma)$,

2) $\nabla u \in C^0(\overline{\Omega \setminus \Gamma} \setminus \{(0, 0)\})$,

3) in the neighborhood of $(0, 0)$ there exist constants $C > 0$ and $\varepsilon > -1$ such that

$$|\nabla u(x)| \leq C|x|^{\varepsilon} \quad \text{as} \quad x \to 0, \tag{8}$$

4) for every $x \in \partial\Omega_{\pm}$ there exists a uniform limit of $(\nu_x, \nabla_{\bar{x}} u(\bar{x}))$ as $\bar{x} \in \Omega \setminus \Gamma$ tends to $x \in \partial\Omega_{\pm}$ along the normal $-\nu_x$.

Furthermore, we introduce the class $\wp = \{u \mid u \to 0 \text{ as } |x| \to \infty\}$.

Next, it is well known that the fundamental matrix of $A(\partial_x)$ is given explicitly by

$$D(x, y) = -\frac{1}{4\pi\mu(\tilde{\mu} + 1)} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{9}$$

$$D_{11} = 2\tilde{\mu} \ln|x - y| + 2\tilde{\mu} - 1 + 2\frac{(x_2 - y_2)^2}{|x - y|^2},$$

$$D_{12} = D_{21} = -2\frac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2},$$

$$D_{22} = 2\tilde{\mu} \ln|x - y| + 2\tilde{\mu} - 1 + 2\frac{(x_1 - y_1)^2}{|x - y|^2},$$

$$\tilde{\mu} = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

In view of (9), $D(x, y) = D(y, x) = D(y, x)^{\mathrm{T}}$.

Along with $D(x, y)$ we consider the matrix of singular solutions

$$P(x, y) = (T(\partial_y) D(y, x))^{\mathrm{T}},$$

which is written explicitly as

$$P(x, y) = -\frac{1}{2\pi}\left(\frac{\partial}{\partial \nu_y} \ln|x-y| I + \frac{\tilde{\mu}-1}{\tilde{\mu}+1}\frac{\partial}{\partial \tau_y} \ln|x-y|\tilde{I} + \frac{2}{\tilde{\mu}+1}\tilde{I}\frac{\partial}{\partial \tau_y}\frac{(x-y)^{\mathrm{T}}(x-y)}{|x-y|^2}\right) \tag{10}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tilde{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tau = (\tau_1, \tau_2)^{\mathrm{T}}$ a unit tangential vector to $\partial\Omega_\pm \cup \Gamma$.

Now we denote by $\tilde{D}$ and $\tilde{P}$ the reflection of $D(x, y)$ and $P(x, y)$ with respect to $\partial\Omega_- = \{(x_1, -a) \mid x_1 \in \mathbf{R}\}$, respectively:

$$\tilde{D}(x, y) = D\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) - D\left(\begin{pmatrix} x_1 \\ -2a - x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right), \tag{11}$$

$$\tilde{P}(x, y) = P\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) - P\left(\begin{pmatrix} x_1 \\ -2a - x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right). \tag{12}$$

Then it is obvious that the columns of $\tilde{D}(x, y)$ and $\tilde{P}(x, y)$ vanish on $\partial\Omega_-$.

Using the potential theory, we will find a solution of problem (4)–(7) in the form

$$u(x_1, x_2) = \tilde{V}_{\partial\Omega_+}(g) + \tilde{V}_\Gamma(f) + \tilde{W}_\Gamma(g), \tag{13}$$

where

$$\tilde{V}_{\partial\Omega_+}(g) = \int_{\partial\Omega_+} \tilde{D}(x, y) g(y) \mathrm{d}y_1,$$

$$\tilde{V}_\Gamma(f) = \int_\Gamma \tilde{D}(x, y) f(y) \mathrm{d}y_1,$$

$$\tilde{W}_\Gamma(g) = \int_\Gamma \tilde{P}(x, y) g(y) \mathrm{d}y_1.$$

Now let us introduce function spaces. By $C^{0,\alpha}(G)$ we denote a Hölder space with exponent $\alpha \in (0, 1)$ of functions defined on a domain G and by $C^{1,\beta}(G)$ the subspace of functions of C^1-class whose first order derivatives belong to $C^{0,\beta}(G)$, $\beta \in (0, 1)$. If $(f, g) \in C^{0,\alpha}(\Gamma) \times (C^{0,\alpha}(\partial\Omega_+) \cap C^{1,\beta}(\Gamma))$, then it is easily seen that u defined by (13) is continuous on $\partial\Omega_+ \cup \Gamma^\pm$ and satisfies (4) and (6). Substituting (13) into (5) and (7), we deduce the integral equations for g (cf. [1], [7]). Then, we can derive the equations

$$f\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \Gamma,$$

$$\left(Z+\frac{1}{2}I\right)g=p \quad \text{on} \quad \partial\Omega_+ \tag{14}$$

with $Z=T(\tilde{V}_{\partial\Omega_+}+\tilde{W}_\Gamma)$.

Now we introduce the new space $C^{0,\alpha}_\gamma(G)$ defined by

$$C^{0,\alpha}_\gamma(G)=\{f(x)\in C^{0,\alpha}(G)\mid f(x)=O(|\,x\,|^{-\gamma})\text{ as }|x|\to\infty\}\quad(1<\gamma)$$

equipped with the norm

$$\|g\|_{\gamma,\alpha}=\|g\|_{\gamma,\infty}+|g|_\alpha\,,$$

$$\|g\|_{\gamma,\infty}=\sup_{x\in G}|\,(1+|x|^\gamma)g(x)\,|\,,\quad |g|_\alpha=\sup_{x,\tilde{x}\in G,x\neq\tilde{x}}\frac{|g(x)-g(\tilde{x})|}{|x-\tilde{x}|^\alpha}\,.$$

We say that g belong to $C^{0,\alpha}_{\gamma,w}(\Gamma)$ if $\chi(x)g(x)\in C^{0,\alpha}(\Gamma)$, where

$$\chi(x)=\begin{cases}|x|^\gamma & \text{if } x_1\geq -1\,,\\ |x|^w & \text{if } x_1<-1\,,\end{cases}$$

$w\in[0,1)$. And its norm is defined by

$$\|g\|_{\gamma,w,\alpha}=\|\,\chi(x)g(x)\,\|_{0,\alpha}\,.$$

As shown in [9], [7], for $(f,g)\in C^{0,\alpha}_{\gamma,w}(\Gamma)\times(C^{0,\alpha}_\gamma(\partial\Omega_+)\cap C^{1,\beta}_{\gamma,w}(\Gamma))$ u in the form (13) belongs to class $\mathcal{K}$. In particular, the inequality (8) holds with $\varepsilon=-w$ if $w\in(0,1)$. Inverting the singular integral operator, we arrive at the integral equation of second kind (cf. [10])

$$(I-Y_1)\left(\frac{\partial}{\partial x_1}g(x)\right)=\frac{1}{\pi^2R(x)}\int_{-R}^0\frac{R(y)\,\mathrm{d}y_1}{y-x}\int_{\partial\Omega_+}T\tilde{D}(y,z)g(z)\mathrm{d}z_1$$
$$\text{as}\quad R\to\infty\,,\quad x\in\Gamma\,, \tag{15}$$

where the integral $Y_1(f)$ on Γ is in the sense of principal value,

$$Y_1(f(x))=\frac{1}{\pi^2R(x)}\int_{-R}^0\frac{R(y)\,\mathrm{d}y_1}{y-x}\int_\Gamma\left(\frac{\partial}{\partial\tau_z}\tilde{Q}(z,y)-\frac{1}{z-y}\right)f(z)\mathrm{d}z_1\,,$$

$$R(x)=\sqrt{(x+R)x}$$

and

$$T\tilde{P}=-\frac{\partial^2}{\partial\tau_x\partial\tau_y}\tilde{Q}$$

with

$$Q(x,y)=-\frac{2\mu}{\pi(\tilde{\mu}+1)}\left(\ln|x-y|I-I+\frac{(x-y)^{\mathrm{T}}(x-y)}{|x-y|^2}\right),$$

$$\tilde{Q}(x, y) = Q\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) - Q\left(\begin{pmatrix} x_1 \\ -2a - x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right).$$

In (14) Z can be represented as $Zg = K_1 g + K_2 g$, where

$$\begin{aligned} K_1 g &= \text{v.p.} \int_{\partial\Omega_+} \frac{1}{x_1 - y_1} g\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 , \\ K_2 g &= (Z - K_1) g . \end{aligned} \tag{16}$$

Then K_1 has a 1-singular kernel and K_2 is a non-singular operator. Applying the operator $(K_1 - \frac{1}{2}I)$ to both sides of (14), we have

$$\left(K_1 K_2 - \frac{1}{2} K_2 - \left(\frac{1}{4} + \pi^2\right) I\right) g = \left(K_1 - \frac{1}{2} I\right) p . \tag{17}$$

Similarly, the operator Y_1 can be decomposed into

$$Y_1 = Y_{11} + Y_{10} ,$$

where Y_{11} has a 1-singular kernel and Y_{10} is a non-singular operator.

In [7] we remarked the following.

REMARK 1. Problem (4)–(7) has a unique solution $u \in \mathcal{K} \cap \wp$ for any $p \in C_{\gamma}^{0,\alpha}(\partial\Omega_+)$ with any $\alpha \in (0, 1)$ and any $\gamma > 1$.

Moreover, we require stronger regularity of g.

REMARK 2. If $p \in C_{\gamma}^{1,\alpha}(\partial\Omega_+)$, then $g \in C_{\gamma}^{1,\alpha}(\partial\Omega_+) \cap C_{\gamma,w+1}^{2,\beta}(\Gamma)$ with $w \in (0, 1)$.

This Remark can be proved in a similar way as in the proof of Theorem 2 in [7].

3. The problem for perturbed crack

In this section we consider a boundary value problem for shifted crack with an angle from the initial crack.

Now let us consider the shifted crack Γ_ε from the initial crack Γ

$$\Gamma_\varepsilon = \{x_\varepsilon \mid x_\varepsilon = x_0 + \varepsilon X, x_0 \in \Gamma\} \tag{18}$$

with $X = (\cos\theta_0, \ \sin\theta_0)$ and the sliding parameter $\varepsilon > 0$. This means that Γ_ε shifts ε with an angle θ_0 from Γ. Then we deduce the boundary value problem with respect to the displacement u_ε

$$(*)\begin{cases} Au_\varepsilon = 0 \quad \text{in} \quad \Omega \setminus \Gamma_\varepsilon\,, \\ Tu_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon^\pm\,, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_-\,, \\ Tu_\varepsilon = p \quad \text{on} \quad \partial\Omega_+\,, \end{cases}$$

where $\Gamma_\varepsilon^\pm$ mean both sides of Γ_ε. We seek a solution u_ε of problem $(*)$ in the form

$$u_\varepsilon = u + \varepsilon\hat{u}\,, \tag{19}$$

where u is a solution of problem (4)–(7). Differentiation of Tu_ε on $\Gamma_\varepsilon^\pm$ with respect to ε yields

$$0 = T\left(\frac{\partial u_\varepsilon}{\partial\varepsilon} + \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial}{\partial\varepsilon}\varepsilon\cos\theta_0 + \frac{\partial u_\varepsilon}{\partial x_2}\frac{\partial}{\partial\varepsilon}\varepsilon\sin\theta_0\right)\Bigg|_{\Gamma_\varepsilon^\pm}.$$

Letting $\varepsilon \to 0$, we get

$$T\left(\hat{u} + \frac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right)\Bigg|_{\Gamma^\pm} = 0\,.$$

In view of (4)–(7), (19) and $(*)$ we obtain the boundary value problem of $\hat{u}$:

$$(**)\begin{cases} A\hat{u} = 0 \quad \text{in} \quad \Omega \setminus \Gamma\,, \\ T\hat{u} = -T\left(\dfrac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right) \quad \text{on} \quad \Gamma^\pm\,, \\ \hat{u} = 0 \quad \text{on} \quad \partial\Omega_-\,, \\ T\hat{u} = 0 \quad \text{on} \quad \partial\Omega_+\,. \end{cases}$$

Similarly for u we can apply the potential theory to problem $(**)$, so that the solution of $(**)$ is described in the form

$$\hat{u}(x_1, x_2) = \tilde{V}_{\partial\Omega_+}(h_1) + \tilde{V}_\Gamma(h_2) + \tilde{W}_\Gamma(h_1)\,, \tag{20}$$

where $(h_2, h_1) \in C^{0,\alpha}_{\gamma,w+1}(\Gamma) \times (C^{0,\alpha}_\gamma(\partial\Omega_+) \cap C^{1,\beta}_{\gamma,w}(\Gamma))$, $\gamma > 1$, $w \in (0, 1)$, have the similar properties as (f, g). In order for $\hat{u}$ in (20) to satisfy the boundary condition in $(**)$ we substitute (20) into $(**)$ and derive the integral equations on $\partial\Omega_+$ and Γ.

It is easily obtained

$$\frac{1}{2}h_1\begin{pmatrix} x_1 \\ a \end{pmatrix} + \text{v.p.}\int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ a \end{pmatrix}\mathrm{d}y_1$$
$$+ \int_\Gamma T\tilde{D}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\mathrm{d}y_1$$

$$+\int_{\Gamma} T\tilde{P}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_1 \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \partial\Omega_+ . \tag{21}$$

It yields

$$\begin{aligned}
&\pm\frac{1}{2}h_2\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1 \begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2 \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&\quad - \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_1 \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \Bigg|_{y_1=-\infty}^{0} \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1} h_1 \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&= -T\left(\frac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right) \quad \text{on} \quad \Gamma^\pm .
\end{aligned} \tag{22}$$

Note that

$$\begin{aligned}
\frac{\partial^2}{\partial x_2 \partial\tau_x}\ln|x-y| &= \frac{\partial^2}{\partial x_1 \partial\nu_x}\ln|x-y| , \\
\frac{\partial^2}{\partial x_2 \partial\nu_x}\ln|x-y| &= -\frac{\partial^2}{\partial x_1 \partial\tau_x}\ln|x-y| .
\end{aligned} \tag{23}$$

Then using integration by parts and Remark 2, we can rewrite (22) to

$$\begin{aligned}
&\pm\frac{1}{2}h_2\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1 \begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2 \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1} h_1 \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&= -\Bigg\{\Bigg(\int_{\partial\Omega_+} \frac{\partial}{\partial x_1} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g \begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial^2}{\partial y_1^2} g \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1\Bigg)\cos\theta_0 \\
&\quad + \Bigg(\int_{\partial\Omega_+} \frac{\partial}{\partial x_2} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g \begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \pm \frac{1}{2}\frac{\partial^2}{\partial x_1^2} g(x) \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial\nu_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial^2}{\partial y_1^2} g \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1\Bigg)\sin\theta_0\Bigg\} ,
\end{aligned} \tag{24}$$

since h_1 vanishes at the crack tip. Subtracting and adding two equations in (24) yield

$$h_2(x) = -\frac{\partial^2}{\partial x_1^2} g(x)\,, \tag{25}$$

$$\begin{aligned}
&\int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.} \int_\Gamma T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.} \int_\Gamma \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1} h_1\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&= -\Bigg\{\Bigg(\int_{\partial\Omega_+} \frac{\partial}{\partial x_1} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.} \int_\Gamma \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial^2}{\partial y_1^2} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1\Bigg) \cos\theta_0 \\
&\quad + \Bigg(\int_{\partial\Omega_+} \frac{\partial}{\partial x_2} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.} \int_\Gamma \frac{\partial}{\partial\nu_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial^2}{\partial y_1^2} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1\Bigg) \sin\theta_0\Bigg\}\,.
\end{aligned} \tag{26}$$

Substituting (25) into (26) leads to the similar formula as (15)

$$\begin{aligned}
(I - Y_1)\frac{\partial}{\partial x_1} h_1(x) &= \frac{1}{\pi^2 R(x)} \int_{-R}^{0} \frac{R(y)}{y - x} \Bigg\{ T\tilde{V}_{\partial\Omega_+} h_1 - T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g \\
&\quad + \cos\theta_0 \left(\frac{\partial}{\partial x_1} T\tilde{V}_{\partial\Omega_+} g + \frac{\partial}{\partial\tau_x} Y_2 g\right) \\
&\quad + \sin\theta_0 \left(\frac{\partial}{\partial x_2} T\tilde{V}_{\partial\Omega_+} g + \frac{\partial}{\partial\nu_x} Y_2 g\right)\Bigg\} \mathrm{d}y_1 \\
&\quad \text{as} \quad R \to \infty\,, \quad x \in \Gamma,
\end{aligned} \tag{27}$$

where

$$Y_2(f) = \text{v.p.} \int_\Gamma \tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial^2}{\partial y_1^2} f \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1\,.$$

Applying Remark 1 and Remark 2 for problem ($**$), we can get a unique solution $\hat{u}$.

4. Shape derivative of energy

In this section we calculate shape derivative of energy functional.

Let Π be the potential energy functional defined by

$$\Pi(u) = \int_{\Omega\setminus\Gamma} E(u,u)\,\mathrm{d}x - \int_{\partial\Omega_\pm} s\cdot u\,\mathrm{d}x_1\,, \tag{28}$$

$E(u,u)$ is the internal energy density

$$E(u,u) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\{\lambda(u_{1,1}+u_{2,2})^2 + 2\mu(u_{1,1}^2+u_{2,2}^2) + \mu(u_{1,2}+u_{2,1})^2\}\,,$$

$$s = (s_i) = (\sigma_{ij}\nu_j) = Tu\,.$$

One can easily verify if $u \in C^2(\Omega\setminus\Gamma)\cap C^1(\overline{\Omega\setminus\Gamma})\cap\wp$ is a solution of (4) in $\Omega\setminus\Gamma$, then

$$2\int_{\Omega\setminus\Gamma} E(u,u)\mathrm{d}x = \int_{\partial\Omega_\pm} u^{\mathrm{T}}Tu\mathrm{d}x_1 + 2\int_{\Gamma} u^{\mathrm{T}}Tu\,\mathrm{d}x_1\,. \tag{29}$$

Indeed, Divergence Theorem and (4) yield that for any $u \in C^2(\Omega\setminus\Gamma)\cap C^1(\overline{\Omega\setminus\Gamma})\cap\wp$

$$0 = \int_{\Omega\setminus\Gamma} u^{\mathrm{T}}Au\,\mathrm{d}x = -2\int_{\Omega\setminus\Gamma} E(u,u)\mathrm{d}x + \int_{\partial\Omega_\pm} u^{\mathrm{T}}Tu\,\mathrm{d}x_1 + 2\int_{\Gamma} u^{\mathrm{T}}Tu\,\mathrm{d}x_1\,.$$

Thus, if u is a solution of problem $(*)$, then $\Pi(u)$ vanishes except on $\partial\Omega_+$. Then from (19), (28) $\Pi(u_\varepsilon)$ is written by

$$\Pi(u_\varepsilon) = -\frac{1}{2}\int_{\partial\Omega_+} p^{\mathrm{T}}\cdot u_\varepsilon\,\mathrm{d}x_1 = \Pi(u) + \varepsilon\Pi(\hat{u})\,. \tag{30}$$

Our purpose is to find the derivative of $\Pi(u_\varepsilon)$ with respect to the sliding parameter ε, namely, from (30)

$$\left.\frac{\mathrm{d}\Pi(u_\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \lim_{\varepsilon\to 0}\frac{\Pi(u_\varepsilon)-\Pi(u)}{\varepsilon} = \Pi(\hat{u})\,. \tag{31}$$

From (15) it implies that

$$\frac{\partial}{\partial x_1}g(x) = Y_3(T\tilde{V}_{\partial\Omega_+}g) \quad \text{on} \quad \Gamma\,, \tag{32}$$

where

$$Y_3 g = \lim_{R\to\infty}((1+\pi^2)I - Y_{11}Y_{10} - Y_{10})^{-1}$$
$$\left\{\left(I + Y_{11}\right)\left(\lim_{R\to\infty}\frac{1}{\pi^2 R(z)}\int_{-R}^{0}\frac{R(y)g}{y-z}\,\mathrm{d}y_1\right)\right\}.$$

Substituting (32) into (17) yields that

$$g(x) = \left(K_1K_2 - \frac{1}{2}K_2 - (\frac{1}{4}+\pi^2)I\right)^{-1}\left\{\left(K_1 - \frac{1}{2}I\right)p\right\} \quad \text{on} \quad \partial\Omega_+\,. \tag{33}$$

Similarly, h_1 is described by g and θ_0. Indeed, from (27) it follows that

$$\frac{\partial}{\partial x_1} h_1(x) = Y_3\Big(T\tilde{V}_{\partial\Omega_+} h_1 - T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g\Big) + A_1 \cos\theta_0 + B_1 \sin\theta_0 \quad \text{on} \quad \Gamma\,, \tag{34}$$

where A_1, B_1 are functions defined by

$$A_1 = Y_3 \left(\frac{\partial}{\partial x_1} T\tilde{V}_{\partial\Omega_+} g + \frac{\partial}{\partial \tau_x} Y_2 g \right) ,$$

$$B_1 = Y_3 \left(\frac{\partial}{\partial x_2} T\tilde{V}_{\partial\Omega_+} g + \frac{\partial}{\partial \nu_x} Y_2 g \right) .$$

Substituting (25), (34) into (21), we have

$$h_1(x) = C + A_2 \cos\theta_0 + B_2 \sin\theta_0 \quad \text{on} \quad \partial\Omega_+\,, \tag{35}$$

where

$$C = \left(K_1 K_2 - \frac{1}{2} K_2 - \left(\frac{1}{4} + \pi^2\right) I \right)^{-1}$$
$$\left\{ \left(K_1 - \frac{1}{2} I \right) \left(I + \int_\Gamma \frac{\partial}{\partial \tau_x} \tilde{Q}(x,y) Y_3 \right) \left(T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g \right) \right\} ,$$

$$A_2 = \left(K_1 K_2 - \frac{1}{2} K_2 - \left(\frac{1}{4} + \pi^2\right) I \right)^{-1}$$
$$\left\{ \left(K_1 - \frac{1}{2} I \right) \left(\int_\Gamma \frac{\partial}{\partial \tau_x} \tilde{Q}(x,y) \right) (-A_1) \right\} ,$$

$$B_2 = \left(K_1 K_2 - \frac{1}{2} K_2 - \left(\frac{1}{4} + \pi^2\right) I \right)^{-1}$$
$$\left\{ \left(K_1 - \frac{1}{2} I \right) \left(\int_\Gamma \frac{\partial}{\partial \tau_x} \tilde{Q}(x,y) \right) (-B_1) \right\} .$$

Since A_i, B_i and C are functions depending on g, h_i depends only on surface force p for $i = 1, 2$. Hence, substituting (25), (34), (35) into (20), we have

$$\hat{u} = \tilde{V}_{\partial\Omega_+} (C + A_2 \cos\theta_0 + B_2 \sin\theta_0) + \tilde{V}_\Gamma \left(- \frac{\partial^2}{\partial x_1^2} g \right)$$
$$+ \tilde{V}^*_\Gamma \Big(Y_3 \Big(T\tilde{V}_{\partial\Omega_+} (C + A_2 \cos\theta_0 + B_2 \sin\theta_0) - T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g \Big)$$
$$+ A_1 \cos\theta_0 + B_1 \sin\theta_0 \Big) ,$$

since (23) leads to

$$\tilde{W}_\Gamma = \frac{\partial}{\partial x_1}\tilde{V}_\Gamma^* .$$

Thus, from (30) $\Pi(\hat{u})$ is written as

$$-2\Pi(\hat{u}) = D + A_3\cos\theta_0 + B_3\sin\theta_0\,, \tag{36}$$

where

$$\begin{aligned} D = \int_{\partial\Omega_+} p^{\mathrm{T}}\cdot\biggl(&\tilde{V}_{\partial\Omega_+}C + \tilde{V}_\Gamma\biggl(-\frac{\partial^2}{\partial x_1^2}g\biggr) \\ &+\tilde{V}_\Gamma^*\biggl(Y_3\biggl(T\tilde{V}_{\partial\Omega_+}C - T\tilde{V}_\Gamma\frac{\partial^2}{\partial x_1^2}g\biggr)\biggr)\biggr)\mathrm{d}x_1\,, \\ A_3 = \int_{\partial\Omega_+} p^{\mathrm{T}}\cdot&(\tilde{V}_{\partial\Omega_+}A_2 + \tilde{V}_\Gamma^*(Y_3(T\tilde{V}_{\partial\Omega_+}A_2) + A_1))\mathrm{d}x_1\,, \\ B_3 = \int_{\partial\Omega_+} p^{\mathrm{T}}\cdot&(\tilde{V}_{\partial\Omega_+}B_2 + \tilde{V}_\Gamma^*(Y_3(T\tilde{V}_{\partial\Omega_+}B_2) + B_1))\mathrm{d}x_1\,. \end{aligned}$$

(31) is equivalent to

$$\left.\frac{\mathrm{d}\Pi(u_\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = -\frac{1}{2}(D + A_3\cos\theta_0 + B_3\sin\theta_0)\,. \tag{37}$$

Hence, summing up the above

THEOREM 1. *Suppose a homogeneous elastic body Ω with a crack Γ is loaded a surface force p. Then, as the crack shifts with an angle θ_0 the derivative of the energy functional with respect to the sliding parameter ε is given by the formula* (37).

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