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Groups Defined by Extended Affine Lie Algebras with Nullity 2

Dedicated to Professor Takeo Yokonuma on the occasion of his 65th birthday

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Abstract. After certain completions, we define adjoint groups of extended affine Lie algebras with nullity 2. Then we show that such groups have Tits systems with affine Weyl groups (Part I). This idea allows us to consider linear groups over some completed quantum tori. By the same argument, we can prove that these linear groups also have Tits systems with affine Weyl groups. Using this fact we will study their universal central extensions as well as associated K_1-groups and K_2-groups (Part II). We will discuss some relationship among our groups constructed here (Part III).

1. Introduction

The classification theory of finite dimensional complex semisimple Lie groups was completed by W. Killing and E. Cartan in the early 20th. It was a very important observation that the classification of finite dimensional complex semisimple Lie groups can be reduced to classify finite dimensional complex semisimple Lie algebras, which is equivalent to classify (finite) reduced root systems. Then, finally such root systems can be described completely in terms of Dynkin diagrams or Cartan matrices (cf. [4]).

Around 1967, a new idea was born. That is, V. Kac and R. Moody independently discovered that there exists a natural and very important generalization of the above theory. They gave the definition of generalized Cartan matrices, and constructed the associated Lie algebras (cf. [8], [10]), and they developed the so-called Kac-Moody (Lie algebra) theory. In general, Kac-Moody Lie algebras can be infinite dimensional. Then, the corresponding groups and root systems were studied systematically (cf. [6], [8], [10], [17], [14]).

In this Kac-Moody theory, there is the most important class called affine Lie algebras and associated groups. Here we sometimes include loop algebras and loop groups as a rough

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explanation. Then many applications are found to other areas in mathematics as well as in mathematical physics. In 1990, two physicists named R. Høegh-Krohn and B. Toresani introduced a new generalization of affine Lie algebras. Such new Lie algebras were originally called quasi-simple Lie algebras. Later, five mathematicians (AABGP) arranged and developed the theory of those new Lie algebras (cf. [2]). Now they are called extended affine Lie algebras. However, the corresponding groups have not yet considered systematically in group theory.

Each extended affine Lie algebra has its nullity. For example, affine Lie algebras have nullity 1. Roughly saying, the nullity is the rank of the lattice generated by imaginary roots or isotropic roots. Then, it is a very important open problem to study the groups corresponding to extended affine Lie algebras with nullity ≥ 2. Again roughly speaking, such Lie algebras have the structure similar to loop algebras with many variables. Then, the corresponding groups look like loop groups with many variables. Those groups are generally rather difficult to study in some sense.

On the other hand, linear groups over several fields and rings have been studied (cf. [1], [5], [6], [7], [9], [12], [13], [15], [21]). Then, these groups can sometimes be considered using the associated K-theory. Also, it is very natural to consider linear groups over quantum tori, since linear Lie algebras over quantum tori can appear as some homomorphic images of extended affine Lie algebras, more precisely their derived subalgebras. The idea of our approach here comes from this point of view.

In this paper, we assume that the nullity is 2, and we need some kind of completion. Extended affine Lie algebras with nullity 2 are theoretically corresponding to elliptic Lie algebras, and our completion is essentially corresponding to K. Saito's marking for his extended affine root systems (cf. [19]). Here, we make a completion of the corresponding groups, and we study them. In fact, we will obtain Bruhat type decompositions, and using such decompositions we will establish group presentations and universal central extensions as well as the structures of the corresponding K_1-groups and K_2-groups.

One may expect, in the next step, some structure theory without completions. Also, one may want to study the higher nullity case. At this moment, it seems to be rather difficult. To do so, we might need another new idea. It is our dream in a future.

We will discuss the groups corresponding to extended affine Lie algebras in Part I. Then, we will study some linear groups over quantum tori and their universal central extensions in Part II. Finally, in Part III we will deal with the relation between the results in Part I and Part II.

Now we can draw the following picture in terms of Lie algebras.

		Kac-Moody Lie Algebras		
	$\nearrow$			
Affine Lie Algebras Loop Algebras	$\rightarrow$	Linear Lie Algebras over Coordinate Rings	$\supset$	Linear Lie Algebras over Quantum Tori with 2 Variables
	$\searrow$			
		Extended Affine Lie Algebras	$\supset$	Some EALA with Nullity 2

Here, we will study the group version of this picture. That is, we will study some groups defined by extended affine Lie algebras with nullity 2 as well as some linear groups over quantum tori, and discuss the relationship among them.

In the above picture, one can find three kinds of topics as further development after affine Lie algebras or loop algebras: namely one is the theory of Kac-Moody Lie algebras, another is the theory of linear Lie algebras over rings, and the third is the theory of extended affine Lie algebras. All of them are very interesting and important in Lie theory. There might be a hidden new observation.

Part I: EALA Groups

We make a completion of some extended affine Lie algebra with nullity 2, and construct and study the associated adjoint group. We suppose here that a field F is of characteristic 0.

1. Extended affine Lie algebras

Let L be an extended affine Lie algebra over a field, F, of characteristic 0, which is studied in [2], [17], [18], for example, and called an EALA sometimes. That is:

(EALA1) L has a nondegenerate symmetric invariant bilinear form

$$\mathfrak{b} : L \times L \longrightarrow F ,$$

(EALA2) L has a finite dimensional toral subalgebra H, consisting of diagonalizable elements under the adjoint representation, with $H \neq 0$ and $C_L(H) = H$,

(EALA3) $\mathrm{ad}_L(x)$ is locally nilpotent for all $x \in L_\alpha$ with $\alpha \in R^\times$,

where $R^\times$ is the set of nonisotropic roots of the root system R defined by (L, H), and where L_α is the root subspace of L corresponding to α,

(EALA4) L is irreducible (in terms of the root system R).

We let $\mathbf{Z}$ denote the ring of rational integers, and $\mathbf{Q}$ the field of rational numbers. The multiplicative group of F is denoted by $F^\times$, namely $F^\times = F \setminus \{0\}$. The definition of extended affine Lie algebras gives a natural generalization of affine Lie algebras (cf. [8], [10]). For each $\alpha \in R^\times$, let $\sigma_\alpha \in GL(H^*)$ be the reflection defined by $\sigma_\alpha(\mu) = \mu - \mu(h_\alpha)\alpha$ for all $\mu \in H^*$, where h_α is the coroot of $\alpha \in R^\times$ defined by $h_\alpha = 2t_\alpha/\mathfrak{b}(t_\alpha, t_\alpha)$ with $t_\alpha \in H$ satisfying $\mathfrak{b}(h, t_\alpha) = \alpha(h)$ for all $h \in H$. Then the Weyl group W is defined to be the subgroup of

$GL(H^*)$ generated by σ_α for all $\alpha \in R^\times$, that is, $W = W(R) = \langle\, \sigma_\alpha \mid \alpha \in R^\times \,\rangle$. Let V be the **Q**-span of R, and V_0 the **Q**-span of R^0, where $R^0 = R \setminus R^\times$ is the set of isotropic roots of R. Then the famous Kac-conjecture says that the induced bilinear form on V is positive semi-definite after a certain neccessary scalar modification (cf. [2], [17]) and V_0 is its radical. Furthermore, the image $\bar{R}^\times$ of $R^\times$ in $\bar{V} = V/V_0$ is a (not neccessarily reduced) finite root system (cf. [2], [4]). We choose a complete set, Φ, of representatives of $\bar{R}^\times$ in $R^\times$. In this note, we suppose the following three conditions (ASS1) – (ASS3).

Assumption.

(ASS1) $R^0 = \mathbf{Z}\xi \oplus \mathbf{Z}\eta$ for some nonzero $\xi, \eta \in R^0$.

(ASS2) $R^\times = \Phi + (\mathbf{Z}\xi \oplus \mathbf{Z}\eta)$.

(ASS3) Φ is a reduced (irreducible finite) root system.

2. Completed adjoint groups

We shall consider, as a formal infinite sum,

$$\sum_{i=k}^{\infty} s_{\alpha+i\eta}\,,$$

where $\alpha \in R$ and $s_{\alpha+i\eta} \in L_{\alpha+i\eta}$. We put $\Phi_a = \Phi + \mathbf{Z}\xi$. For $a = \dot{\alpha} + m\xi \in \Phi_a$ with $\dot{\alpha} \in \Phi$ and $m \in \mathbf{Z}$, we put $\Gamma_a = \{\sum_{i=k}^{\infty} s_{a+i\eta} \mid k \in \mathbf{Z},\ s_{a+i\eta} \in L_{a+i\eta}\}$, and for $m \in \mathbf{Z}$ we put $\Gamma_{m\xi} = \{\sum_{i=k}^{\infty} s_{m\xi+i\eta} \mid k \in \mathbf{Z},\ s_{m\xi+i\eta} \in L_{m\xi+i\eta}\}$. Then we define

$$\hat{L} = \Big(\bigoplus_{a\in\Phi_a} \Gamma_a\Big) \oplus \Big(\bigoplus_{m\in\mathbf{Z}} \Gamma_{m\xi}\Big),$$

which naturally becomes a Lie algebra and is called the completion of L along with η. Also we put $\Gamma_0' = \{\sum_{i=1}^{\infty} s_{i\eta} \mid s_{i\eta} \in L_{i\eta}\}$. For each $s \in \Gamma_a$ we define

$$x_a(s) = \exp \operatorname{ad}(s) \in \operatorname{Aut}(\hat{L})\,,$$

and for each $s' \in \Gamma_0'$ we define

$$x_0(s') = \exp \operatorname{ad}(s') \in \operatorname{Aut}(\hat{L})\,.$$

Then we can construct $\mathfrak{G}$ to be the subgroup of $\operatorname{Aut}(\hat{L})$ generated by $x_a(s)$ and $x_0(s')$ for all $a \in \Phi_a$ and for all $s \in \Gamma_a$ and $s' \in \Gamma_0'$. One may symbolically call $\mathfrak{G}$ an EALG or an EALA group, otherwise one may call it the completed adjoint group defined by L. Let W_a $(= W(\Phi_a) = W_a(\Phi))$ be the subgroup of $W(R)$ generated by σ_a for all $a \in \Phi_a$, and then W_a is called the affine Weyl group of Φ or the Weyl group of Φ_a (cf. [4], [17]).

3. Some relations

We fix a simple system Π of Φ, and we let denote by Φ^+ (resp. Φ^-) the set of positive (resp. negative) roots of Φ with respect to Π. Then we put $\Phi_a^+ = (\Phi^+ + \mathbf{Z}_{\geq 0}\xi) \cup (\Phi^- + \mathbf{Z}_{>0}\xi)$ and $\Phi_a^- = (\Phi^+ + \mathbf{Z}_{<0}\xi) \cup (\Phi^- + \mathbf{Z}_{\leq 0}\xi)$, and we set $\Pi_a = \{\dot{\alpha},\ -\dot{\alpha}_0 + \xi \mid \dot{\alpha} \in \Pi\}$, where $\dot{\alpha}_0$ is the highest root of Φ relative to Π. For $a = \dot{\alpha} + m\xi \in \Phi_a$, we define $\mathfrak{U}_a = \langle\, x_a(s) \mid s \in \Gamma_a \rangle$. For $\{\pm\alpha\} \subset R^\times$, we choose elements $e_\alpha \in L_\alpha$ and $e_{-\alpha} \in L_{-\alpha}$ such that $\{e_\alpha, h_\alpha, e_{-\alpha}\}$ is an sl_2-triplet with $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $[h_\alpha, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}$. If $\alpha \in R^\times$, then $L_\alpha = Fe_\alpha$, since $\dim L_\alpha = 1$. For $\alpha \in R^\times$ and for $r \in F$ and $t \in F^\times$, we put

$$
\begin{aligned}
x_\alpha(r) &= \exp \mathrm{ad}(re_\alpha)\,,\\
w_\alpha(t) &= x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)\,,\\
h_\alpha(t) &= w_\alpha(t)w_\alpha(-1)\,.
\end{aligned}
$$

Then we have the following standard relations (cf. [2], [10], [21]):

(R1) $w_\alpha(t)x_\beta(r)w_\alpha(-t) = x_{\sigma_\alpha\beta}(\eta_{\alpha\beta}t^{-\beta(h_\alpha)}r)$,

(R2) $w_\alpha(t)w_\beta(u)w_\alpha(-t) = w_{\sigma_\alpha\beta}(\eta_{\alpha\beta}t^{-\beta(h_\alpha)}u)$,

(R3) $w_\alpha(t)h_\beta(u)w_\alpha(-t) = h_{\sigma_\alpha\beta}(u)$,

(R4) $h_\alpha(t)x_\beta(r)h_\alpha(t)^{-1} = x_\beta(t^{\beta(h_\alpha)}r)$,

(R5) $h_\alpha(t)w_\beta(u)h_\alpha(t)^{-1} = w_\beta(t^{\beta(h_\alpha)}u)$,

(R6) $h_\alpha(t)h_\beta(u)h_\alpha(t)^{-1} = h_\beta(u)$,

(R7) $w_\alpha(t)x_a(s)w_\alpha(-t) = x_{\sigma_\alpha a}(v)$,

(R8) $w_\alpha(t)x_0(s')w_\alpha(-t) = x_0(v')$

for all $\alpha, \beta \in R^\times, r \in F, t, u \in F^\times, a \in \Phi_a, s \in \Gamma_a$ and $s' \in \Gamma_0'$. Here $w_\alpha(1)(e_\beta) = \eta_{\alpha\beta}e_{\sigma_\alpha\beta}$ with $\eta_{\alpha\beta} \in F^\times$, and $w_\alpha(t)(s) = v \in \Gamma_{\sigma_\alpha a}$ and $w_\alpha(t)(s') = v' \in \Gamma_0'$.

4. Subgroups and Tits systems

For $a \in \Phi_a$, $i \in \mathbf{Z}$ and $t \in F^\times$, we set

$$\theta_{a,i}(t) = w_{a+i\eta}(t)w_a(-1)\,.$$

Then, we define

$$
\begin{aligned}
\mathfrak{U}^\pm &= \langle\, \mathfrak{U}_a \mid a \in \Phi_a^\pm \rangle\,,\\
\mathfrak{T}_0 &= \langle\, h_\alpha(t),\ \theta_{a,i}(t),\ x_0(s') \mid \\
&\qquad \alpha \in R^\times,\ t \in F^\times, \quad a \in \Phi_a,\ i \in \mathbf{Z},\ s' \in \Gamma_0' \rangle\,,\\
\mathfrak{B}^\pm &= \langle\, \mathfrak{T}_0, \mathfrak{U}^\pm \rangle\,,\\
\mathfrak{N} &= \langle\, w_a(t),\ \mathfrak{T}_0 \mid a \in \Phi_a,\ t \in F^\times \rangle\\
&= \langle\, w_\alpha(t),\ x_0(s') \mid \alpha \in R^\times,\ t \in F^\times,\ s' \in \Gamma_0' \rangle\,,
\end{aligned}
$$

$$\mathfrak{S} = \{w_a(1) \bmod \mathfrak{T}_0 \mid a \in \Pi_a\}\,.$$

Sometimes we identify $\mathfrak{S}$ with $\{w_a(1) \mid a \in \Pi_a\}$. Also we put

$$\mathfrak{Y}_{\pm a} = \langle\, x_{\pm a}(r)\mathfrak{U}_b x_{\pm a}(-r) \mid b \in \Phi_a^{\pm} \setminus \{\pm a\},\ r \in F\rangle$$

for each $a \in \Pi_a$. Then we obtain the following (cf. [2], [7], [21]).

(Q1) $w_a(t)\mathfrak{U}_b w_a(-t) = \mathfrak{U}_{\sigma_a(b)}$ for all $a, b \in \Phi_a$ and $t \in F^{\times}$:
this follows from the standard relations among the $x_\alpha(r)$, $w_\alpha(t)$ and $h_\alpha(t)$.

(Q2) $\mathfrak{T}_0$ normalizes $\mathfrak{U}_a$ and $\mathfrak{T}_0\mathfrak{U}_a = \mathfrak{U}_a\mathfrak{T}_0$ for all $a \in \Phi_a$:
by the definitions of $\mathfrak{U}_a$ and $\mathfrak{T}_0$ together with (Q1), this can be obtained.

(Q3) $\mathfrak{B}^{\pm} = \mathfrak{U}^{\pm}\mathfrak{T}_0 = \mathfrak{T}_0\mathfrak{U}^{\pm}$, $\mathfrak{U}^{\pm} \lhd \mathfrak{B}^{\pm}$:
by (Q2), we find that $\mathfrak{U}^{\pm}$ is normalized by $\mathfrak{T}_0$, which implies that $\mathfrak{U}^{\pm}$ is a normal subgroup of $\mathfrak{B}^{\pm}$, hence $\mathfrak{B}^{\pm}$ is a product of both $\mathfrak{U}^{\pm}$ and $\mathfrak{T}_0$.

(Q4) $\mathfrak{B}^{\pm} \cap \mathfrak{N} = \mathfrak{T}_0 \lhd \mathfrak{N}$ and $\mathfrak{N}/\mathfrak{T}_0 \simeq W_a$:
by the standard relations among the $x_\alpha(r)$, $w_\alpha(t)$ and $h_\alpha(t)$, we have that $\mathfrak{T}_0$ is normal in $\mathfrak{N}$. Considering the action of $\mathfrak{N}$ on the set $\Omega = \{\Gamma_a \mid a \in \Phi_a\}$, there is a natural homomorphism of $\mathfrak{N}$ onto W_a, and $\mathfrak{T}_0$ acts on Ω trivially. Hence, it induces a homomorphism of $\mathfrak{N}/\mathfrak{T}_0$ onto W_a. On the other hand, the fact that W_a is a Coxeter group implies that $\mathfrak{N}/\mathfrak{T}_0$ is isomorphic to W_a. By the definitions of $\mathfrak{B}^{\pm}$ and $\mathfrak{N}$, we see $\mathfrak{B}^{\pm} \cap \mathfrak{N} \supset \mathfrak{T}_0$. If $x \in \mathfrak{B}^{\pm} \cap \mathfrak{N}$, then x must stabilize $\Omega^{\pm} = \{\Gamma_a \mid a \in \Phi_a^{\pm}\}$ and $\bar{x} \in \mathfrak{N}/\mathfrak{T}_0$ is corresponding to $1 \in W_a$. This means that $x \in \mathfrak{T}_0$, which shows that $\mathfrak{B}^{\pm} \cap \mathfrak{N}$ coincides with $\mathfrak{T}_0$.

(Q5) $\mathfrak{Y}_{\pm a} \lhd \mathfrak{U}^{\pm}$ and $\mathfrak{U}^{\pm} = \mathfrak{Y}_{\pm a}\mathfrak{U}_{\pm a} = \mathfrak{U}_{\pm a}\mathfrak{Y}_{\pm a}$ for all $a \in \Phi_a$:
we see that $\mathfrak{Y}_{\pm a}$ is a normal subgroup of $\mathfrak{U}$ by the definition of $\mathfrak{Y}_{\pm a}$, which leads to the fact that $\mathfrak{U}^{\pm}$ is a product of $\mathfrak{Y}_{\pm a}$ and $\mathfrak{U}_{\pm a}$.

(Q6) $w_{\pm a}(t)\mathfrak{Y}_{\pm a} w_{\pm a}(-t) = \mathfrak{Y}_{\pm a}$ for all $a \in \Pi_a$ and $t \in F^{\times}$:
this can be established (cf. [1], [6]).

Hence, using the standard argument, we can show the following theorem.

THEOREM 1. *Notation is as above. Then,* $(\mathfrak{G}, \mathfrak{B}^{\pm}, \mathfrak{N}, \mathfrak{S})$ *is a Tits system with the corresponding affine Weyl group* W_a.

PROOF OF THEOREM 1. We will show our result in case of $\mathfrak{B}$. For $\mathfrak{B}^{-}$, the proof can similarly be given. It is easy to show that $\mathfrak{G}$ is generated by $\mathfrak{B}$ and $\mathfrak{N}$, and the above (Q4) says that $\mathfrak{T}_0$ is a normal subgroup of $\mathfrak{N}$ with $\mathfrak{B} \cap \mathfrak{N} = \mathfrak{T}_0$ and $\mathfrak{N}/\mathfrak{T}_0$ is isomorphic to the affine Weyl group, W_a, of Φ (cf. [4], [17], [12]). Also by (Q3) one can obtain that $\mathfrak{B}$ is a product of $\mathfrak{U}$ and $\mathfrak{T}_0$ satisfying that $\mathfrak{U}$ is a normal subgroup of $\mathfrak{B}$. The main part of the proof should be to prove:

$$w_a(1)\mathfrak{B}w_a(-1) \subset \mathfrak{B} \cup \mathfrak{B}w_a(1)\mathfrak{B}$$

for all $a \in \Pi_a$. Since $w_a(1)\mathfrak{B}w_a(-1) \subset \mathfrak{U}_{-a}\mathfrak{B}$, which can be established as in the standard method using (Q1) – (Q6), as in [1], [6], [7], [21], it is enough to show

$$\mathfrak{U}_{-a} \subset \mathfrak{B} \cup \mathfrak{B}w_a(1)\mathfrak{B}\,.$$

Let $x \in \mathfrak{U}_{-a}$. If $x = 1$, then x is in $\mathfrak{B}$. We suppose $x \neq 1$. Then, we can write $x = x_{-a}(s)$ for some $s = \sum_{i=k}^{\infty} s_{-a+i\eta} \in \Gamma_{-a}$ with $s_{-a+k\eta} = te_{-a+k\eta} \neq 0$ for some $t \in F^{\times}$. Then there is a suitable element $s' \in \Gamma_0'$ such that

$$x_0(s')(s_{-a+k\eta}) = x_0(s')(te_{-a+k\eta}) = s\,.$$

Hence, we obtain

$$\begin{aligned}
x &= x_{-a}(s)\\
&= x_0(s')x_{-a+k\eta}(t)x_0(-s')\\
&= x_0(s')x_{a-k\eta}(t^{-1})w_{a-k\eta}(-t^{-1})x_{a-k\eta}(t^{-1})x_0(-s')\\
&= x_0(s')x_{a-k\eta}(t^{-1})(w_{a-k\eta}(-t^{-1})w_a(-1))w_a(1)x_{a-k\eta}(t^{-1})x_0(-s')\\
&= x_0(s')x_{a-k\eta}(t^{-1})\theta_{a,-k}(-t^{-1})w_a(1)x_{a-k\eta}(t^{-1})x_0(-s')\\
&\in \mathfrak{T}_0\mathfrak{U}_a\mathfrak{T}_0 w_a(1)\mathfrak{U}_a\mathfrak{T}_0\\
&\subset \mathfrak{B}w_a(1)\mathfrak{B}\,.
\end{aligned}$$

Other remaining part of the proof can also be established easily. Q.E.D.

COROLLARY. *Notation is as above. Then, we have*:

(1) $\mathfrak{G} = \bigcup_{w \in W_a} \mathfrak{B}^{\pm} w \mathfrak{B}^{\pm}$ (*Bruhat decomposition*),
(2) $\mathfrak{G} = \bigcup_{w \in W_a} \mathfrak{B}^{\pm} w \mathfrak{B}^{\mp}$ (*Birkhaff decomposition*),
(3) $\mathfrak{G} = \mathfrak{U}^{\pm}\mathfrak{U}^{\mp}\mathfrak{T}_0\mathfrak{U}^{\pm}$ (*Gauss decomposition*).

Part II: Groups over Quantum Tori

As a typical example, we can take a certain central extension of $\mathfrak{sl}_n$ over a quantum torus with some derivation part. Our idea in Part I can be applied to this example, which allows us to consider, in a similar way, a linear group defined by $\mathfrak{sl}_n$ over a quantum torus. According to the story of Part I, one may expect to discuss several subgroups of PGL_n. However here we will choose and study GL_n instead of PGL_n. One can argue about PGL_n in the same way as in case of GL_n. Here we suppose that F is a field of any characteristic.

5. Completed quantum tori

Let F be a field (of any characteristic). We fix an element q of $F^{\times}$. Let $K = F((X_1))$ be the field of the formal power series in X_1 over F, that is,

$$K = \left\{ \sum_{j=m}^{\infty} a_j X_1^j \,\middle|\, m \in \mathbf{Z},\ a_j \in F \right\},$$

and let $K_q = K[X_2, X_2^{-1}]$ be the (not necessarily commutative) ring of Laurent polynomials in X_2 over K with $X_2X_1 = qX_1X_2$, that is,

$$K_q = \left\{ \sum_{i=k}^{\ell} a_i(X_1)X_2^i \,\middle|\, k, \ell \in \mathbf{Z},\ k \le \ell,\ a_i(X_1) \in K \right\}.$$

We call K_q the completed quantum torus associated with $q \in F^\times$. If $a(X_1) \in K$ and $i \in \mathbf{Z}$, then we have $X_2^i\, a(X_1) = a(q^i X_1)\, X_2^i$. In general, we obtain

$$\begin{aligned}\left(\sum_{i=k_1}^{\ell_1} a_i(X_1)X_2^i\right)\left(\sum_{j=k_2}^{\ell_2} b_j(X_1)X_2^j\right) &= \sum_{i=k_1}^{\ell_1}\sum_{j=k_2}^{\ell_2} a_i(X_1)b_j(q^iX_1)X_2^{i+j} \\ &= \sum_{m=k_1+k_2}^{\ell_1+\ell_2}\left(\sum_{i=k_1}^{m-k_2} a_i(X_1)b_{m-i}(q^iX_1)\right)X_2^m .\end{aligned}$$

Using the spread of degrees in X_2, we find that K_q is a Euclidean ring and that K_q has no (nonzero) zero-divisor.

6. General linear groups and Tits systems

Let $M(n, K_q)$ be the ring of $n \times n$ matrices whose entries are in K_q, and we set $GL(n, K_q) = M(n, K_q)^\times$, the multiplicative group of $M(n, K_q)$.

Let $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}$ be a root system of type A_{n-1} in the sense of [4], where the ε_i are an orthonormal basis of a certain Euclidean space with an inner product $(\cdot,\cdot)$, and let $\Pi = \{\dot\alpha_1, \ldots, \dot\alpha_{n-1}\}$ be a simple system of Φ, where $\dot\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. We put $\Phi^+ = \{\dot\alpha_i + \dot\alpha_{i+1} + \cdots + \dot\alpha_j \mid 1 \le i \le j \le n-1\}$, the set of positive roots, and $\Phi^- = -\Phi^+$, the set of negative roots, and hence $\Phi = \Phi^+ \cup \Phi^-$. Then $\alpha_0 = \dot\alpha_1 + \dot\alpha_2 + \cdots + \dot\alpha_{n-1}$ is the highest root of Φ with respect to Π. The associated abstruct affine (real) root system is defined by $\Phi_a = \Phi \times \mathbf{Z}$. As simple roots, we choose $a_1 = (\alpha_1, 0), a_2 = (\alpha_2, 0), \ldots, a_{n-1} = (\alpha_{n-1}, 0), a_n = (-\alpha_0, 1)$, that is, $\Pi_a = \{a_1, a_2, \ldots, a_n\}$ is a simple system of Φ_a. Let $\Phi_a^+ = (\Phi^+ \times \mathbf{Z}_{\ge 0}) \cup (\Phi^- \times \mathbf{Z}_{>0})$ and $\Phi_a^- = (\Phi^+ \times \mathbf{Z}_{<0}) \cup (\Phi^- \times \mathbf{Z}_{\le 0})$, which are called positive roots and negative roots of Φ_a respectively. For each $\dot\alpha \in \Phi$, we define

$$e_{\dot\alpha} = \begin{cases} E_{i,j+1} & \text{if } \dot\alpha = \dot\alpha_i + \dot\alpha_{i+1} + \cdots + \dot\alpha_j\,, \\ E_{j+1,i} & \text{if } \dot\alpha = -(\dot\alpha_i + \dot\alpha_{i+1} + \cdots + \dot\alpha_j)\,, \end{cases}$$

where E_{ij} is the matrix unit with 1 in the (i, j) position and 0 elsewhere. For $\dot\alpha = \varepsilon_i - \varepsilon_j \in \Phi$ and $f \in K_q$, we put

$$x_{\dot\alpha}(f) = x_{ij}(f) = I + fe_{\dot\alpha}\,,$$

where $I = E_{11} + E_{22} + \cdots + E_{nn}$ is the identity matrix. Then the elementary subgroup $E(n, K_q)$ is defined to be the subgroup of $GL(n, K_q)$ generated by $x_{\dot{\alpha}}(f)$ for all $\dot{\alpha} \in \Phi$ and $f \in K_q$.

In a standard way, the Weyl group $\dot{W}$ of Φ is generated by $\sigma_{\dot{\alpha}}$ for all $\dot{\alpha} \in \Phi$, where $\sigma_{\dot{\alpha}}$ is the reflection along with $\dot{\alpha}$. Then the associated affine Weyl group W_a is generated by σ_a for all $a = (\dot{\alpha}, m) \in \Phi_a$, where

$$\sigma_a(b) = \left(\sigma_{\dot{\alpha}}\dot{\beta}, \ell - \frac{2(\dot{\alpha}, \dot{\beta})}{(\dot{\alpha}, \dot{\alpha})}m\right)$$

for $a = (\dot{\alpha}, m),\ b = (\dot{\beta}, \ell) \in \Phi_a$. We call W_a $(= W_a(\Phi) = W(\Phi_a))$ the affine Weyl group of Φ or the Weyl group of Φ_a. Usually Φ is identified with $\Phi \times \{0\}$ in Φ_a.

For $a = (\dot{\alpha}, m) \in \Phi_a$, $r \in K$ and $t \in K^\times$, we define

$$\begin{aligned} x_a(r) &= x_{\dot{\alpha}}(rX_2^m)\,, \\ w_a(t) &= x_a(t)x_{-a}(-t^{-1})x_a(t)\,, \\ h_a(t) &= w_a(t)w_a(-1)\,. \end{aligned}$$

Then we put

$$\begin{aligned} E &= E(n, K_q)\,, \\ U_a &= \langle x_a(r) \mid r \in K\rangle\,, \\ U^\pm &= \langle U_a \mid a \in \Phi_a^\pm\rangle\,, \\ T &= \langle h_a(t) \mid a \in \Phi_a,\ t \in K^\times\rangle\,, \\ N &= \langle w_a(t) \mid a \in \Phi_a,\ t \in K^\times\rangle\,. \end{aligned}$$

If $h \in T$ is expressed as

$$h = \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_n \end{pmatrix}$$

with $u_, \ldots, u_n \in K_q^\times$, then we define, for each $i = 1, \ldots, n$,

$$\deg_i(h) = \deg(u_i) = m_i\,,$$

where $u_i = t_i X_2^{m_i}$ with $t_i \in K^\times$ and $m_i \in \mathbf{Z}$, Then we set:

$$\begin{aligned} T_0 &= \langle h \mid h \in T,\ \deg_i(h) = 0 \text{ for all } i = 1, \ldots, n\rangle\,, \\ B^\pm &= \langle U^\pm,\ T_0\rangle\,, \\ S &= \{w_a(1) \bmod T_0 \mid a \in \Pi_a\}\,. \end{aligned}$$

Sometimes we identify S with $\{w_a(1) \mid a \in \Pi_a\}$. As one can imagine, we will establish the following result.

THEOREM 2. *Notation is as above. Then,* $(E, B^{\pm}, N, S)$ *is a Tits system with the corresponding affine Weyl group* W_a.

The proof is essentially the same as in [12]. The only difference is that our ring K_q is not necessarily commutative. Hence, we sometimes need special relations in our group E in the noncommutative case. For example,

(A) $x_{\dot\alpha}(f)\, x_{\dot\alpha}(g) = x_{\dot\alpha}(f+g)\,,$

(B) $$[x_{\dot\alpha}(f), x_{\dot\beta}(g)] = \begin{cases} x_{\dot\alpha+\dot\beta}(fg) & \text{if } \dot\alpha+\dot\beta \in \Phi,\ j=k\,, \\ x_{\dot\alpha+\dot\beta}(-gf) & \text{if } \dot\alpha+\dot\beta \in \Phi,\ i=\ell\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

where $f, g \in K_q$ and $\dot\alpha, \dot\beta \in \Phi$ satisfying that $\dot\alpha = \varepsilon_i - \varepsilon_j$, $\dot\beta = \varepsilon_k - \varepsilon_\ell$ and $\dot\alpha + \dot\beta \neq 0$. Furthermore, we put

$$w_{\dot\alpha}(u) = x_{\dot\alpha}(u) x_{-\dot\alpha}(-u^{-1}) x_{\dot\alpha}(u)\,,$$
$$h_{\dot\alpha}(u) = w_{\dot\alpha}(u) w_{\dot\alpha}(-1)$$

for each $u \in K_q^{\times}$. Then, we obtain:

(P1) $w_{\dot\alpha}(u)^{-1} = w_{\dot\alpha}(-u),\ w_{\dot\alpha}(u) = w_{-\dot\alpha}(-u^{-1})\,,$

(P2) $$\begin{cases} w_{\dot\alpha}(u) x_{\pm\dot\alpha}(f) w_{\dot\alpha}(-u) = x_{\mp\dot\alpha}(-u^{\mp1} f u^{\mp1})\,, & \\ w_{\dot\alpha}(u) x_{\dot\beta}(f) w_{\dot\alpha}(-u) = x_{\dot\beta}(f) & \text{if } (\dot\alpha, \dot\beta) = 0\,, \\ w_{\dot\alpha}(u) x_{\dot\beta}(f) w_{\dot\alpha}(-u) = x_{\sigma_{\dot\alpha}\dot\beta}(uf) & \text{if } \dot\alpha \pm \dot\beta \neq 0\,, \quad j=k\,, \\ w_{\dot\alpha}(u) x_{\dot\beta}(f) w_{\dot\alpha}(-u) = x_{\sigma_{\dot\alpha}\dot\beta}(-fu) & \text{if } \dot\alpha \pm \dot\beta \neq 0\,, \quad i=\ell\,, \\ w_{\dot\alpha}(u) x_{\dot\beta}(f) w_{\dot\alpha}(-u) = x_{\sigma_{\dot\alpha}\dot\beta}(-u^{-1}f) & \text{if } \dot\alpha \pm \dot\beta \neq 0\,, \quad i=k\,, \\ w_{\dot\alpha}(u) x_{\dot\beta}(f) w_{\dot\alpha}(-u) = x_{\sigma_{\dot\alpha}\dot\beta}(fu^{-1}) & \text{if } \dot\alpha \pm \dot\beta \neq 0\,, \quad j=\ell\,, \end{cases}$$

$$
(\mathrm{P3})\quad \begin{cases}
w_{\dot\alpha}(u)w_{\pm\dot\alpha}(v)w_{\dot\alpha}(-u) = w_{\mp\dot\alpha}(-u^{\mp1}vu^{\mp1})\,,\\
w_{\dot\alpha}(u)w_{\dot\beta}(v)w_{\dot\alpha}(-u) = w_{\dot\beta}(v) \\
\qquad \text{if } (\dot\alpha,\dot\beta)=0\,,\\
w_{\dot\alpha}(u)w_{\dot\beta}(v)w_{\dot\alpha}(-u) = w_{\sigma_{\dot\alpha}\dot\beta}(uv) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad j=k\,,\\
w_{\dot\alpha}(u)w_{\dot\beta}(v)w_{\dot\alpha}(-u) = w_{\sigma_{\dot\alpha}\dot\beta}(-vu) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad i=\ell\,,\\
w_{\dot\alpha}(u)w_{\dot\beta}(v)w_{\dot\alpha}(-u) = w_{\sigma_{\dot\alpha}\dot\beta}(-u^{-1}v) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad i=k\,,\\
w_{\dot\alpha}(u)w_{\dot\beta}(v)w_{\dot\alpha}(-u) = w_{\sigma_{\dot\alpha}\dot\beta}(vu^{-1}) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad j=\ell\,,
\end{cases}
$$

(P4) $h_{\dot\alpha}(u) = h_{-\dot\alpha}(u^{-1}) = h_{-\dot\alpha}(u)^{-1}$,

$$
(\mathrm{P5})\quad \begin{cases}
w_{\dot\alpha}(u)h_{\pm\dot\alpha}(v)w_{\dot\alpha}(-u) = h_{\mp\dot\alpha}(u^{\mp1}vu^{\mp1})h_{\mp\dot\alpha}(u^{\pm2})\,,\\
w_{\dot\alpha}(u)h_{\dot\beta}(v)w_{\dot\alpha}(-u) = h_{\dot\beta}(v) \\
\qquad \text{if } (\dot\alpha,\dot\beta)=0\,,\\
w_{\dot\alpha}(u)h_{\dot\beta}(v)w_{\dot\alpha}(-u) = h_{\sigma_{\dot\alpha}\dot\beta}(uv)h_{\sigma_{\dot\alpha}\dot\beta}(u^{-1}) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad j=k\,,\\
w_{\dot\alpha}(u)h_{\dot\beta}(v)w_{\dot\alpha}(-u) = h_{\sigma_{\dot\alpha}\dot\beta}(vu)h_{\sigma_{\dot\alpha}\dot\beta}(u^{-1}) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad i=\ell\,,\\
w_{\dot\alpha}(u)h_{\dot\beta}(v)w_{\dot\alpha}(-u) = h_{\sigma_{\dot\alpha}\dot\beta}(u^{-1}v)h_{\sigma_{\dot\alpha}\dot\beta}(u) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad i=k\,,\\
w_{\dot\alpha}(u)h_{\dot\beta}(v)w_{\dot\alpha}(-u) = h_{\sigma_{\dot\alpha}\dot\beta}(vu^{-1})h_{\sigma_{\dot\alpha}\dot\beta}(u) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad j=\ell\,,
\end{cases}
$$

$$
(\mathrm{P6})\quad \begin{cases}
h_{\dot\alpha}(u)x_{\pm\dot\alpha}(f)h_{\dot\alpha}(u)^{-1} = x_{\pm\dot\alpha}(u^{\pm1}fu^{\pm1})\,,\\
h_{\dot\alpha}(u)x_{\dot\beta}(f)h_{\dot\alpha}(u)^{-1} = x_{\dot\beta}(f) \\
\qquad \text{if } (\dot\alpha,\dot\beta)=0\,,\\
h_{\dot\alpha}(u)x_{\dot\beta}(f)h_{\dot\alpha}(u)^{-1} = x_{\dot\beta}(u^{-1}f) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad j=k\,,\\
h_{\dot\alpha}(u)x_{\dot\beta}(f)h_{\dot\alpha}(u)^{-1} = x_{\dot\beta}(fu^{-1}) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad i=\ell\,,\\
h_{\dot\alpha}(u)x_{\dot\beta}(f)h_{\dot\alpha}(u)^{-1} = x_{\dot\beta}(uf) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad i=k\,,\\
h_{\dot\alpha}(u)x_{\dot\beta}(f)h_{\dot\alpha}(u)^{-1} = x_{\dot\beta}(fu) \\
\qquad \text{if } \dot\alpha\pm\dot\beta\neq 0\,, \quad j=\ell\,,
\end{cases}
$$

for all $\dot\alpha, \dot\beta \in \Phi$, and for all $f \in K_q$ and $u, v \in K_q^\times$, where $\dot\alpha$ and $\dot\beta$ are written as

$$\dot{\alpha} = \varepsilon_i - \varepsilon_j \quad \text{and} \quad \dot{\beta} = \varepsilon_k - \varepsilon_\ell\,.$$

As a special case, we have the following relation:

(B)′ $w_{\dot{\alpha}}(u)x_{\dot{\alpha}}(f)w_{\dot{\alpha}}(-u) = x_{-\dot{\alpha}}(-u^{-1}fu^{-1})$

for all $\dot{\alpha} \in \Phi$ and for all $f \in K_q$ and $u \in K_q^{\times}$. Then, using those relations, we can establish the following lemma (cf. [1], [7], [12]).

LEMMA 3. *Notation is as above. Then*:

(1) $B^{\pm} = U^{\pm} \rtimes T_0$.

(2) $T \lhd N$ *and* $T_0 \lhd N$.

(3) $B^{\pm} \cap N = T_0$.

(4) $N/T_0 \simeq W_a$.

(5) $(N/T_0, S)$ *is a Coxeter system.*

For our purpose, we need a little bit more explicit discussion. Put

$$Y_{\pm a} = \langle x_{\pm a}(r)U_b x_{\pm a}(-r) \mid r \in K,\ b \in \Phi_a^{\pm} \setminus \{\pm a\}\rangle$$

for each $a \in \Pi_a$. Then, we obtain the following (cf. [1], [7], [10]).

PROPOSITION 4. *Let* $a \in \Pi_a$. *Then*:

(1) $U^{\pm} = Y_{\pm a} \rtimes U_{\pm a}$.

(2) $w_{\pm a}(t)Y_{\pm a}w_{\pm a}(-t) = Y_{\pm a}$ *for all* $t \in K^{\times}$.

(3) $B^{\pm} \cup B^{\pm}w_a(1)B^{\pm}$ *is a subgroup.*

(4) $sB^{\pm}s \not\subset B^{\pm}$ *for all* $s \in S$.

Hence, it is easy now to establish Theorem 2 in a standard way.

PROOF OF THEOREM 2. We should check the axiom of "Tits System." We will consider the case of B. We can easily confirm that E is generated by B and N. By Lemma 3 (2) and (3), we see $B \cap N = T_0 \lhd N$. By Lemma 3 (5), we can find $W_a = N/T_0 = \langle S\rangle$. For $w \in W_a$, we define the length $l(w)$ of w with respect to S as usual. If $w \in W_a$ and $s = w_a(1) \in S$ with $l(w) < l(sw)$, then $w(a) \in \Phi^+$ and

$$\begin{aligned} wBs &= wU_aY_aT_0s \\ &= (wU_aw^{-1})ws(s^{-1}Y_as)(s^{-1}T_0s) \\ &= U_{w(a)}wsY_aT_0 \\ &\subset BwsB\,. \end{aligned}$$

If $w \in W_a$ and $s = w_a(1) \in S$ with $l(ws) < l(w)$, then $w(a) \in \Phi_a^-$. Put $w' = ws$. Then $l(w') < l(w''s)$ and

$$\begin{aligned} wBs &= w'sBs \\ &\subset w'(B \cup BsB) \\ &= w'B \cup w'BsB \end{aligned}$$

$$\subset w'B \cup Bw'sB$$
$$\subset BwsB \cup BwB\,.$$

In any case, we obtain

$$wBs \subset BwsB \cup BwB\,.$$

By Proposition 4 (4), we have $sBs \not\subset B$ for all $s \in S$. Q.E.D.

COROLLARY. *Notation is as above. Then, we have*:
(1) $E = \bigcup_{w \in W_a} B^{\pm} w B^{\pm}$ (*Bruhat decomposition*),
(2) $E = \bigcup_{w \in W_a} B^{\mp} w B^{\pm}$ (*Birkhoff decomposition*),
(3) $E = U^{\pm} U^{\mp} T_0 U^{\pm}$ (*Gauss decomposition*).

7. Steinberg groups

Let $St(n, K_q)$ be the Steinberg group of type (A_{n-1}) over K_q, which is defined by the generators

$$\hat{x}_{ij}(f)$$

for all $1 \leq i \neq j \leq n$ and $f \in K_q$ and the defining relations

(A) $\hat{x}_{ij}(f)\hat{x}_{ij}(g) = \hat{x}_{ij}(f+g)$

(B) $\big[\hat{x}_{ij}(f), \hat{x}_{k\ell}(g)\big] = \begin{cases} \hat{x}_{i\ell}(fg) & \text{if } j = k\,, \\ \hat{x}_{kj}(-gf) & \text{if } i = \ell\,, \\ 1 & \text{otherwise} \end{cases}$

for all $1 \leq i \neq j \leq n$ and $1 \leq k \neq \ell \leq n$ with $(i, j) \neq (\ell, k)$, and for all $f, g \in K_q$. Exactly this definition is valid for $n \geq 3$. If $n = 2$, then we should replace (B) by the following (B)$'$:

(B)$'$ $\hat{w}_{ij}(u)\hat{x}_{ij}(f)\hat{w}_{ij}(-u) = \hat{x}_{ji}(-u^{-1} f u^{-1})$

for all i, j with $\{i, j\} = \{1, 2\}$ and for all $f \in K_q$ and $u \in K_q^{\times}$, where

$$\hat{w}_{ij}(u) = \hat{x}_{ij}(u)\hat{x}_{ji}(-u^{-1})\hat{x}_{ij}(u)\,.$$

Then, there is a natural homomorphism of ϕ of $St(n, K_q)$ onto $E(n, K_q)$ with $\phi(\hat{x}_{ij}(f)) = x_{ij}(f)$ for all $1 \leq i \neq j \leq n$ and $f \in K_q$. Similarly we put

$$\hat{h}_{ij}(u) = \hat{w}_{ij}(u)\hat{w}_{ij}(-1)\,.$$

We note that (B)$'$ holds in $St(n, K_q)$ for $n \geq 3$. In fact, if we choose an index k different from i, j, then

$$\begin{aligned} &\hat{w}_{ij}(u)\hat{x}_{ij}(f)\hat{w}_{ij}(-u) \\ &\quad = \hat{x}_{ij}(u)\hat{x}_{ji}(-u^{-1})\hat{x}_{ij}(u)[\hat{x}_{ik}(1), \hat{x}_{kj}(f)]\hat{x}_{ij}(-u)\hat{x}_{ji}(u^{-1})\hat{x}_{ij}(-u) \end{aligned}$$

$$
\begin{aligned}
&= \hat{x}_{ij}(u)\hat{x}_{ji}(-u^{-1})[\hat{x}_{ik}(1), \hat{x}_{kj}(f)]\hat{x}_{ji}(u^{-1})\hat{x}_{ij}(-u)\\
&= \hat{x}_{ij}(u)[\hat{x}_{jk}(-u^{-1})\hat{x}_{ik}(1), \hat{x}_{ki}(fu^{-1})\hat{x}_{kj}(f)]\hat{x}_{ij}(-u)\\
&= [\hat{x}_{ik}(-1)\hat{x}_{jk}(-u^{-1})\hat{x}_{ik}(1), \hat{x}_{kj}(-f)\hat{x}_{ki}(fu^{-1})\hat{x}_{kj}(f)]\\
&= [\hat{x}_{jk}(-u^{-1}), \hat{x}_{ki}(fu^{-1})]\\
&= \hat{x}_{ji}(-u^{-1}fu^{-1})\,.
\end{aligned}
$$

Here we also use the same idea of notation:

$$\hat{x}_{\dot\alpha}(f) = \hat{x}_{ij}(f)\,, \quad \hat{w}_{\dot\alpha}(u) = \hat{w}_{ij}(u)\,, \quad \hat{h}_{\dot\alpha}(u) = \hat{h}_{ij}(u)$$

if $\dot\alpha = \varepsilon_i - \varepsilon_j \in \Phi$. Then, in $St(n, K_q)$ we observe the relations ($\hat{\text{P}}$1) – ($\hat{\text{P}}$6) similar to (P1) – (P6), where $x,\ w,\ h$ should be changed into $\hat{x},\ \hat{w},\ \hat{h}$ respectively. More precisely we need to change a little bit, namely:

($\hat{\text{P}}$4) $\hat{h}_{\dot\alpha}(u) = \hat{h}_{-\dot\alpha}(u)^{-1}\,,$
and

$$
(\hat{\text{P}}5)\quad
\begin{cases}
\hat{w}_{\dot\alpha}(u)\hat{h}_{\pm\dot\alpha}(v)\hat{w}_{\dot\alpha}(-u) = \hat{h}_{\mp\dot\alpha}(-u^{\mp1}vu^{\mp1})\hat{h}_{\mp\dot\alpha}(-u^{\mp2})^{-1}, & \\
\hat{w}_{\dot\alpha}(u)\hat{h}_{\dot\beta}(v)\hat{w}_{\dot\alpha}(-u) = \hat{h}_{\dot\beta}(v) & \text{if } (\dot\alpha,\dot\beta) = 0\,,\\
\hat{w}_{\dot\alpha}(u)\hat{h}_{\dot\beta}(v)\hat{w}_{\dot\alpha}(-u) = \hat{h}_{\sigma_{\dot\alpha}\dot\beta}(uv)\hat{h}_{\sigma_{\dot\alpha}\dot\beta}(u)^{-1} & \text{if } \dot\alpha \pm \dot\beta \neq 0\,,\quad j = k\,,\\
\hat{w}_{\dot\alpha}(u)\hat{h}_{\dot\beta}(v)\hat{w}_{\dot\alpha}(-u) = \hat{h}_{\sigma_{\dot\alpha}\dot\beta}(-vu)\hat{h}_{\sigma_{\dot\alpha}\dot\beta}(-u)^{-1} & \text{if } \dot\alpha \pm \dot\beta \neq 0\,,\quad i = \ell\,,\\
\hat{w}_{\dot\alpha}(u)\hat{h}_{\dot\beta}(v)\hat{w}_{\dot\alpha}(-u) = \hat{h}_{\sigma_{\dot\alpha}\dot\beta}(-u^{-1}v)\hat{h}_{\sigma_{\dot\alpha}\dot\beta}(-u^{-1})^{-1} & \text{if } \dot\alpha \pm \dot\beta \neq 0\,,\quad i = k\,,\\
\hat{w}_{\dot\alpha}(u)\hat{h}_{\dot\beta}(v)\hat{w}_{\dot\alpha}(-u) = \hat{h}_{\sigma_{\dot\alpha}\dot\beta}(vu^{-1})\hat{h}_{\sigma_{\dot\alpha}\dot\beta}(u^{-1})^{-1} & \text{if } \dot\alpha \pm \dot\beta \neq 0\,,\quad j = \ell\,,
\end{cases}
$$

which are slightly different from (P4) and (P5).

Now we define the subgroups named $\hat{N}$ and $\hat{T}$ of $St(n, K_q)$ by

$$\hat{N} = \langle \hat{w}_{\dot\alpha}(u) \mid \dot\alpha \in \Phi,\ u \in K_q^\times \rangle\,,$$

$$\hat{T} = \langle \hat{h}_{\dot\alpha}(u) \mid \dot\alpha \in \Phi,\ u \in K_q^\times \rangle\,.$$

Then we put

$$\hat{T}_0 = \langle \hat{h} \in \hat{T} \mid \deg_i(\phi(\hat{h})) = 0 \ \text{ for all } \ i = 1, \dots, n \rangle\,.$$

Then we shall define:

$$\hat{U}_a = \langle \hat{x}_{\dot\alpha}(rX_2^m) \mid r \in K \rangle \quad \text{for } \ a = (\dot\alpha, m) \in \Phi_a\,,$$

$$\hat{U}^{\pm} = \langle \hat{U}_a \mid a \in \Phi_a^{\pm} \rangle\,,$$
$$\hat{B}^{\pm} = \langle \hat{U}^{\pm},\ \hat{T}_0 \rangle\,,$$
$$\hat{S} = \{\hat{w}_a(1) \bmod \hat{T}_0 \mid a \in \Pi_a\}\,.$$

Then, similarly we can show the following (cf. Lemma 3, [13]).

LEMMA 5. *Notation is as above. Then*:

(1) $\hat{U}^{\pm} \lhd \hat{B}^{\pm} = \hat{U}^{\pm}\hat{T}_0$.

(2) $\hat{T} \lhd \hat{N}$ *and* $\hat{T}_0 \lhd \hat{N}$.

(3) $\hat{B}^{\pm} \cap \hat{N} = \hat{T}_0$.

(4) $\hat{N}/\hat{T}_0 \simeq W_a$.

(5) $(\hat{N}/\hat{T}_0, \hat{S})$ *is a Coxeter system.*

We put $\hat{Y}_{\pm a} = \langle x\hat{U}_b x^{-1} \mid x \in \hat{U}_{\pm a},\ b \in \Phi_a^{\pm} \setminus \{\pm a\}\rangle$ for each $a \in \Pi_a$. Then, we obtain the following (cf. [1], [12], [21]).

PROPOSITION 6. *Let* $a \in \Pi_a$. *Then*:

(1) $\hat{Y}_{\pm a} \lhd \hat{U}^{\pm} = \hat{Y}_{\pm a}\hat{U}_{\pm a}$.

(2) $\hat{w}_{\pm a}(t)\hat{Y}_{\pm a}\hat{w}_{\pm a}(-t) = \hat{Y}_{\pm a}$ *for all* $t \in K^{\times}$.

(3) $\hat{B}^{\pm} \cup \hat{B}^{\pm}\hat{w}_a(1)\hat{B}^{\pm}$ *is a subgroup.*

(4) $\hat{s}\hat{B}^{\pm}\hat{s} \not\subset \hat{B}^{\pm}$ *for all* $\hat{s} \in \hat{S}$.

Hence, we can reach the next result, which is similar to the proof of Theorem 2.

THEOREM 7. *Notation is as above. Then,* $(St(n, K_q), \hat{B}^{\pm}, \hat{N}, \hat{S})$ *is a Tits system with the corresponding affine Weyl group* W_a.

COROLLARY. *Notation is as above. Then, we have*:

(1) $St(n, K_q) = \bigcup_{w \in W_a} \hat{B}^{\pm} w \hat{B}^{\pm}$ (*Bruhat decomposition*),

(2) $St(n, K_q) = \bigcup_{w \in W_a} \hat{B}^{\mp} w \hat{B}^{\pm}$ (*Birkhoff decomposition*),

(3) $St(n, K_q) = \hat{U}^{\pm}\hat{U}^{\mp}\hat{T}_0\hat{U}^{\pm}$ (*Gauss decomposition*).

8. K_2-groups and presentations (rank 1)

Here we suppose that $n = 2$, that is, the rank of Φ is 1 in the sense of root systems. Namely we assume $\Phi = \{\pm\dot{\alpha}\}$. Then we put $K_2(2, K_q) = \operatorname{Ker} \phi$, where ϕ is the canonical homomorphism of $St(2, K_q)$ onto $E(2, K_q)$. Let $\tilde{E}(2, K_q)$ be the group defined by generators $\tilde{x}_{ij}(f)$ for all $\{i, j\} = \{1, 2\}$ and for all $f \in K_q$ and the defining relations (A) and (B)$'$ together with the following relation:

(C) $\tilde{c}(u_1, v_1)\tilde{c}(u_2, v_2)\cdots\tilde{c}(u_p, v_p) = 1$

for all $u_1, \dots, u_p, v_1, \dots, v_p \in K_q^\times$ satisfying

$$[u_1, v_1][u_2, v_2] \cdots [u_p, v_p] = 1\,,$$

where for $u, v \in K_q^\times$ we put

$$\begin{aligned} \tilde{w}_{ij}(u) &= \tilde{x}_{ij}(u)\tilde{x}_{ji}(-u^{-1})\tilde{x}_{ij}(u)\,,\\ \tilde{h}_{ij}(u) &= \tilde{w}_{ij}(u)\tilde{w}_{ij}(-1)\,,\\ \tilde{c}(u, v) &= \tilde{h}_{12}(u)\tilde{h}_{12}(v)\tilde{h}_{12}(vu)^{-1}\,, \end{aligned}$$

and where x, w in the relations (A) and (B)$'$ should be changed into $\tilde{x}, \tilde{w}$ respectively. Using the above discussion, we obtain the following theorem.

THEOREM 8. *Notation is as above. Then, we have $\tilde{E}(2, K_q) \simeq E(2, K_q)$.*

PROOF OF THEOREM 8. The homomorphism $\phi : St(2, K_q) \to E(2, K_q)$ induces two canonical homomorphisms called $\hat{\phi}$ and $\tilde{\phi}$, that is,

$$\begin{aligned} \hat{\phi} &: St(2, K_q) \to \tilde{E}(2, K_q)\,,\\ \tilde{\phi} &: \tilde{E}(2, K_q) \to E(2, K_q)\,, \end{aligned}$$

which are defined by

$$\hat{\phi}(\hat{x}_{ij}(f)) = \tilde{x}_{ij}(f) \quad \text{and} \quad \tilde{\phi}(\tilde{x}_{ij}(f)) = x_{ij}(f)$$

with the following diagram.

$$\begin{tikzcd} & \tilde{E}(2, K_q) \arrow[dr, "\tilde{\phi}"] & \\ St(2, K_q) \arrow[ur, "\hat{\phi}"] \arrow[rr, "\phi"'] & & E(2, K_q) \end{tikzcd}$$

We use the same notation of subgroups of $\tilde{E}(2, K_q)$ as in $St(2, K_q)$ changing $\hat{}$ into $\tilde{}$, namely $\hat{\phi}(\hat{\ }) = \tilde{\ }$. Then, we find two kinds of Bruhat decompositions:

$$\begin{array}{ccccccc} \tilde{E}(2, K_q) & = & \bigcup_{w \in W_a} \tilde{B}w\tilde{B} & \supset & \tilde{B} & = & \tilde{U}\tilde{T}_0\,,\\ \downarrow & & & & \downarrow & & \\ E(2, K_q) & = & \bigcup_{w \in W_a} BwB & \supset & B & = & U \rtimes T_0\,. \end{array}$$

Therefore, by these decompositions, we can obtain $\operatorname{Ker}\tilde{\phi} \subset \tilde{B}$. We take an element $\tilde{x} \in \operatorname{Ker}\tilde{\phi}$. Then, we write $\tilde{x}$ as $\tilde{x} = \tilde{y}\tilde{z}$ for some $\tilde{y} \in \tilde{U}$ and $\tilde{z} \in \tilde{T}_0$. Put $y = \tilde{\phi}(\tilde{y})$ and $z = \tilde{\phi}(\tilde{z})$. Since $\tilde{x} \in \operatorname{Ker}\tilde{\phi}$, we have

$$1 = \tilde{\phi}(\tilde{x}) = \tilde{\phi}(\tilde{y})\tilde{\phi}(\tilde{z}) = yz \in U \rtimes T_0\,,$$

which implies $y = 1$ and $z = 1$. Hence, in particular, $\tilde{y}$ and $\tilde{z}$ belong to Ker $\tilde{\phi}$.

CLAIM 1. $\tilde{y} = 1$.

PROOF OF CLAIM 1. Since $\tilde{y} \in \tilde{T}_0 \subset \tilde{T}$, we can write $\tilde{y}$ as

$$\tilde{y} = \tilde{h}_{12}(u_1)^{\pm 1}\tilde{h}_{12}(u_2)^{\pm 1} \cdots \tilde{h}_{12}(u_p)^{\pm 1}$$

for some $u_1, \dots, u_p \in K_q^\times$ using the relation ($\hat{\mathrm{P}}4$). On the other hand, $[u, u^{-1}] = 1$ for all $u \in K_q^\times$ implies $1 = \tilde{c}(u, u^{-1}) = \tilde{h}_{12}(u)\tilde{h}_{12}(u^{-1})$ by (C), which leads to $\tilde{h}_{12}(u)^{-1} = \tilde{h}_{12}(u^{-1})$. Hence, one can write

$$\tilde{y} = \tilde{h}_{12}(v_1)\tilde{h}_{12}(v_2) \cdots \tilde{h}_{12}(v_p)$$

for some $v_1, \dots, v_p \in K_q^\times$. Since $\tilde{\phi}(\tilde{y}) = 1$, we see that

$$v_1 v_2 \cdots v_p = v_1^{-1} v_2^{-1} \cdots v_p^{-1} = 1\,.$$

Then, we can compute, by (C),

$$\begin{aligned}
\tilde{y} &= \tilde{c}(v_1, v_2)\, \tilde{h}_{12}(v_2 v_1)\, \tilde{h}_{12}(v_3)\ \cdots\ \tilde{h}_{12}(v_p) \\
&= \tilde{c}(v_1, v_2)\, \tilde{c}(v_2 v_1, v_3)\, \tilde{h}_{12}(v_3 v_2 v_1)\, \tilde{h}_{12}(v_4)\ \cdots\ \tilde{h}_{12}(v_p) \\
&= \tilde{c}(v_1, v_2)\, \tilde{c}(v_2 v_1, v_3)\ \cdots\ \tilde{c}(v_{p-1} \cdots v_2 v_1, v_p)\, \tilde{h}_{12}(v_p \cdots v_1) \\
&= \tilde{c}(v_1, v_2)\, \tilde{c}(v_2 v_1, v_3)\ \cdots\ \tilde{c}(v_{p-1} \cdots v_2 v_1, v_p)
\end{aligned}$$

and furthermore the last part of this equation should be 1 by (C), since

$$[v_1, v_2][v_2 v_1, v_3] \cdots [v_{p-1} \cdots v_2 v_1, v_p] = 1\,.$$

CLAIM 2. $\tilde{z} = 1$.

PROOF OF CLAIM 2. Using the degree map of $K[X_2]$ in X_2, we can establish that U is the free product of

$$\begin{pmatrix} 1 & K[X_2] \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ X_2 K[X_2] & 1 \end{pmatrix}.$$

Hence, $\tilde{U}$ is isomorphic to U, and $\tilde{z} = 1$.

Therefore, we just reached $\tilde{x} = 1$, which implies that $\tilde{E}(2, K_q) \simeq E(2, K_q)$. Q.E.D.

We need to know a set of generators for $K_2(2, K_q)$, which we will use later, as follows.

PROPOSITION 9. *Notation is as above. Then we have*:

$$\begin{aligned}
K_2(2, K_q) = \langle \hat{c}(u_1, v_1)\hat{c}(u_2, v_2) \cdots \hat{c}(u_k, v_k) \mid k \geq 1\,, \\
u_1, v_1, \dots, u_k, v_k \in K_q^\times,\ [u_1, v_1][u_2, v_2] \cdots [u_k, v_k] = 1 \rangle\,,
\end{aligned}$$

where $\hat{c}(u, v) = \hat{h}_{12}(u)\hat{h}_{12}(v)\hat{h}_{12}(vu)^{-1}$, *and* $K_2(2, K_q)$ *is a central subgroup of* $St(2, K_q)$.

PROOF OF PROPOSITION 9. It is enough to confirm that

$$\hat{c}(u_1, v_1)\hat{c}(u_2, v_2)\cdots\hat{c}(u_k, v_k)$$

is central if $[u_1, v_1][u_2, v_2]\cdots[u_k, v_k] = 1$, which is easily checked by a direct computation:

$$\begin{aligned}
&\{\hat{c}(u_1,v_1)\hat{c}(u_2, v_2)\cdots\hat{c}(u_k, v_k)\}\hat{x}_{12}(f)\{\hat{c}(u_1, v_1)\hat{c}(u_2, v_2)\cdots\hat{c}(u_k, v_k)\}^{-1}\\
&\qquad = \hat{x}_{12}([u_1, v_1][u_2, v_2]\cdots[u_k, v_k]f)\\
&\qquad = \hat{x}_{12}(f)
\end{aligned}$$

and

$$\begin{aligned}
&\{\hat{c}(u_1,v_1)\hat{c}(u_2, v_2)\cdots\hat{c}(u_k, v_k)\}\hat{x}_{21}(f)\{\hat{c}(u_1, v_1)\hat{c}(u_2, v_2)\cdots\hat{c}(u_k, v_k)\}^{-1}\\
&\qquad = \hat{x}_{21}(f[v_k, u_k]\cdots[v_2, u_2][v_1, u_1])\\
&\qquad = \hat{x}_{21}(f\{[u_1, v_1][u_2, v_2]\cdots[u_k, v_k]\}^{-1})\\
&\qquad = \hat{x}_{21}(f)
\end{aligned}$$

for all $f \in K_q$. Q.E.D.

9. K_2-groups and presentations (higher rank)

We suppose $n \geq 3$. Put $K_2(n, K_q) = \mathrm{Ker}\ \phi$. Then, in general, there exists a canonical homomorphism of $K_2(2, K_q)$ into $K_2(n, K_q)$, which is induced from the following diagram (cf. [9]):

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_2(2, K_q) & \longrightarrow & St(2, K_q) & \longrightarrow & E(2, K_q) & \longrightarrow & 1 \ \text{(exact)}\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_2(n, K_q) & \longrightarrow & St(n, K_q) & \longrightarrow & E(n, K_q) & \longrightarrow & 1 \ \text{(exact)}
\end{array}$$

Since K_q is a Euclidean ring, the homomorphism of $K_2(2, K_q)$ into $K_2(n, K_q)$ is surjective by [5]. Hence, we have the following.

THEOREM 10. *Suppose $n \geq 3$. Let $\tilde{E}(n, K_q)$ be the group generated by $\tilde{x}_{ij}(f)$ for all $1 \leq i \neq j \leq n$ and $f \in K_q$ with the defining relations* (A), (B) *and* (C). *Then, $\tilde{E}(n, K_q)$ is isomorphic to $E(n, K_q)$.*

On the other hand, by a similar computation appeared in the proof of Proposition 9, we see that

$$\hat{c}(u_1, v_1)\hat{c}(u_2, v_2)\cdots\hat{c}(u_k, v_k)$$

is central in $St(n, K_q)$. Therefore, we obtain the following.

PROPOSITION 11. *Suppose $n \geq 3$. Then,*

$$\begin{aligned}
K_2(n, K_q) = \langle\, \hat{c}(u_1, v_1)\hat{c}(u_2, v_2)\cdots\hat{c}(u_k, v_k) \mid \quad & k \geq 1\,,\\
& u_1, v_1, \ldots, u_k, v_k \in K_q^{\times},\ [u_1, v_1][u_2, v_2]\cdots[u_k, v_k] = 1\rangle\,,
\end{aligned}$$

where $\hat{c}(u, v) = \hat{h}_{12}(u)\hat{h}_{12}(v)\hat{h}_{12}(vu)^{-1}$, *and* $K_2(n, K_q)$ *is a central subgroup of* $St(n, K_q)$.

10. Universal central extensions

We first fix four central elements $a^*, b^*, c^*, d^* \in K^\times$ as follows. If $|F| \geq 4$, then we choose $a^* \in F^\times$ such that $a^{*2} - 1 \neq 0$, and put $b^* = 1/(a^{*2} - 1) \in F^\times$. In case of $|F| \leq 3$, we set $a^* = X_1^2 \in K^\times$ and $b^* = 1/(a^{*2} - 1) \in K^\times$. Furthermore, we choose $c^*, d^* \in F^\times$ such that

$$c^* \neq 0\,, \quad c^* - 1 \neq 0\,, \quad c^{*2} - c^* + 1 \neq 0\,, \quad d^{*3} - 1 \neq 0$$

if $|F| \geq 5$, and we define $c^* = d^* = X_1^6 \in K^\times$ if $|F| \leq 4$.

We note that $E(n, K_q)$ is perfect, since

$$[h_{ij}(a^*), x_{ij}(b^* f)] = x_{ij}(f)$$

for $1 \leq i \neq j \leq n$ and $f \in K_q$. Hence, there is a (unique up to isomorphism) universal central extension of $E(n, K_q)$. We will establish that $St(n, K_q)$ is a universal central extension of $E(n, K_q)$ for all $n \geq 2$. By the same reason as in case of $E(n, K_q)$, we can see that $St(n, K_q)$ is also perfect.

Let $\phi^* : E^* \to E(n, K_q)$ be a central extension. For $z \in E(n, K_q)$, we put

$$M(z) = \{z^* \in E^* \mid \phi^*(z^*) = z\} = \phi^{*-1}(z)\,,$$

and hence, in particular, we have $M(1) = \operatorname{Ker} \phi^*$. Then, for $1 \leq i \neq j \leq n$ and $f \in K_q$, we define

$$x_{ij}^*(f) = \left[h^*, x^*\right] \in E^*\,,$$

where we choose $h^* \in M(h_{ij}(a^*))$ and $x^* \in M(x_{ij}(b^* f))$. This $x_{ij}^*(f)$ is well-defined by the so-called central trick. Also we put

$$w_{ij}^*(u) = x_{ij}^*(u)x_{ji}^*(-u^{-1})x_{ij}^*(u)\,,$$
$$h_{ij}^*(u) = w_{ij}^*(u)w_{ij}^*(-1)\,.$$

Then we obtain the following.

LEMMA 12. *Let* $1 \leq i \neq j \leq n$, *and let* $f \in K_q$ *and* $u \in K_q^\times$. *Then*:

(1) $w_{ij}^*(u)x_{ij}^*(f)w_{ij}^*(-u) = x_{ji}^*(-u^{-1}fu^{-1})$.

(2) $h_{ij}^*(u)x_{ij}^*(f)h_{ij}^*(u)^{-1} = x_{ji}^*(ufu)$.

PROOF OF LEMMA 12. By the definition, we have

$$\begin{aligned} w_{ij}^*(u)x_{ij}^*(f)w_{ij}^*(-u) &= w_{ij}^*(u)[h_{ij}^*(a^*), x_{ij}^*(b^* f)]w_{ij}^*(-u) \\ &= [w_{ij}^*(u)h_{ij}^*(a^*)w_{ij}^*(-u), w_{ij}^*(u)x_{ij}^*(b^* f)w_{ij}^*(-u)] \end{aligned}$$

$$
\begin{aligned}
&= [h^*_{ji}(a^*), x^*_{ji}(-b^* u^{-1} f u^{-1})] \\
&= x^*_{ji}(-u^{-1} f u^{-1})
\end{aligned}
$$

and

$$
\begin{aligned}
h^*_{ij}(u) x^*_{ij}(f) h^*_{ij}(u)^{-1} &= h^*_{ij}(u)[h^*_{ij}(a^*), x^*_{ij}(b^* f)] h^*_{ij}(u)^{-1} \\
&= [h^*_{ij}(u) h^*_{ij}(a^*) h^*_{ij}(u)^{-1}, h^*_{ij}(u) x^*_{ij}(b^* f) h^*_{ij}(u)^{-1}] \\
&= [h^*_{ij}(a^*), x^*_{ij}(b^* u f u)] \\
&= x^*_{ij}(u f u)\,,
\end{aligned}
$$

which implies the lemma. Q.E.D.

We suppose $(i, j) \neq (\ell, k)$ with $1 \leq i \neq j \leq n$. For i, j, k, ℓ and for $f, g \in K_q$, we define $\pi(f, g) = \pi_{i,j,k,\ell}(f, g)$ by

$$
\pi(f, g) = \pi_{i,j,k,\ell}(f, g) = \begin{cases} [x^*_{ij}(f), x^*_{k\ell}(g)] x^*_{i\ell}(fg)^{-1} & \text{if } j = k\,, \\[2ex] [x^*_{ij}(f), x^*_{k\ell}(g)] x^*_{kj}(-gf)^{-1} & \text{if } i = \ell\,, \\[2ex] [x^*_{ij}(f), x^*_{k\ell}(g)] & \text{otherwise}\,. \end{cases}
$$

Then we can show the following (cf. [21]).

LEMMA 13. *Let $f, f', g, g' \in K_q$ and $1 \leq i \neq j \leq n$ with $(i, j) \neq (\ell, k)$. Then*:
(1) $\pi_{i,j,k,\ell}(f + f', g) = \pi_{i,j,k,\ell}(f, g)\pi_{i,j,k,\ell}(f', g)$,
(2) $\pi_{i,j,k,\ell}(f, g + g') = \pi_{i,j,k,\ell}(f, g)\pi_{i,j,k,\ell}(f, g')$,
(3) $\pi_{i,j,k,\ell}(f, g) = 1$,
(4) $x^*_{ij}(f) x^*_{ij}(g) = x^*_{ij}(f + g)$.

PROOF OF LEMMA 13. To prove (3), we need to show (1) and (2). We will proceed dividing into three cases.

(Case 1) $j \neq k,\ i \neq \ell$: By the definition, we have

$$
\begin{aligned}
\pi(f + f', g) &= [x^*_{ij}(f + f'), x^*_{k\ell}(g)] \\
&= [x^*_{ij}(f) x^*_{ij}(f'), x^*_{k\ell}(g)] \\
&= x^*_{ij}(f) x^*_{ij}(f') x^*_{k\ell}(g) x^*_{ij}(f')^{-1} x^*_{ij}(f)^{-1} x^*_{k\ell}(g)^{-1} \\
&= x^*_{ij}(f) \{x^*_{ij}(f') x^*_{k\ell}(g) x^*_{ij}(f')^{-1} x^*_{k\ell}(g)^{-1}\} \\
&\qquad x^*_{k\ell}(g) x^*_{ij}(f)^{-1} x^*_{k\ell}(g)^{-1} \\
&= x^*_{ij}(f) [x^*_{ij}(f'), x^*_{k\ell}(g)] x^*_{k\ell}(g) x^*_{ij}(f)^{-1} x^*_{k\ell}(g)^{-1} \\
&= x^*_{ij}(f) \pi(f', g) x^*_{k\ell}(g) x^*_{ij}(f)^{-1} x^*_{k\ell}(g)^{-1} \\
&= \pi(f, g)\pi(f', g)
\end{aligned}
$$

and

$$\begin{aligned}\pi(f, g+g') &= [x^*_{ij}(f), x^*_{k\ell}(g+g')] \\ &= [x^*_{ij}(f), x^*_{ij}(g)x^*_{k\ell}(g')] \\ &= x^*_{ij}(f)x^*_{k\ell}(g)x^*_{k\ell}(g')x^*_{ij}(f)^{-1}x^*_{k\ell}(g')^{-1}x^*_{k\ell}(g)^{-1} \\ &= x^*_{ij}(f)x^*_{k\ell}(g)x^*_{ij}(f)^{-1} \\ &\quad \{x^*_{ij}(f)x^*_{k\ell}(g')x^*_{ij}(f)^{-1}x^*_{k\ell}(g')^{-1}\}x^*_{k\ell}(g)^{-1} \\ &= x^*_{ij}(f)x^*_{k\ell}(g)x^*_{ij}(f)^{-1}[x^*_{k\ell}(f), x^*_{k\ell}(g')]x^*_{k\ell}(g)^{-1} \\ &= x^*_{ij}(f)x^*_{k\ell}(g)x^*_{ij}(f)^{-1}\pi(f, g')x^*_{k\ell}(g)^{-1} \\ &= \pi(f, g)\pi(f, g')\,.\end{aligned}$$

If $i \neq k$ and $j \neq \ell$, then

$$\begin{aligned}\pi(b^*f, g) &= h^*_{ij}(a^*)\pi(b^*f, g)h^*_{ij}(a^*)^{-1} \\ &= h^*_{ij}(a^*)[x^*_{ij}(b^*f), x^*_{k\ell}(g)]h^*_{ij}(a^*)^{-1} \\ &= [x^*_{ij}(a^{*2}b^*f), x^*_{k\ell}(g)] \\ &= \pi(a^{*2}b^*f, g)\end{aligned}$$

and

$$\begin{aligned}\pi(f, g) &= \pi(a^{*2}b^*f - b^*f, g) \\ &= \pi(a^{*2}b^*f, g)\pi(b^*f, g)^{-1} \\ &= 1\,.\end{aligned}$$

If $i = k$ and $j \neq \ell$, then we obtain

$$\begin{aligned}\pi(f, g) &= h^*_{ij}(d^*)\pi(f, g)h^*_{ij}(d^*)^{-1} \\ &= \pi(d^{*2}, d^*g)\end{aligned}$$

and

$$\begin{aligned}\pi(f, g) &= h^*_{\ell j}(d^*)\pi(f, g)h^*_{\ell j}(d^*)^{-1} \\ &= \pi(d^*f, d^{*-1}g)\,.\end{aligned}$$

Therefore,

$$\pi(f, g) = \pi(d^{*2}f, d^*g) = \pi(d^{*3}f, g)\,,$$

which implies $\pi((d^{*3}-1)f, g) = 1$ and $\pi(f, g) = 1$.

If $i \neq k$ and $j = \ell$, we similarly obtain, using the same d^*,

$$\pi(f, g) = \pi(d^{*2}, d^* f) = \pi(d^{*3}, g)\,,$$

which gives $\pi((d^{*3} - 1)f, g) = 1$ and $\pi(f, g) = 1$.

If $i = k$ and $j = \ell$, then, by

$$\begin{aligned}\pi(f, g) &= h^*_{ij}(c^*)\pi(f, g)h^*_{ij}(c^*)^{-1} \\ &= h^*_{ij}(c^*)[x^*_{ij}(f), x^*_{ij}(g)]h^*_{ij}(c^*)^{-1} \\ &= [x^*_{ij}(c^{*2} f), x^*_{ij}(c^{*2} g)] \\ &= \pi(c^{*2} f, c^{*2} g)\,,\end{aligned}$$

we obtain

$$\begin{aligned}\pi(f, g) &= \pi(c^* f + (1 - c^*)f, g) \\ &= \pi(c^* f, g)\pi((1 - c^*)f, g) \\ &= \pi(f, g/c^*)\pi(f, g/(1 - c^*)) \\ &= \pi(f, (g/c^*) + (g/(1 - c^*))) \\ &= \pi(f, g/\{c^*(1 - c^*)\}) \\ &= \pi(c^*(1 - c^*)f, g)\end{aligned}$$

and

$$\pi((c^{*2} - c^* + 1)f, g) = 1\,.$$

Therefore, we reached $\pi(f, g) = 1$ for all $f, g \in K_q$. In particular, we notice that

$$x^*_{ij}(f)x^*_{ij}(g) = x^*_{ij}(g)x^*_{ij}(f)$$

for all $1 \leq i \neq j \leq n$ and $f, g \in K_q$.

Now, we put

$$\tau(f, g) = \tau_{ij}(f, g) = x^*_{ij}(f)x^*_{ij}(g)x^*_{ij}(f + g)^{-1}$$

for each $1 \leq i \neq j \leq n$ and $f, g \in K_q$. Then, we have

$$\begin{aligned}\tau(b^* f, b^* g) &= h^*_{ij}(a^*)\tau(b^* f, b^* g)h^*_{ij}(a^*)^{-1} \\ &= h^*_{ij}(a^*)x^*_{ij}(b^* f)x^*_{ij}(b^* g)x^*_{ij}(b^* f + b^* g)^{-1}h^*_{ij}(a^*)^{-1} \\ &= [h^*_{ij}(a^*), x^*_{ij}(b^* f)]x^*_{ij}(b^* f) \\ &\quad\ [h^*_{ij}(a^*), x^*_{ij}(b^* g)]x^*_{ij}(b^* g) \\ &\quad\ [h^*_{ij}(a^*), x^*_{ij}(b^* f + b^* g)^{-1}]x^*_{ij}(b^* f + b^* g)^{-1} \\ &= x^*_{ij}(f)x^*_{ij}(b^* f)x^*_{ij}(g)x^* ij(b^* g)\end{aligned}$$

$$
\begin{aligned}
&x^*_{ij}(f+g)^{-1}x^*_{ij}(b^*f+b^*g)^{-1}\\
&= \tau(f,g)\tau(b^*f,b^*g)
\end{aligned}
$$

and $\tau(f,g)=1$, which implies (4). Hence, we see $x^*_{ij}(f)^{-1}=x^*_{ij}(-f)$.

(Case 2) $j=k$: By the definition $\pi(f,g)=[x^*_{ij}(f),x^*_{j\ell}(g)]x^*_{i\ell}(-fg)$, we have

$$
\begin{aligned}
\pi(f+f',g) &= [x^*_{ij}(f+f'),x^*_{j\ell}(g)]x^*_{i\ell}(-(f+f')g)\\
&= x^*_{ij}(f)x^*_{ij}(f')x^*_{j\ell}(g)x^*_{ij}(-f')x^*_{ij}(-f)x^*_{j\ell}(-g)\\
&\quad\ x^*_{i\ell}(-fg)x^*_{i\ell}(-f'g)\\
&= x^*_{ij}(f)\{x^*_{ij}(f')x^*_{j\ell}(g)x^*_{ij}(-f')x^*_{j\ell}(-g)x^*_{i\ell}(-f'g)\}\\
&\quad\ x^*_{j\ell}(g)x^*_{ij}(-f)x^*_{j\ell}(-g)x^*_{i\ell}(-fg)\\
&= x^*_{ij}(f)\pi(f',g)x^*_{j\ell}(g)x^*_{ij}(-f)x^*_{j\ell}(-g)x^*_{i\ell}(-fg)\\
&= \pi(f,g)\pi(f',g)
\end{aligned}
$$

and we similarly have

$$
\begin{aligned}
\pi(f,g+g') &= [x^*_{ij}(f),x^*_{j\ell}(g+g')]x^*_{i\ell}(-f(g+g'))\\
&= x^*_{ij}(f)x^*_{j\ell}(g+g')x^*_{ij}(-f)x^*_{j\ell}(-g-g')x^*_{i\ell}(-fg-fg')\\
&= x^*_{ij}(f)x^*_{j\ell}(g)x^*_{ij}(-f)\\
&\quad\ \{x^*_{ij}(f)x^*_{j\ell}(g')x^*_{ij}(-f)x^*_{j\ell}(-g')x^*_{i\ell}(-fg')\}\\
&\quad\ x^*_{j\ell}(-g)x^*_{i\ell}(-fg)\\
&= x^*_{ij}(f)x^*_{j\ell}(g)x^*_{ij}(-f)\pi(f,g')x^*_{j\ell}(-g)x^*_{i\ell}(-fg)\\
&= \pi(f,g)\pi(f,g')\,.
\end{aligned}
$$

On the other hand, we obtain

$$
\begin{aligned}
\pi(f,g) &= h^*_{ij}(d^*)\pi(f,g)h^*_{ij}(d^*)^{-1}\\
&= h^*_{ij}(d^*)[x^*_{ij}(f),x^*_{j\ell}(g)]x^*_{i\ell}(-fg)h^*_{ij}(d^*)^{-1}\\
&= [x^*_{ij}(d^{*2}f),x^*_{j\ell}(d^{*-1}g)]x^*_{i\ell}(-d^*fg)\\
&= \pi(d^{*2}f,d^{*-1}g)
\end{aligned}
$$

and

$$
\begin{aligned}
\pi(f,g) &= h^*_{i\ell}(d^*)\pi(f,g)h^*_{i\ell}(d^*)^{-1}\\
&= h^*_{i\ell}(d^*)[x^*_{ij}(f),x^*_{j\ell}(g)]x^*_{i\ell}(-fg)h^*_{i\ell}(d^*)^{-1}\\
&= [x^*_{ij}(d^*f),x^*_{j\ell}(d^*g)]x^*_{i\ell}(-d^{*2}fg)
\end{aligned}
$$

$$= \pi(d^* f, d^* g)\,.$$

Therefore, we have

$$\pi(f, g) = \pi(d^{*2} f, d^{*-1} g) = \pi(d^{*3} f, g)$$

and

$$\pi((d^{*3} - 1) f, g) = 1\,.$$

Hence, we just showed $\pi(f, g) = 1$ for all $f, g \in K_q$.

(Case 3) $i = \ell$: By the definition $\pi(f, g) = [x^*_{ij}(f), x^*_{ki}(g)]$, we can similarly reach

$$\pi(f + f', g) = \pi(f, g)\pi(f', g)\,,$$
$$\pi(f, g + g') = \pi(f, g)\pi(f, g')\,.$$

Using the same d^*, we can establish $\pi(f, g) = 1$ for all $f, g \in K_q$. Q.E.D.

THEOREM 14. *Notation is as above. Then, $St(n, K_q)$ is a universal central extension of $E(n, K_q)$.*

PROOF OF THEOREM 14. Using a given central extension $\phi^* : E^* \to E(n, K_q)$, we constructed the elements $x^*_{ij}(f)$ for all $1 \leq i \neq j \leq n$ and $f \in K_q$. The relations which we obtained above give a homomorphism, called $\hat{\phi}^*$, of $St(n, K_q)$ into E^* such that $\hat{\phi}^*(\hat{x}_{ij}(f)) = x^*_{ij}(f)$ for all $1 \leq i \neq j \leq n$ and $f \in K_q$. This $\hat{\phi}^*$ is a desired homomorphism. Q.E.D.

In case of $q = 1$, the group structure of $K_2(n, K_q)$ has been discussed in terms of Witt rings (cf. [14], [15]). However, for general q, it might be rather difficult to determine its group structure.

11. Normalities and K_1-groups

Since K_q is a Euclidean ring, we see

$$\begin{aligned} GL(n, K_q) &= \langle E(n, K_q),\ D(n, K_q)\rangle \\ &= \langle E(n, K_q),\ D_1(n, K_q)\rangle \\ &\rhd E(n, K_q)\,, \end{aligned}$$

where

$$D(n, K_q) = \left\{ \left. \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_n \end{pmatrix} \right| u_1, \ldots, u_n \in K_q^\times \right\}$$

and $D_1(n, K_q) = \{ d(u) \mid u \in K_q^\times \}$ with

$$d(u) = \begin{pmatrix} u & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We put $K_1(n, K_q) = GL(n, K_q)/E(n, K_q)$ as in [9]. Hence, we obtain

$$\begin{aligned} K_1(n, K_q) &= E(n, K_q)D_1(n, K_q)/E(n, K_q) \\ &\simeq D_1(n, K_q)/(E(n, K_q) \cap D_1(n, K_q)). \end{aligned}$$

Using our Bruhat decomposition:

$$E(n, K_q) = \bigcup_{w \in W_a} BwB$$

as well as

$$BwB = UwT_0U = U\dot{w} \begin{pmatrix} X_2^{m_1} & 0 & \cdots & 0 \\ 0 & X_2^{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_2^{m_n} \end{pmatrix} T_0U$$

for some $\dot{w} \in \dot{W}$ and for some $m_1, m_2, \ldots, m_n \in \mathbf{Z}$ with $m_1 + m_2 + \cdots + m_n = 0$, we obtain:

$$\begin{aligned} & BwB \cap D_1(n, K_q) \neq \emptyset \\ \Longrightarrow \quad & U\dot{w} \begin{pmatrix} X_2^{m_1} & 0 & \cdots & 0 \\ 0 & X_2^{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_2^{m_n} \end{pmatrix} T_0U \cap D_1(n, K_q) \neq \emptyset \\ \Longrightarrow \quad & U \cap D_1(n, K_q)UT_0 \begin{pmatrix} X_2^{-m_1} & 0 & \cdots & 0 \\ 0 & X_2^{-m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_2^{-m_n} \end{pmatrix} \dot{w}^{-1} \neq \emptyset. \end{aligned}$$

If we set

$$\begin{cases} K^\times &= K \setminus \{0\}, \\ K_q^{\geq 0} &= K[X_2], \\ K_q^{>0} &= K[X_2]X_2, \end{cases}$$

then we have

$$U \subset \begin{pmatrix} 1+K_q^{>0} & K_q^{\geq 0} & \cdots & K_q^{\geq 0} \\ K_q^{>0} & 1+K_q^{>0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & K_q^{\geq 0} \\ K_q^{>0} & \cdots & K_q^{>0} & 1+K_q^{>0} \end{pmatrix}.$$

We suppose $BwB \cap D_1(n, K_q) \neq \emptyset$. Then, we take an element $x = d(u)x_+x_0y_0$ with $d(u) \in D_1(n, K_q)$, $x_+ \in U$, $x_0 \in T_0$ and

$$y_0 = \begin{pmatrix} X_2^{-m_1} & 0 & \cdots & 0 \\ 0 & X_2^{-m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_2^{-m_n} \end{pmatrix}$$

such that $x\dot{w}^{-1}$ lies in U and

$$\begin{pmatrix} tX_2^{m-m_1}(1+K_q^{>0}) & X_2^{m-m_2}K_q^{\geq 0} & \cdots & X_2^{m-m_n}K_q^{\geq 0} \\ X_2^{-m_1}K_q^{>0} & X_2^{-m_2}(1+K_q^{>0}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & X_2^{-m_n}K_q^{\geq 0} \\ X_2^{-m_1}K_q^{>0} & \cdots & X_2^{-m_{n-1}}K_q^{>0} & X_2^{-m_n}(1+K_q^{>0}) \end{pmatrix} \dot{w}^{-1},$$

where $u = tX_2^m \in K_q^\times$ with $t \in K^\times$ and $m \in \mathbf{Z}$. If $m_i > 0$ for some $2 \leq i \leq n$, then x has an entry with a negative power of X_2. This is a contradiction. Therefore $m_i \leq 0$ for all $2 \leq i \leq n$. In particular, we have $m_1 \geq 0$. Now we suppose that as a permutation of columns $\dot{w}^{-1}$ takes j to 1. Then $2 \leq \dot{w}^{-1}(k) \leq n$ for all $2 \leq k \leq n$ with $k \neq j$. Since every diagonal entry of x has a nonzero constant term, we have $m_k = 0$ for all $2 \leq k \leq n$ with $k \neq j$. Furthermore $\dot{w}^{-1}(k) = k$ for all such k, which follows from the fact that $\dot{w}^{-1}(k)$ cannot be not only greater than k but also less than k by checking the positions of $1 + K_q^{>0}$ and $K_q^{>0}$. Hence, we just falled into two cases, namely the case when $\dot{w}$ is the transposition of the 1-st column and the j-th column with $j \neq 1$ or the case when $\dot{w} = 1$. Suppose that $\dot{w} = (1, j)$ is a transposition with $1 \neq j$. Then, we have $tX_2^{m-m_1}(1+K_q^{>0}) \cap K_q^{\geq 0} \neq \emptyset$ since the 1-st column is going to the j-th column, which implies $m - m_1 \geq 0$. On the other hand, we have $X_2^{m-m_j}K_q^{\geq 0} \cap (1 + K_q^{\geq 0}) \neq \emptyset$ since the j-th column is going to the 1-st column, which implies $m - m_j \leq 0$. Combining these inequalities and $m_1 + m_j = 0$, we obtain

$$m \leq m_j = -m_1 \leq 0 \leq m_1 \leq m\,,$$

which leads to $m = m_1 = m_j = 0$. However, the $1 + K_q^{>0}$ of the j-column can not move to the 1-st column, since $1 + K_q^{>0} \cap K_q^{>0} = \emptyset$. This is a contradiction. Therefore, $\dot{w} = 1$. We just proved that $BwB \cap D_1(n, K_q) \neq \emptyset$ implies $w = 1$ and $u \in K^\times$, and we also showed

$$B \cap D_1(n, K_q) \subset \left\{ \left. \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right| t \in K^\times \right\}.$$

Thus,

$$\begin{aligned} E(n, K_q) \cap D_1(n, K_q) &= B \cap D_1(n, K_q) \\ &= (U \rtimes T_0) \cap D_1(n, K_q) \\ &= T_0 \cap D_1(n, K_q) \\ &\subset T \cap D_1(n, K_q)\,. \end{aligned}$$

Since $T = \langle h_{ij}(u) \mid 1 \leq i \neq j \leq n,\ u \in K_q^\times \rangle$, we find $T = \langle h_{1j}(u) \mid 2 \leq j \leq n,\ u \in K_q^\times \rangle$ by the relation $h_{ij}(u) = h_{1i}(u^{-1}) h_{1j}(u)$. Hence, any element $h \in T \cap D_1(n, K_q)$ can be expressed as

$$h = h_{1,\ell_1}(u_1) h_{1,\ell_2}(u_2) \cdots h_{1,\ell_k}(u_k)$$

with $2 \leq \ell_1, \ldots, \ell_k \leq n$ and $u_1, \ldots, u_k \in K_q^\times$ satisfying

$$\prod_{i \in \Delta_j} u_i^{-1} = 1\,,$$

where $\Delta_j = \{i \mid 1 \leq i \leq n,\ \ell_i = j\}$ for each $2 \leq j \leq n$. We note $\{1, 2, \ldots, k\} = \Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_n$. Then we put $\delta_j = \prod_{i \in \Delta_j} u_{\ell_i}^{-1}$. By $h \in D_1(n, K_q)$, we see $\delta_j = 1$ for all $2 \leq j \leq n$. If we write

$$h = \begin{pmatrix} u & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

with $u \in K_q^\times$, then we obtain

$$u = u_1 u_2 \cdots u_k = u_1 u_2 \cdots u_k \delta_2 \delta_3 \cdots \delta_n\,.$$

Therefore, we can rewrite u as

$$u = u_1 u_2 \cdots u_k u_{i_1}^{-1} u_{i_2}^{-1} \cdots u_{i_k}^{-1}$$

with $\{1, 2, \ldots, k\} = \{i_1, i_2, \ldots, i_k\}$. If $i_\ell = 1$, then putting

$$v_1 = u_2 \cdots u_k u_{i_1}^{-1} \cdots u_{i_{\ell-1}}^{-1}$$

we have

$$u = [u_1, v_1] v_1 u_{i_{\ell+1}}^{-1} \cdots u_{i_k}^{-1} .$$

If $i_{\ell'} = 2$, then putting

$$v_2 = u_3 \cdots u_k u_{i_1}^{-1} \cdots u_{i_{\ell'-1}}^{-1} ,$$

where $u_{i_\ell}^{-1}$ is missing in case of $\ell < \ell'$, we have

$$u = [u_1, v_1][u_2, v_2] v_2 u_{i_{\ell'+1}}^{-1} \cdots u_{i_k}^{-1} ,$$

where $u_{i_\ell}^{-1}$ is missing in case of $\ell > \ell'$. Continuing this we finally reach

$$u = [u_1, v_1][u_2, v_2] \cdots [u_k, v_k]$$

for some $v_1, \ldots, v_k \in K_q^\times$. Thus, we have

$$T \cap D_1(n, K_q) \subset \begin{pmatrix} [K_q^\times, K_q^\times] & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} .$$

On the other hand, we can see

$$\begin{pmatrix} [u, v] & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = h_{12}(u) h_{12}(v) h_{12}(u^{-1} v^{-1})$$

for all $u, v \in K_q^\times$, which implies

$$\begin{pmatrix} [K_q^\times, K_q^\times] & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \subset E(n, K_q) .$$

Hence, we have

$$\begin{aligned} E(n, K_q) \cap D_1(n, K_q) &\subset T \cap D_1(n, K_q) \\ &\subset \begin{pmatrix} [K_q^\times, K_q^\times] & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \\ &\subset E(n, K_q) \cap D_1(n, K_q) \end{aligned}$$

and

$$E(n, K_q) \cap D_1(n, K_q) = \begin{pmatrix} [K_q^\times, K_q^\times] & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Therefore, we obtain the following.

THEOREM 15. *Notation is as above. Then, we have*

$$K_1(n, K_q) \simeq K_q^\times / [K_q^\times, K_q^\times]$$

for all $n \geq 2$.

Part III: Homomorphisms and isomorphisms

We will discuss a relationship between the group $\mathfrak{G}$ in Part I and the group E in Part II. Here we suppose again that F is a field of characteristic 0.

12. Homomorphisms of Lie algebras

There is an EALA L such that the core $L_c = \langle L_\alpha \mid \alpha \in R^\times \rangle$, which is the subalgebra of L generated by L_α for all $\alpha \in R^\times$, is a universal central extension of

$$\mathfrak{sl}_n(F_q) = \left\{ \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{n-1,n} \\ f_{n1} & \cdots & f_{n,n-1} & f_{nn} \end{pmatrix} \middle| \; f_{11} + f_{22} + \cdots + f_{nn} = 0 \right\}$$
$$\bigoplus \quad [F_q, F_q]\, I\,,$$

where $F_q = F[X_1, X_2]$ is a quantum torus over F defined by the relation $X_2 X_1 = q X_1 X_2$. That is, $L = (\mathfrak{sl}_n(F_q) \oplus \mathfrak{z}) \oplus \mathfrak{d}$, where $\mathfrak{sl}_n(F_q) \oplus \mathfrak{z}$ is a universal central extension of $\mathfrak{sl}_n(F_q)$ and

$\mathfrak{d}$ gives a certain derivation part, and hence the core L_c of L just coincides with $\mathfrak{sl}_n(F_q) \oplus \mathfrak{z}$. Therefore, we have the following exact sequence:

$$0 \longrightarrow \mathfrak{z} \longrightarrow L_c \xrightarrow{\psi} \mathfrak{sl}_n(F_q) \longrightarrow 0\,.$$

Then, we can choose $e_\alpha,\ e_{-\alpha} \in L$ for $\alpha \in R^\times$ satisfying

$$\psi(e_\alpha) = X_1^\ell X_2^m E_{ij}\,,$$
$$\psi(e_{-\alpha}) = X_2^{-m} X_1^{-\ell} E_{ji}\,,$$

where $\alpha = \dot\alpha + m\xi + \ell\eta$ with $\dot\alpha = \varepsilon_i - \varepsilon_j \in \Phi^+$ and $\ell, m \in \mathbf{Z}$. Then, for $a = (\dot\alpha, m) \in \Phi_a$, we see that each

$$s = \sum_{k=k_0}^{\infty} r_k e_{\dot\alpha + m\xi + k\eta} \in \Gamma_a$$

with $r_k \in F$ is corresponding to some

$$\sum_{k=k_0}^{\infty} r_k' X_1^k X_2^m E_{ij}$$

with $r_k' \in F$. Hence, Γ_a is corresponding to

$$K X_2^m E_{ij}\,.$$

To establish a relation between Part I and Part II, from now on, we shall take the core L_c instead of the full Lie algebra L. Even if we select L_c and make its completion $\hat{L}_c$, then we use the same notation for groups and group elements. For example, we can define $\Gamma_0'(L_c)$ for L_c in the same way as in Part I. Then, $\Gamma_0'(L_c)$ is corresponding to

$$\left\{ \begin{pmatrix} \sum_{i=1}^{\infty} r_i X_1^i & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^{\infty} r_i' X_1^i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sum_{i=1}^{\infty} r_i'' X_1^i \end{pmatrix} \in \mathfrak{sl}_n(K_q) \;\middle|\; r_i, r_i', \ldots, r_i'' \in F \right\}.$$

We note that as formal functions we have two bijective maps:

$$F[[X_1]]X_1 \;\underset{\text{Log}}{\overset{\text{Exp}}{\rightleftarrows}}\; 1 + F[[X_1]]X_1$$

between $F[[X_1]]X_1$ and $1 + F[[X_1]]X_1$. Then the group $\mathfrak{G}(\hat{L}_c)$ is generated by $x_a(s)$ and $x_0(s')$ for all $a \in \Phi_a$ and for all $s \in \Gamma_a(L_c)$ and $s' \in \Gamma_0'(L_c)$. We note that the same method works for $\mathfrak{G}(\hat{L}_c)$. Since the subalgebra $\hat{L}_c$ is invariant under the action of $\mathfrak{G}$, by restriction there is a natural homomorphism $\mathfrak{G} \to \mathfrak{G}|_{\hat{L}_c}$, whose image $\mathfrak{G}|_{\hat{L}_c}$ contains $\mathfrak{G}(\hat{L}_c)$.

13. Homomorphisms of groups

We discussed $E(n, K_q)$ as a subgroup of $GL(n, K_q)$. Now we need to consider the center, $Z(E(n, K_q))$, of $E(n, K_q)$. We note that every element of $Z(E(n, K_q))$ must be a scalar matrix with diagonal entries in the center, $Z(K_q^\times)$, of $K_q^\times$ (cf. Section 14). Usually we put $PE(n, K_q) = E(n, K_q)/Z(E(n, K_q))$ and $PGL(n, K_q) = GL(n, K_q)/Z$, where $Z = Z(GL(n, K_q))$. Since $E(n, K_q) \cap Z = Z(E(n, K_q))$, there is a natural injection of $PE(n, K_q)$ into $PGL(n, K_q)$. Then, the notion $x \bmod Z$ for $x \in E(n, K_q)$ makes sense as an element of $PE(n, K_q)$. Therefore, there exists a canonical surjective homomorphism, called $\bar{\psi}$, of $\mathfrak{G}(\hat{L}_c)$ onto $PE(n, K_q)$ with

$$\mathfrak{G}(\hat{L}_c) \xrightarrow{\bar{\psi}} PE(n, K_q)$$

using the above central extension $L_c \xrightarrow{\psi} \mathfrak{sl}_n(F_q)$ and their completions. Hence, mod Z we obtain the following correspondence:

$$x_a(\Gamma_a(L_c)) \longrightarrow x_a(K) \mod Z;$$

$$x_0(\Gamma_0'(L_c)) \longrightarrow \left\langle \left\{ \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r_n \end{pmatrix} \middle| \begin{array}{c} r_1, r_2, \ldots, r_n \\ \in (1 + F[[X_1]]X_1), \\ r_1 r_2 \cdots r_n = 1 \end{array} \right\}, \right.$$

$$\left. \left\{ \begin{pmatrix} e^f & 0 & \cdots & 0 \\ 0 & e^f & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^f \end{pmatrix} \middle| \begin{array}{c} f = f(X_1) \\ \in (F[[X_1]]X_1 \cap [K_q, K_q]), \\ e^f = Exp(f) \end{array} \right\} \right\rangle \mod Z;$$

$$\mathfrak{U}_a \longrightarrow U_a \mod Z;$$

$$x_\alpha(F) \longrightarrow x_{\dot{\alpha}}(F X_1^\ell X_2^m) \mod Z;$$

$$w_\alpha(F^\times) \longrightarrow w_{\dot{\alpha}}(F^\times X_1^\ell X_2^m) \mod Z;$$

$$h_\alpha(F^\times) \longrightarrow h_{\dot{\alpha}}(F^\times) \mod Z;$$

$$\begin{aligned}
\theta_{a,i}(F^\times) &\longrightarrow h_{\dot\alpha}(F^\times X_1^i X_2^m) h_{\dot\alpha}(X_2^m)^{-1} \quad \mathrm{mod} \quad Z\,;\\
\mathfrak{U}^\pm(\hat L_c) &\longrightarrow U^\pm \quad \mathrm{mod} \quad Z\,;\\
\mathfrak{T}_0(\hat L_c) &\longrightarrow T_0' = \bar\psi(\mathfrak{T}_0(\hat L_c)) \subset T_0 \quad \mathrm{mod} \quad Z\,;\\
\mathfrak{B}^\pm(\hat L_c) &\longrightarrow B'^\pm = \bar\psi(\mathfrak{B}^\pm(\hat L_c)) \subset B^\pm \quad \mathrm{mod} \quad Z\,;\\
\mathfrak{N}(\hat L_c) &\longrightarrow N' = \bar\psi(\mathfrak{N}(\hat L_c)) \subset N \quad \mathrm{mod} \quad Z\,;\\
\mathfrak{S}(\hat L_c) &\longrightarrow S \quad \mathrm{mod} \quad Z\,;\\
\mathfrak{Y}_{\pm a}(\hat L_c) &\longrightarrow Y_{\pm a} \quad \mathrm{mod} \quad Z\,.
\end{aligned}$$

We note that $Z(E(n, K_q))$ is contained in T_0 (cf. Section 14). Using the following Bruhat decompositions:

$$\mathfrak{G}(\hat L_c) = \bigcup_{w \in W_a} \mathfrak{B}^\pm(\hat L_c) w \mathfrak{B}^\pm(\hat L_c)$$

and

$$PE(n, K_q) = \bigcup_{w \in W_a} B^\pm w B^\pm \quad \mathrm{mod} \quad Z\,,$$

we observe the following:

$$(*) \qquad \begin{cases} T_0' = T_0 \\ B'^\pm = B^\pm \\ N' = N \end{cases}$$

modulo Z. To confirm this in another way, we will check the generators of T_0' explicitly. We need to write down the matrices of the generators of $\mathfrak{T}_0$ under the homomorphism $\bar\psi$. That is, T_0' is generated by, modulo Z,

$$\begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r_n \end{pmatrix}, \quad \begin{pmatrix} e^f & 0 & \cdots & 0 \\ 0 & e^f & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^f \end{pmatrix}, \quad \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & t_n \end{pmatrix},$$

$$\begin{pmatrix} X_1^{m_1} & 0 & \cdots & 0 \\ 0 & X_1^{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_1^{m_n} \end{pmatrix}, \quad \begin{pmatrix} q^{\ell_1} & 0 & \cdots & 0 \\ 0 & q^{\ell_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q^{\ell_n} \end{pmatrix}$$

for all $r_1, r_2, \ldots, r_n \in (1 + F[[X_1]]X_1)$ with $r_1 r_2 \cdots r_n = 1$, for all $f \in (F[[X_1]]X_1 \cap [K_q, K_q])$ with $e^f = Exp(f)$, for all $t_1, t_2, \ldots, t_n \in F^\times$ with $t_1 t_2 \cdots t_n = 1$, for all

$m_1, m_2, \ldots, m_n \in \mathbf{Z}$ with $m_1 + m_2 + \cdots + m_n = 0$, and for all $\ell_1, \ell_2, \ldots, \ell_n \in \mathbf{Z}$. This means that T_0' is generated by, modulo Z,

$$\begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q^\ell e^f & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

for all $u_1, u_2, \ldots, u_n \in K^\times$ with $u_1 u_2 \cdots u_n = 1$, for all $\ell \in \mathbf{Z}$ and for all $f \in (F[[X_1]]X_1 \cap [K_q, K_q])$. Then, we obtain

$$\begin{aligned} &\left\langle q^\ell e^f \,\middle|\, \ell \in \mathbf{Z},\ f \in F[[X_1]]X_1 \cap [K_q, K_q] \right\rangle \\ &= \left\langle q^\ell,\ e^{[fX_2^m, gX_2^{-m}]} \,\middle|\, \ell, m \in \mathbf{Z},\ f, g \in K,\ fg \in F[[X_1]] \right\rangle \\ &= \left\langle q^\ell,\ \frac{e^{f(X_1)g(q^m X_1)}}{e^{g(X_1)f(q^{-m}X_1)}} \,\middle|\, \ell, m \in \mathbf{Z},\ f(X_1), g(X_1) \in (1 + F[[X_1]]) \right\rangle \\ &= \left\langle q^\ell,\ \frac{e^{f(X_1)}}{e^{f(q^m X_1)}} \,\middle|\, \ell, m \in \mathbf{Z},\ f(X_1) \in (1 + F[[X_1]]) \right\rangle \\ &= [K_q^\times, K_q^\times]\,. \end{aligned}$$

Therefore, T_0' is generated by, modulo Z,

$$\begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

for all $u_1, u_2, \ldots, u_n \in K^\times$ with $u_1 u_2 \cdots u_n = 1$ and for all $v \in [K_q^\times, K_q^\times]$. This leads to the fact that T_0' coincides with T_0 modulo Z, which implies a proof of (*) by direct matrix computation without using Bruhat decompositions.

14. Remark on $Z(E(n, K_q))$

Let $z = (z_{ij}) \in Z(E(n, K_q))$. Then, z commutes with $x_{ij}(f)$ for all $1 \le i \ne j \le n$ and for all $f \in K_q$, which means that z is a scalar matrix λI for some nonzero central element

$\lambda \in K_q^\times \cap Z(K_q)$. Then, we see

$$z = \lambda I = \begin{pmatrix} \lambda^n & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} h_{12}(\lambda^{-(n-1)})h_{23}(\lambda^{-(n-2)}) \cdots h_{n-1,n}(\lambda^{-1})$$

and

$$\begin{pmatrix} \lambda^n & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \in E(n, K_q)\,.$$

Therefore, by the same discussion as in Section 11, we have $\lambda^n \in [K_q^\times, K_q^\times]$. On the other hand, by direct calculation, we obtain

$$[K_q^\times, K_q^\times] \subset \langle q \rangle\,(1 + F[[X_1]]X_1) \subset K^\times\,.$$

If we write $\lambda = fX_2^m$ for some $f \in K^\times$ and $m \in \mathbf{Z}$, then $\lambda^n = gX_2^{mn} \in K^\times$ with $g \in K^\times$. Hence, $m = 0$, which implies $z \in T_0$, and

$$Z(E(n, K_q)) \subset T_0\,.$$

In particular, both Bruhat decompositions:

$$\mathfrak{G}(\hat{L}_c) = \bigcup_{w \in W_a} \mathfrak{B}^\pm(\hat{L}_c) w \mathfrak{B}^\pm(\hat{L}_c)$$

and

$$PE(n, K_q) = \bigcup_{w \in W_a} \bar{B} w \bar{B}$$

are compatible with the homomorphism $\bar{\psi}$, where $\bar{B} = B$ modulo Z.

15. Isomorphisms of groups

We should confess first that we made some redundant discussion in the previous section. It might be so usefull to understand an explicit relationship between both groups $\mathfrak{G}(\hat{L}_c)$ and $PE(n, K_q)$, but we will show here that they are isomorphic.

From the following exact sequence

$$0 \longrightarrow \mathfrak{z} \longrightarrow L_c \xrightarrow{\psi} \mathfrak{sl}_n(F_q) \longrightarrow 0\,,$$

we constructed its completed version:

$$0 \longrightarrow \hat{\mathfrak{z}} \longrightarrow \hat{L}_c \xrightarrow{\hat{\psi}} \mathfrak{sl}_n(K_q) \longrightarrow 0\,,$$

which is also a central extension of Lie algebras over F. Then, the groups $\mathfrak{G}(\hat{L}_c)$ and $PE(n, K_q)$ are subgroups of $Aut(\hat{L}_c)$ and $Aut(\mathfrak{sl}_n(K_q))$ respectively. To study both subgroups, we need the next lemmas.

LEMMA 16. *Notation is as above. Then, we have* $\hat{L}_c = [\hat{L}_c, \hat{L}_c]$.

PROOF OF LEMMA 16. The central extension ψ can be reduced to the following skew-symmetric F-bilinear mapping (cf. [3], [16]):

$$\{\cdot\,,\,\cdot\} : F_q \times F_q \to \mathfrak{z}\,,$$

where $\mathfrak{z}$ is given by

$$\mathfrak{z} = Fz_0^{(1)} \oplus Fz_0^{(2)}$$

if q is generic, that is, q is not a root of 1, and

$$\mathfrak{z} = \left(\bigoplus_{v \in (\mathbf{Z}\nu)^2} Fz_v^{(1)} \right) \oplus Fz_0^{(2)}$$

if q is singular, that is, q is a root of 1 with ν as the minimal positive power satisfying $q^\nu = 1$, and the definition of the mapping { , } is given by

$$\{X_1^{r_1}X_2^{r_2}, X_1^{s_1}X_2^{s_2}\} = \begin{cases} r_1 z_0^{(1)} + r_2 z_0^{(2)} & \text{if } r_1 + s_1 = r_2 + s_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

if q is generic, and

$$\{X_1^{r_1}X_2^{r_2}, X_1^{s_1}X_2^{s_2}\} = \begin{cases} r_1 z_0^{(1)} + r_2 z_0^{(2)} & \text{if } \quad r_1 + s_1 = r_2 + s_2 = 0 \\ r_2 z^{(1)}_{(r_1+s_1,0)} & \text{if } \begin{cases} r_1 + s_1 \neq 0 \\ r_1 + s_1 \equiv 0 \ (\mathrm{mod}\ \nu) \\ r_2 + s_2 = 0 \end{cases} \\ \dfrac{r_1 s_2 - s_1 r_2}{r_2 + s_2} \ z^{(1)}_{(r_1+s_1, r_2+s_2)} & \text{if } \begin{cases} r_1 + s_1 \equiv 0 \ (\mathrm{mod}\ \nu) \\ r_2 + s_2 \neq 0 \\ r_2 + s_2 \equiv 0 \ (\mathrm{mod}\ \nu) \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

if q is singular. Then, $\hat{\mathfrak{z}} = \oplus_{m \in \mathbf{Z}\nu} \Gamma_{m\xi}(\mathfrak{z})$ and the above explicit construction of $\mathfrak{z}$ imply that $\hat{L}_c$ is perfect. In fact, for example, we suppose that q is singular, and we take

$$\hat{z} = \sum_{i=1}^{\infty} c_i z^{(1)}_{(i\nu, m)} \in \Gamma_{m\xi}(\mathfrak{z})\,.$$

Then we can compute in a standard way

$$\left[\sum_{i=1}^{\infty} \frac{c_i}{i\cdot\nu}\, X_1^{i\nu} h_{\dot\alpha}\ ,\ X_2^m h_{\dot\alpha}\right] = \sum_{i=1}^{\infty} \frac{c_i}{i\cdot\nu}\{X_1^{i\nu}, X_2^m\} = \hat{z}$$

if $m \neq 0$, and

$$\left[\sum_{i=1}^{\infty} c_i X_1^{i\nu} X_2 h_{\dot\alpha}\ ,\ X_2^{-1} h_{\dot\alpha}\right] = \sum_{i=1}^{\infty} c_i\{X_1^{i\nu} X_2, X_2^{-1}\} = \hat{z}$$

if $m = 0$. Here we choose one $\dot\alpha \in \Phi$ and $h_{\dot\alpha}$ is identified with its image $\psi(h_{\dot\alpha})$ by ψ. Hence, the main infinite sum parts of elements belonging to $\Gamma_{m\xi}(\mathfrak{z})$ can be written as elements of $[\hat{L}_c, \hat{L}_c]$. We also see that the remaining finite sum parts can be expressed as elements of $[\hat{L}_c, \hat{L}_c]$. Q.E.D.

Let

$$0 \longrightarrow \mathcal{Z} \longrightarrow \mathcal{L} \xrightarrow{\mu} \mathcal{L}' \longrightarrow 0$$

be an exact sequence of Lie algebras with a characteristic ideal $\mathcal{Z}$ of $\mathcal{L}$. Then, μ induces a natural group homomorphism, $\bar\mu$, of $Aut(\mathcal{L})$ into $Aut(\mathcal{L}')$.

LEMMA 17. *Notation is as above. Suppose that $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ and $\mathcal{Z}$ is the center of $\mathcal{L}$. Then, the homomorphism $\bar\mu$ is injective.*

PROOF OF LEMMA 17. Let $g \in \operatorname{Ker}\bar\mu$. Then, we note that for every $x \in \mathcal{L}$ there is a central element $z(x) \in \mathcal{Z}$ such that $g(x) = x + z(x)$. On the other hand, we can express x as a finite linear combination: $\sum_{i=1}^{k}[x_i, y_i]$ with $x_i, y_i \in \mathcal{L}$, since $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$. Hence we obtain $g(x) = x$ by

$$\begin{aligned} g([x_i, y_i]) &= [g(x_i), g(y_i)] \\ &= [x_i + z(x_i), y_i + z(y_i)] \\ &= [x_i, y_i]\,. \end{aligned}$$

This means $g = 1$ and $\operatorname{Ker}\bar\mu = 1$. Q.E.D.

Then, using these lemmas, we obtain the following.

THEOREM 18. *Notation is as above. Then, $\mathfrak{G}(\hat{L}_c)$ is isomorphic to $PE(n, K_q)$. In particular, $St(n, K_q)$ is a universal central extension of $\mathfrak{G}(\hat{L}_c)$.*

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