TOKYO J. MATH.
VOL. 29, NO. 2, 2006

Mixed Multiplicities of Modules over Noetherian Local Rings

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(Communicated by R. Tsushima)

Abstract. The relationship between mixed multiplicities of arbitrary ideals in local rings and Hilbert-Samuel multiplicities was solved by Viêt in [8]. In this paper, we extend some important results of Viêt in [8, 9, 10, 11] to modules. We build the concept of (FC)-sequences of modules and use this notion to study reductions of ideals with respect to modules, mixed multiplicities of modules and multiplicities of Rees modules.

1. Introduction

Throughout this paper, $(A, \mathfrak{m})$ denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue field $k = A/\mathfrak{m}$; M a finitely generated A-module with Krull dimension $\dim M = d > 0$.

Let J be an $\mathfrak{m}$-primary ideal and $(I_1, I_2, \ldots, I_s)$ be a set of ideals of A such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann} M}$. Set

$$\mathfrak{a} : \mathfrak{b}^\infty = \bigcup_{n \geq 0} (\mathfrak{a} : \mathfrak{b}^n);\ \ M^* = \frac{M}{0_M : I^\infty} \quad \text{and} \quad q = \dim M^* .$$

Then using the same argument as in the proof of Proposition 3.1 of Viêt [8], we have that there exists a positive integer u such that the Bhattacharya function [2]

$$l_A\left(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M}\right)$$

is a polynomial of degree $q - 1$ for all values of $n, n_1, n_2, \ldots, n_s \geq u$.

Now, we write the terms of total degree $q - 1$ in this polynomial in the form

$$B(n, n_1, \ldots, n_s) = \sum_{d_0 + d_1 + \ldots + d_s = q-1} e_A(J^{[d_0+1]}, I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) \frac{n^{d_0} n_1^{d_1} \cdots n_s^{d_s}}{d_0! d_1! \cdots d_s!} ,$$

Received March 7, 2005

then $e_A(J^{[d_0+1]}, I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ are non-negative integers and are called the mixed multiplicity of the set of ideals $(J, I_1, \ldots, I_s)$ with respect to M of the type $(d_0 + 1, d_1, \ldots, d_s)$ ($B(n, n_1, \ldots, n_s)$ is called the Bhattacharya polynomial [2]).

It has long been known that mixed multiplicity is an important object of algebraic geometry and commutative algebra. In the case that local rings, first, Risler and Teissier in 1973 showed that mixed multiplicities of two $\mathfrak{m}$-primary ideals are multiplicities of an ideal generated by elements chosen sufficiently generally [4]. Rees in 1984 proved that each mixed multiplicity of a set of $\mathfrak{m}$-primary ideals is the multiplicity of a joint reduction of them [3]. In general, the relationship between mixed multiplicities of a set of arbitrary ideals and Hilbert-Samuel multiplicities was solved by Viêt in 2000 [8]. In answer to this problem, he built a sequence of elements called a (FC)-sequence. The results of Viêt in [8, 9, 10, 11] showed that (FC)-sequences carry the important information on mixed multiplicities and reductions of ideals ...

The aim of this note is to show that one can extend many results of Viêt in [8, 9, 10, 11] to modules. Although most of the argument of [8, 9, 10, 11] can apply to modules, there are some critical places which have to be dealt with.

The important key to statements and proofs of our results is the concept of (FC)-sequences in modules. As in [8], the proof of the existence of (FC)-sequences in modules is based on our generalized Rees' Lemma (Lemma 2.2, Section 2).

This paper is divided into 4 sections.

In Section 2, first, we construct the concept of (FC)-sequences in modules. Next, we show a lemma called generalized Rees' Lemma (Lemma 2.2) and use this lemma to prove the existence of weak-(FC)-sequences (Proposition 2.3). Last, we get a result concerning reductions of ideals with respect to modules: maximal weak-(FC)-sequences of modules generate generalized joint reductions of ideals with respect to modules (Theorem 2.9).

In Section 3, we link the mixed mutiplicities of modules and Hilbert-Samuel multiplicity via (FC)-sequences of modules. Main results of this section are Theorem 3.4, Theorem 3.6.

In Section 4, as applying the results on mixed multiplicities of Section 3, we establish multiplicity formulas of Rees modules with repect to arbitrary ideals (Theorem 4.2). In particular, we get interesting results concerning multiplicities of Rees modules of $\mathfrak{m}$-primary ideals (Theorem 4.3, Theorem 4.4).

2. (FC)-sequences of modules

In this section, first, we build the concept of (FC)-sequences in modules and show some important properties of (FC)-sequences. Last, we use these sequences to study reductions of ideals with respect to modules.

We now turn to the definition of (FC)-sequences of modules.

DEFINITION 2.1. Let $U = (I_1, \ldots, I_s)$ be a set of ideals of A such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$. Set $M^* = \dfrac{M}{0_M : I^\infty}$. We say that an element $x \in A$ is an

(FC)-element of M with respect to U if there exists an ideal I_i of U and a positive integer n'_i such that

(FC$_1$) : $x \in I_i \setminus \mathfrak{m} I_i$ and

$$I_1^{n_1} \cdots I_s^{n_s} M^* \cap x M^* = x I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_s^{n_s} M^*$$

for all $n_i \geq n'_i$ and all non-negative integers $n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_s$.

(FC$_2$) : $0_M : x \subseteq 0_M : I^\infty$.

(FC$_3$) : $\dim\left(\dfrac{M}{xM : I^\infty}\right) = \dim M^* - 1$.

We call x a weak-(FC)-element of M with respect to U if x satisfies the conditions (FC$_1$) and (FC$_2$) with respect to U.

Let $x_1, \ldots, x_t$ be a sequence in A. For each $i = 0, 1, \ldots, t-1$, set $\overline{M} = \dfrac{M}{(x_1, \ldots, x_i)M}$; $\bar{A} = \dfrac{A}{(x_1, \ldots, x_i)}$ and $\bar{I}_1 = I_1 \bar{A}, \ldots, \bar{I}_s = I_s \bar{A}$. Let $\bar{x}_{i+1}$ denote the image of x_{i+1} in $\bar{A}$. Then

(i) $x_1, \ldots, x_t$ is said to be an (FC)-sequence of M with respect to U if $\bar{x}_{i+1}$ is an (FC)-element of $\overline{M}$ with respect to $(\bar{I}_1, \ldots, \bar{I}_s)$ for $i = 0, 1, \ldots, t-1$.

(ii) $x_1, \ldots, x_t$ is called a weak-(FC)-sequence of M with respect to U if $\bar{x}_{i+1}$ is a weak-(FC)-element of $\overline{M}$ with respect to $(\bar{I}_1, \ldots, \bar{I}_s)$ for $i = 0, 1, \ldots, t-1$.

The following lemma will play a crucial role for showing the existence of weak-(FC)-sequences.

LEMMA 2.2 (Generalized Rees'Lemma). *Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue field $k = A/\mathfrak{m}$. Let M be a finitely generated A-module and $U = (I_1, \ldots, I_s)$ be a set of ideals of A. Let Σ be a finite set of prime ideals non containing $I_1 \cdots I_s$. Then for each $i = 1, \ldots, s$, there exists an element $x_i \in I_i \setminus \mathfrak{m} I_i$, x_i not contained in any prime ideal in Σ, and a positive integer k_i such that*

$$I_1^{r_1} \cdots I_i^{r_i} \cdots I_s^{r_s} M \cap x_i M = x_i I_1^{r_1} \cdots I_{i-1}^{r_{i-1}} I_i^{r_i - 1} I_{i+1}^{r_{i+1}} \cdots I_s^{r_s} M$$

for any $r_i \geq k_i$ and all non-negative integers $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_s$.

PROOF. Set

$$R(U) = \bigoplus_{r_1, \ldots, r_s \in Z} I_1^{r_1} \cdots I_s^{r_s} t_1^{r_1} \cdots t_s^{r_s} \quad \text{and} \quad M(U) = \bigoplus_{r_1, \ldots, r_s \in Z} I_1^{r_1} \cdots I_s^{r_s} M t_1^{r_1} \cdots t_s^{r_s},$$

where $(t_1, \ldots, t_s)$ is a set of indeterminates and $I_i^{r_i} = A$ for $r_i \leq 0$. Then $R(U)$ is a Noetherian graded ring and $M(U)$ is a Noetherian graded $R(U)$-module. Set $u_1 = t_1^{-1}, \ldots, u_s = t_s^{-1}$. It is easily seen that $u_1 \cdots u_s$ is a non-zero-divisor in $M(U)$. By the corollary of Lemma 2.7 in [3], the set of prime associated with $(u_1 \cdots u_s)^n M(U)$ is independent of n and so is finite. We divide this set into two subsets: $\mathfrak{S}_1$ consisting of those containing $I_i t_i$ and $\mathfrak{S}_2$ those that do not.

From $I_i/\mathfrak{m}I_i$ is a vector space over the infinite field k and the sets Σ, $\mathfrak{S}_2$ are both finite, we can choose $x_i \in I_i \setminus \mathfrak{m}I_i$ such that x_i is not contained in any prime ideal belonging to Σ and $x_i t_i$ is not contained in any prime ideal belonging to $\mathfrak{S}_2$.

Set

$$M_n = \frac{(u_1 \cdots u_s)^n M(U) : x_i t_i}{(u_1 \cdots u_s)^n M(U)} .$$

We will show that M_n is annihilated by $(I_i t_i)^N$ if N is sufficiently large. If $P \in \mathrm{Ass}_{R(U)}(M_n)$, then there exists $z \in (u_1 \cdots u_s)^n M(U) : x_i t_i$ such that $P = (u_1 \cdots u_s)^n M(U) : z$. Since $x_i t_i \in P$, we have $P \in \mathfrak{S}_1$. Consequently, $I_i t_i$ is contained in P. This implies that $I_i t_i$ is contained in $\sqrt{\mathrm{Ann}_{R(U)}(M_n)}$. Since I_i is finitely generated, there exists an integer $N > 0$ such that

$$(I_i t_i)^N \subseteq \mathrm{Ann}_{R(U)}(M_n) .$$

In addition, M_n is a finitely generated graded $R(U)$-module, this implies that if r_i is sufficiently large then any element of M_n of degree $(r_1, \ldots, r_s)$ is zero.

Now, let $B(M)$ denote the submodule of $M(U)$ consisting of all finite sums:

$$\sum c_{r_1 \ldots r_s} t_1^{r_1} \cdots t_s^{r_s}, \text{ where } c_{r_1 \ldots r_s} \in x_i M \cap I_1^{r_1} \cdots I_s^{r_s} M .$$

Then $B(M)$ has a finite generating set consisting of elements of the form $x_i b_i t_1^{r_1} \cdots t_s^{r_s}$, where $b_i \in M$. We can find an integer q such that

$$(u_1 \cdots u_s)^q b_i t_1^{r_1} \cdots t_{i-1}^{r_{i-1}} t_i^{r_i - 1} t_{i+1}^{r_{i+1}} \cdots t_s^{r_s} \in M(U)$$

for all elements $x_i b_i t_1^{r_1} \cdots t_s^{r_s}$. Hence

$$B(M) \subseteq x_i t_i M(U) : (u_1 \cdots u_s)^q .$$

Suppose that $z t_1^{r_1} \cdots t_s^{r_s} \in B(M)$, where $z \in x_i M \cap I_1^{r_1} \cdots I_s^{r_s} M$. We have

$$(u_1 \cdots u_s)^q z t_1^{r_1} \cdots t_s^{r_s} = x_i t_i w ,$$

where w is a homogeneous element of $M(U)$ whose i-th degree equal to $r_i - q - 1$. By the first part of the proof, w will belong to $(u_1 \cdots u_s)^q M(U)$ for sufficiently large r_i. Since the generating set of $B(M)$ is finite, we can choose a positive integer k_i such that if $r_i \geq k_i$, then $w \in (u_1 \cdots u_s)^q M(U)$. Hence $w = (u_1 \cdots u_s)^q w'$ for $w' \in M(U)$. Note that

$$(u_1 \cdots u_s)^q z t_1^{r_1} \cdots t_s^{r_s} = x_i t_i w,\ (u_1 \cdots u_s)^q z t_1^{r_1} \cdots t_s^{r_s} = x_i t_i (u_1 \cdots u_s)^q w' .$$

Since $(u_1 \cdots u_s)^q$ is a non-zero-divisor, $z t_1^{r_1} \cdots t_s^{r_s} = x_i t_i w'$ and so $z t_1^{r_1} \cdots t_s^{r_s} \in x_i t_i M(U)$. Therefore, $z \in x_i I_1^{r_1} \cdots I_{i-1}^{r_{i-1}} I_i^{r_i - 1} I_{i+1}^{r_{i+1}} \cdots I_s^{r_s} M$. Hence if $r_i \geq k_i$, then

$$x_i M \cap I_1^{r_1} \cdots I_s^{r_s} M \subseteq x_i I_1^{r_1} \cdots I_{i-1}^{r_{i-1}} I_i^{r_i - 1} I_{i+1}^{r_{i+1}} \cdots I_s^{r_s} M \subseteq x_i M \cap I_1^{r_1} \cdots I_s^{r_s} M$$

for all non-negative integers $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_s$. That means for any $r_i \geq k_i$, we get

$$I_1^{r_1} \cdots I_s^{r_s} M \cap x_i M = x_i I_1^{r_1} \cdots I_{i-1}^{r_{i-1}} I_i^{r_i - 1} I_{i+1}^{r_{i+1}} \cdots I_s^{r_s} M$$

for all non-negative integers $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_s$.
The proof is complete. ∎

The following proposition will show the existence of weak-(FC)-sequences.

PROPOSITION 2.3. *Let $(I_1, \dots, I_s)$ be a set of ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$. Then for any $1 \leq i \leq s$, there exists a weak-(FC)-element $x_i \in I_i$ with respect to $(I_1, \dots, I_s)$.*

PROOF. Set $\mathcal{F} = \mathrm{Ass}_A\left(\dfrac{M}{0_M : I^\infty}\right)$. Since I is not contained in $\sqrt{\mathrm{Ann}M}$, we have $\mathcal{F} \neq \emptyset$. It is easily seen that $\mathcal{F}$ is finite and $\mathcal{F} = \{P \in \mathrm{Ass}(M) \mid P \not\supseteq I\}$. By Lemma 2.2, for each $i = 1, \dots, s$, there exists an element $x_i \in I_i \setminus \mathfrak{m} I_i$ such that x_i satisfies the condition (FC_1) and $x_i \notin P$ for all $P \in \mathcal{F}$. Since $x_i \notin P$ for any $P \in \mathcal{F}$, x_i also satisfies the condition (FC_2). Hence x_i is a weak-(FC)-element of M with respect to $(I_1, \dots, I_s)$. ∎

(FC)-sequences in modules have some important properties as follows.

PROPOSITION 2.4. *If $x \in I_i$ is a weak-(FC)-element of M with respect to $(I_1, \dots, I_s)$, then*

$$I_1^{n_1} \cdots I_s^{n_s} M \cap xM = x I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_s^{n_s} M \tag{2.1}$$

for all sufficiently large $n_1, \dots, n_s$.

PROOF. Since x satisfies the condition (FC_1), we have

$$(I_1^{n_1} \cdots I_s^{n_s} M + 0_M : I^\infty) \cap (xM + 0_M : I^\infty) = x I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M + (0_M : I^\infty)$$

or $I_1^{n_1} \cdots I_s^{n_s} M \cap (xM + 0_M : I^\infty) + (0_M : I^\infty) = x I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M + (0_M : I^\infty)$ for all sufficiently large $n_1, \dots, n_s$. Since

$$I_1^{n_1} \cdots I_s^{n_s} M \cap xM + (0_M : I^\infty) \subseteq I_1^{n_1} \cdots I_s^{n_s} M \cap (xM + 0_M : I^\infty) + (0_M : I^\infty)$$

and $x I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M + (0_M : I^\infty) \subseteq I_1^{n_1} \cdots I_s^{n_s} M \cap xM + (0_M : I^\infty)$, it follows that

$$x I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M + (0_M : I^\infty) = I_1^{n_1} \cdots I_s^{n_s} M \cap xM + (0_M : I^\infty)$$

for all sufficiently large $n_1, \dots, n_s$. Therefore,

$$\begin{aligned} & x I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M + (0_M : I^\infty) \cap I_1^{n_1} \cdots I_s^{n_s} M \\ &\quad = I_1^{n_1} \cdots I_s^{n_s} M \cap xM + (0_M : I^\infty) \cap I_1^{n_1} \cdots I_s^{n_s} M . \end{aligned}$$

for all sufficiently large $n_1, \dots, n_s$. By Artin-Rees lemma, $(0_M : I^\infty) \cap I_1^{n_1} \cdots I_s^{n_s} M = 0_M$ for all sufficiently large $n_1, \dots, n_s$. Hence we get (2.1) for all sufficiently large $n_1, \dots, n_s$. ■

In the case that d $\mathfrak{m}$-primary ideals, we have the following interesting result.

PROPOSITION 2.5. *Let $U = (I_1, \dots, I_d)$ be a set of $\mathfrak{m}$-primary ideals. Then there exists a weak-(FC)-sequence $x_1, \dots, x_d$ in $\bigcup_{i=1}^d I_i$ with respect to U. And if $x_1, \dots, x_d$ is a weak-(FC)-sequence in $\bigcup_{i=1}^d I_i$ with respect to U, then $x_1, \dots, x_{d-1}$ is an (FC)-sequence in $\bigcup_{i=1}^d I_i$ with respect to U.*

PROOF. Set $I = I_1 \cdots I_d$ and $M^* = \dfrac{M}{0_M : I^\infty}$. By Proposition 2.3, there exists a weak-(FC)-element $x_1 \in \bigcup_{i=1}^d I_i$ with respect to U. From $d = \dim M = \operatorname{ht}(I + \operatorname{Ann}M/\operatorname{Ann}M) > 0$ and $I = I_1 \cdots I_d$ is $\mathfrak{m}$-primary, it is easily seen that $\dim M^* = \dim M$. We will show that

$$\dim(M/x_1M) = d - 1\,.$$

Since x_1 satisfies the condition (FC_2), x_1 is a non zero-divisor in M^*. Thus, we have

$$\dim(M^*/x_1M^*) = \dim M^* - 1 = d - 1\,.$$

It follows that

$$d - 1 \ \le \ \dim(M/x_1M) \ \le \ d\,.$$

If $\dim(M/x_1M) = d$, then x_1 belongs to some minimal prime ideal of $\operatorname{Ass}M^*$. This implies that x_1 is a zero-divisor in M^* (contradiction). Therefore, $\dim(M/x_1M) = d - 1$. By induction, there exists a weak-(FC)-sequence $x_1, \dots, x_d$ in $\bigcup_{i=1}^d I_i$ with respect to U. And if $x_1, \dots, x_d$ is a weak-(FC)-sequence in $\bigcup_{i=1}^d I_i$ with respect to U, then

$$\begin{aligned}\dim\left(\frac{M}{(x_1, \dots, x_t)M : I^\infty}\right) &= \dim\left[\frac{M/(x_1, \dots, x_t)M}{((x_1, \dots, x_t)M : I^\infty)/(x_1, \dots, x_t)M}\right] \\ &= \dim\left(\frac{M}{(x_1, \dots, x_t)M}\right) = d - t\end{aligned}$$

for all $t < d$. Hence $x_1, \dots, x_{d-1}$ is an (FC)-sequence of M in $\bigcup_{i=1}^d I_i$ with respect to U. ■

PROPOSITION 2.6. *Let $U = (I_1, \dots, I_s)$ be a set of ideals. Assume that $x \in I_i$ is a weak-(FC)-element of M with respect to U. Then there exists an integer c such that*

$$[I_1^{n_1} \cdots I_s^{n_s} M : xA] \cap I_1^{n_1} \cdots I_i^{c} \cdots I_s^{n_s} M = I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M$$

for all $n_j > c,\ j = 1, \dots, s$.

PROOF. Set $I = I_1 \cdots I_s$. By Proposition 2.4 and Artin-Rees lemma, there exists an integer c such that

$$I_1^{n_1} \cdots I_s^{n_s} M \cap xM = xI_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_s^{n_s} M$$

and $(0_M : I^\infty) \cap I_1^{n_1} \cdots I_i^c \cdots I_s^{n_s} M = 0_M$ for all $n_j > c,\ \ j = 1, \ldots, s$. Consequently, we have

$$\begin{aligned}
&[I_1^{n_1} \cdots I_s^{n_s} M : xA] \cap I_1^{n_1} \cdots I_i^c \cdots I_s^{n_s} M \\
&\quad = [(I_1^{n_1} \cdots I_s^{n_s} M \cap xM) : xA] \cap I_1^{n_1} \cdots I_i^c \cdots I_s^{n_s} M \\
&\quad = [xI_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M : xA] \cap I_1^{n_1} \cdots I_i^c \cdots I_s^{n_s} M \\
&\quad = [I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M + 0_M : x] \cap I_1^{n_1} \cdots I_i^c \cdots I_s^{n_s} M \\
&\quad \subseteq [I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M + 0_M : I^\infty] \cap I_1^{n_1} \cdots I_i^c \cdots I_s^{n_s} M \\
&\quad = I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M + (0_M : I^\infty) \cap I_1^{n_1} \cdots I_i^c \cdots I_s^{n_s} M \\
&\quad = I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M
\end{aligned}$$

for all $n_j > c,\ \ j = 1, \ldots, s$. Hence

$$[I_1^{n_1} \cdots I_s^{n_s} M : xA] \cap I_1^{n_1} \cdots I_i^c \cdots I_s^{n_s} M = I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s} M$$

for all $n_j > c,\ \ j = 1, \ldots, s$. ∎

It is easily seen that if J is $\mathfrak{m}$-primary ideal and I arbitrary ideal, then $0_M : (JI)^\infty = 0_M : I^\infty$ and $xM : (JI)^\infty = xM : I^\infty$. This fact immediately gives the following result.

LEMMA 2.7. *Let $(I_1, \ldots, I_s)$ be a set of ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\operatorname{Ann} M}$ and $J_1, \ldots, J_t$ be $\mathfrak{m}$-primary ideals. Suppose that $x \in I_i$ be an (FC)-element of M with respect to $(J_1, \ldots, J_t, I_1, \ldots, I_s)$. Then we can replace the condition* (FC_2) *by the condition*

$$0_M : x \subseteq 0_M : I^\infty$$

and the condition (FC_3) *by the condition*

$$\dim\left(\frac{M}{xM : I^\infty}\right) = \dim\left(\frac{M}{0_M : I^\infty}\right) - 1\,.$$

Viêt in [11] gave the concept of generalized joint reductions of ideals in local rings. This notion is a generalization of joint reductions of Rees in [3]. Now, we extend generalized joint reductions of ideals in [11] to modules as follows:

DEFINITION 2.8. Let $I_1, \ldots, I_s$ be ideals. A set of ideals $(\Im_1, \ldots, \Im_t)$ such that $\Im_i \subseteq I_i, i = 1, \ldots, t \leq s$ is called a generalized joint reduction of $I_1, \ldots, I_s$ with respect to M if

$$I_1^{n_1+1} \cdots I_s^{n_s+1} M = \sum_{j=1}^{t} \Im_j I_1^{n_1+1} \cdots I_j^{n_j} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$.

The relationship between maximal weak-(FC)-sequences and generalized joint reductions was determined by Theorem 3.4 of [11] in local rings. We extend this result to modules by the following theorem.

THEOREM 2.9. *Let $I_1, \ldots, I_s$ be ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$ and J be an $\mathfrak{m}$-primary ideal. Suppose that*

$$\Im_1 = (x_{11}, \ldots, x_{1m}) \subseteq I_1$$

$$\Im_2 = (x_{21}, \ldots, x_{2n}) \subseteq I_2$$

$$\cdots$$

$$\Im_t = (x_{t1}, \ldots, x_{tp}) \subseteq I_t$$

and $x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2n}, \ldots, x_{t1}, \ldots, x_{tp}$ is a maximal weak-(FC)-sequence of M in $\bigcup_{i=1}^{s} I_i$ with respect to $(J, I_1, \ldots, I_s)$. Then the following statements hold.

(i) *For any $k \leq t$, we have*

$$(\Im_1, \ldots, \Im_k)M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = \sum_{j=1}^{k} \Im_j I_1^{n_1+1} \cdots I_j^{n_j} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$.

(ii) *$\Im_1, \ldots, \Im_t$ is a generalized joint reduction of $I_1, \ldots, I_s$ with respect to M.*

PROOF. The proof of (i): Using induction on $k \leq t$. For $k = 1$, we will show that

$$(x_{11}, \ldots, x_{1i})M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = (x_{11}, \ldots, x_{1i}) I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$ by induction on $i \leq m$. For $i = 0$, the result is trivial. Suppose that the result is true for $i - 1 \geq 0$. As the next step, we will prove that the result is also true for $i \leq m$. Set $N = (x_{11}, \ldots, x_{1(i-1)})M : I^\infty$. Since $x_{11}, \ldots, x_{1i}$ is a weak-(FC)-sequence of M in I_1 with respect to $(J, I_1, \ldots, I_s)$,

$$(x_{1i}M + N) \cap (I_1^{n_1+1} \cdots I_s^{n_s+1} M + N) = x_{1i} I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M + N$$

for all large $n_1, \ldots, n_s$. Therefore,

$$(x_{1i}M + N) \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M$$

$$= I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap (I_1^{n_1+1} \cdots I_s^{n_s+1} M + N) \cap (x_{1i} M + N)$$
$$= I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap (x_{1i} I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M + N)$$
$$= x_{1i} I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M + I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap N.$$

for all large $n_1, \ldots, n_s$. By Artin-Rees lemma, we have

$$I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap N \subseteq I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap (x_{11}, \ldots, x_{1(i-1)}) M$$

for all large $n_1, \ldots, n_s$. By inductive assumption,

$$I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap N \subseteq (x_{11}, \ldots, x_{1(i-1)}) I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$. Consequently,

$$I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap N = (x_{11}, \ldots, x_{1(i-1)}) I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$. Hence we get

$$(x_{1i} M + N) \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M$$
$$= x_{1i} I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M + (x_{11}, \ldots, x_{1(i-1)}) I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M$$
$$= (x_{11}, \ldots, x_{1i}) I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$. By Artin-Rees lemma,

$$(x_{1i} M + N) \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = (x_{11}, \ldots, x_{1i}) M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$. Thus,

$$(x_{11}, \ldots, x_{1i}) M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = (x_{11}, \ldots, x_{1i}) I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$ and $i \leq m$. From this it follows that

$$(x_{11}, \ldots, x_{1m}) M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = (x_{11}, \ldots, x_{1m}) I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M$$

or

$$\Im_1 M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = \Im_1 I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$. The result is proved for $k = 1$.

Suppose that the result has been proved for $k - 1$. As the next step, we claim that the result is true for k. Set $\mathfrak{N} = (\Im_1, \ldots, \Im_{k-1}) M : I^{\infty}$. By Artin-Rees lemma, we have

$$\mathfrak{N} \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M \subseteq I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap (\Im_1, \ldots, \Im_{k-1}) M$$

and

$$(\mathfrak{N} + \Im_k M) \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M \subseteq I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap (\Im_1, \ldots, \Im_k) M$$

for all large $n_1, \ldots, n_s$. Hence

$$\mathfrak{N} \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap (\Im_1, \ldots, \Im_{k-1}) M$$

and

$$(\mathfrak{N} + \Im_k M) \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap (\Im_1, \ldots, \Im_k) M \tag{2.2}$$

for all large $n_1, \ldots, n_s$. By the result is true for $k = 1$,

$$(\mathfrak{N} + \Im_k M) \cap (I_1^{n_1+1} \cdots I_s^{n_s+1} M + \mathfrak{N}) = \Im_k (I_1^{n_1+1} \cdots I_k^{n_k} \cdots I_s^{n_s+1}) M + \mathfrak{N}$$

for all large $n_1, \ldots, n_s$. From the above facts, we get

$$\begin{aligned}
(\mathfrak{N} + \Im_k M) \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M &= I_1^{n_1+1} \cdots I_s^{n_s+11} M \cap (\mathfrak{N} + \Im_k M) \cap (I_1^{n_1+1} \cdots I_s^{n_s+1} M + \mathfrak{N}) \\
&= I_1^{n_1+1} \cdots I_s^{n_s+1} M \cap (\Im_k I_1^{n_1+1} \cdots I_k^{n_k} \cdots I_s^{n_s+1} M + \mathfrak{N}) \\
&= \Im_k I_1^{n_1+1} \cdots I_k^{n_k} \cdots I_s^{n_s+1} M + \mathfrak{N} \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M \\
&= \Im_k I_1^{n_1+1} \cdots I_k^{n_k} \cdots I_s^{n_s+1} M + (\Im_1, \ldots, \Im_{k-1}) M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M
\end{aligned} \tag{2.3}$$

for all large $n_1, \ldots, n_s$. By inductive asumption applied to $(k-1)$, we see that

$$(\Im_1, \ldots, \Im_{k-1}) M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = \sum_{j=1}^{k-1} \Im_j I_1^{n_1+1} \cdots I_j^{n_j} \cdots I_s^{n_s+1} M \tag{2.4}$$

for all large $n_1, \ldots, n_s$. Hence by (2.2), (2.3) and (2.4), we get

$$(\Im_1, \ldots, \Im_k) M \cap I_1^{n_1+1} \cdots I_s^{n_s+1} M = \sum_{j=1}^{k} \Im_j I_1^{n_1+1} \cdots I_j^{n_j} \cdots I_s^{n_s+1} M$$

for all large $n_1, \ldots, n_s$.

The proof of (ii): Since $x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2n}, \ldots, x_{t1}, \ldots, x_{tp}$ is a maximal weak-(FC)-sequence of M in $\cup_{i=1}^{s} I_i$ with respect to $(J, I_1, \ldots, I_s)$. Hence by Proposition 2.3, we have

$$I_1^{n_1+1} \cdots I_s^{n_s+1} M \subseteq (\Im_1, \ldots, \Im_t) M$$

for all large $n_1, \ldots, n_s$. By applying the part (i), we get (ii).

The proof of Theorem 2.9 is complete. ∎

In the case that all ideals are $\mathfrak{m}$-primary, we have the following collorary.

COROLLARY 2.10. *Let $I_1, \ldots, I_d$ be $\mathfrak{m}$-primary ideals, where $d = \dim M > 0$. Then there exists a set of elements $x_1, \ldots, x_d$, where $x_i \in I_i$ such that $x_1, \ldots, x_d$ is a joint reduction of $I_1, \ldots, I_d$ with respect to M.*

PROOF. By Proposition 2.3, there exists a maximal weak-(FC)-sequence $x_1, \ldots, x_d$ with respect to $(I_1, \ldots, I_d)$, where $x_i \in I_i$. Hence by Theorem 2.9, we get Corollary 2.10. ∎

3. Mixed multiplicities of modules

In this section, we will give some important results on mixed multiplicities of modules by using the concept of (FC)-sequences. Most of all results are based on extending the results of Viêt in [8, 9, 10, 11] to modules.

By applying the argument as in [8, Proposition 3.1] to modules, we get the following proposition.

PROPOSITION 3.1. *Let $(I_1, \ldots, I_s)$ be a set of ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann} M}$ and let J be an $\mathfrak{m}$-primary ideal. Set $M^* = \dfrac{M}{0_M : I^\infty}$. Then*

(i) $l_A\left(\dfrac{J^n I_1^{n_1} \cdots I_s^{n_s} M}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M}\right) = l_A\left(\dfrac{J^n I_1^{n_1} \cdots I_s^{n_s} M^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M^*}\right)$ *for all sufficiently large* $n, n_1, \ldots, n_s$.

(ii) $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M) = e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M^*)$.

To establish the relationship between mixed multiplicities of modules and Hilbert-Samuel multiplicities, we need the following lemma.

LEMMA 3.2. *Let $(I_1, \ldots, I_s)$ be a set of ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann} M}$ and let J be an $\mathfrak{m}$-primary ideal. Set $M^* = \dfrac{M}{0_M : I^\infty}$ and $q = \dim M^*$. Then the following statements hold.*

(i) $e_A(J^{[k_0+1]}, I_1^{[0]}, \ldots, I_s^{[0]}; M) \neq 0$ *and* $e_A(J^{[k_0+1]}, I_1^{[0]}, \ldots, I_s^{[0]}; M) = e(J; M^*)$.

(ii) *If* $\mathrm{ht}(I + \mathrm{Ann} M/\mathrm{Ann} M) > 0$, *then* $e_A(J^{[k_0+1]}, I_1^{[0]}, \ldots, I_s^{[0]}; M) = e(J; M)$.

PROOF. The proof of (i): By using the argument as in [8, Lemma 3.2], we have

$$e(J; I^u M^*) = e_A(J^{[q]}, I_1^{[0]}, \ldots, I_s^{[0]}; M), \tag{2.5}$$

where u being an integer such that $l_A\left(\dfrac{J^{n_0} I_1^{n_1} \cdots I_s^{n_s} M}{J^{n_0+1} I_1^{n_1} \cdots I_s^{n_s} M}\right)$ is a polynomial of degree $q - 1$ for all values of $n_0, n_1, \ldots, n_s \geq u$. Consider the exact sequence

$$0 \to I^u M^* \to M^* \to M^*/I^u M^* \to 0 .$$

Since I is not contained in $\sqrt{\mathrm{Ann} M}$, there exists an element $x \in I^u$ such that x is a non-zero-divisor in M^*. Therefore,

$$\dim(I^u M^*) = \dim M^* > \dim(M^*/I^u M^*) .$$

Hence by [1], we have

$$e(J; I^u M^*) = e(J; M^*)\,. \tag{2.6}$$

By (2.5) and (2.6), we get (i).

The proof of (ii): Consider the exact sequence

$$0 \to 0_M : I^\infty \to M \to M^* \to 0\,.$$

It is easily seen that $\mathrm{Ass}_A(M) = \mathrm{Ass}_A(0_M : I^\infty) \cup \mathrm{Ass}_A(M^*)$ and $\mathrm{Ass}_A(0_M : I^\infty) \cap \mathrm{Ass}_A(M^*) = \emptyset$. From these facts and note that $\mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M) > 0$, any prime ideal $P \in \mathrm{Ass}_A(0_M : I^\infty)$ is a non-minimal element in $\mathrm{Ass}_A(M)$. This follows that

$$\dim(0_M : I^\infty) < \dim M = \dim M^*\,.$$

By [1], $e(J; M^*) = e(J; M)$. Hence by (i), we get

$$e_A(J^{[q]}, I_1^{[0]}, \ldots, I_s^{[0]}; M) = e(J; M^*) = e(J; M)\,.$$

The proof of Lemma 3.2 is complete. ∎

Now, from the above results and applying the argument as in [8, 9, 10], we can extend the results in [8, 9, 10] to modules.

The following proposition is an important result to establish the relationship between mixed multiplicities and Hilbert-Samuel multiplicities. Moreover, this proposition also characterizes the length of maximal weak-(FC)-sequences in modules.

PROPOSITION 3.3. *Let J be an $\mathfrak{m}$-primary ideal and $(I_1, \ldots, I_s)$ be a set of ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$. Set $U = (J, I_1, \ldots, I_s)$. Then the following statements hold.*

(i) *If $x \in I_j$ is an (FC)-element of M with respect to U, then*

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_j^{[k_j]}, \ldots, I_s^{[k_s]}; M) = e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_j^{[k_j-1]}, \ldots, I_s^{[k_s]}; \overline{M})\,,$$

where k_j is a positive integer and $\overline{M} = M/xM$.

(ii) *If $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M) \neq 0$, then for any j such that $k_j > 0$, there exists an (FC)-element $x \in I_j$ with respect to U.*

(iii) *If $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M) \neq 0$ and $x_1, \ldots, x_t$ $(t = k_1 + \cdots + k_s)$ is a weak-(FC)-sequence with respect to U consisting of k_1 elements of $I_1, \ldots, k_s$ elements of I_s, then $x_1, \ldots, x_t$ is an (FC)-sequence with respect to U.*

(iv) *Let $x_1, \ldots, x_f$ be a weak-(FC)-sequence in $\cup_{i=1}^s I_i$ with respect to U. Then*

$$\dim\left(\frac{M}{(x_1, \ldots, x_f)M : I^\infty}\right) \leq \dim\left(\frac{M}{0_M : I^\infty}\right) - f$$

with equality if and only if $x_1, \ldots, x_f$ is an (FC)-sequence with respect to U.

(v) *For any* $1 \leq j \leq s$, *the length of maximal weak-*(FC)*-sequences in* I_j *with respect to* U *is an invariant.*

(vi) *Let* $x_1, \ldots, x_p$ *is a maximal weak-*(FC)*-sequence in* $\cup_{i=1}^{s} I_i$ *with respect to* U. *If* $\mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M) = h > 0$, *then* $h \leq p$ *and* $x_1, \ldots, x_{h-1}$ *is an* (FC)*-sequence with respect to* U.

PROOF. The proof of (i): Set $M^* = \dfrac{M}{0_M : I^\infty}$; $q = \dim M^*$; $\overline{M}^* = \dfrac{\overline{M}}{0_{\overline{M}} : I^\infty}$ and $N = 0_M : I^\infty$. It is easy to see that $\overline{M}^* \simeq \dfrac{M}{xM : I^\infty}$. For all large $n, n_1, \ldots, n_s$, we have

$$l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}\Big) = l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M + (xM : I^\infty)}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + (xM : I^\infty)}\Big)$$
$$= l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M + xM + N}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + (J^n I_1^{n_1} \cdots I_s^{n_s} M + xM + N) \cap (xM : I^\infty)}\Big).$$

By Artin-Rees lemma, $J^n I_1^{n_1} \cdots I_s^{n_s} M \cap (xM : I^\infty) \subseteq xM$ for all large $n, n_1, \ldots, n_s$. Thus,

$$l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}\Big) = l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M + xM + N}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + xM + N}\Big)$$
$$= l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M + N}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + (xM + N) \cap (J^n I_1^{n_1} \cdots I_s^{n_s} M + N)}\Big)$$
$$= l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M + N}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + xJ^n I_1^{n_1} \cdots I_j^{n_j - 1} \cdots I_s^{n_s} M + N}\Big)$$
$$= l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M + N}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + N}\Big)$$
$$- l_A\Big(\frac{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + xJ^n I_1^{n_1} \cdots I_j^{n_j - 1} \cdots I_s^{n_s} M + N}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + N}\Big)$$
$$= l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M^*}\Big)$$
$$- l_A\Big(\frac{xJ^n I_1^{n_1} \cdots I_j^{n_j - 1} \cdots I_s^{n_s} M + N}{(J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + N) \cap (xJ^n I_1^{n_1} \cdots I_j^{n_j - 1} \cdots I_s^{n_s} M + N)}\Big)$$
$$= l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M^*}\Big)$$
$$- l_A\Big(\frac{xJ^n I_1^{n_1} \cdots I_j^{n_j - 1} \cdots I_s^{n_s} M + N}{(J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M + N) \cap (J^n I_1^{n_1} \cdots I_j^{n_j} \cdots I_s^{n_s} M + N) \cap (xM + N)}\Big)$$

$$= l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}M^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}M^*}\right) - l_A\left(\frac{xJ^n I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M+N}{xJ^{n+1}I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M+N}\right)$$

$$= l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}M^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}M^*}\right) - l_A\left(\frac{xJ^n I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{xJ^{n+1}I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right)$$

for all large $n, n_1, \ldots, n_s$. Since x is a non-zero-divisor in M^*, it follows that

$$l_A\left(\frac{J^n I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{J^{n+1}I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right) = l_A\left(\frac{xJ^n I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{xJ^{n+1}I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right).$$

Hence

$$l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}\right) = l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}M^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}M^*}\right) - l_A\left(\frac{J^n I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{J^{n+1}I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right)$$

for all large $n, n_1, \ldots, n_s$. Since x is an (FC)-sequence of M with respect to U, we have

$$\dim \overline{M}^* = \dim\left(\frac{M}{xM : I^\infty}\right) = \dim M^* - 1 = q - 1\,.$$

It can be verified that $l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}\right)$ is a polynomial of degree $q-2$ for all large $n, n_1, \ldots, n_s$ and the terms of total degree $q-2$ in this polynomial, i.e., the Bhattacharya polynomial of function $l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}\right)$ is equal to the terms of total degree $q-2$ in the polynomial

$$B(n, n_1, \ldots, n_j, \ldots, n_s) - B(n, n_1, \ldots, n_j - 1, \ldots, n_s)\,,$$

where $B(n, n_1, \ldots, n_j, \ldots, n_s)$ is the Bhattacharya polynomial of function $l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}M^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}M^*}\right)$. The above facts show that

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots I_j^{[k_j]}, \ldots, I_s^{[k_s]}; M) = e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_j^{[k_j-1]}, \ldots, I_s^{[k_s]}; \overline{M})\,.$$

The proof of (ii): Applying Proposition 2.3, for any j such that $k_j > 0$, there exists a weak-(FC)-element $x \in I_j$ with respect to U.

Set $\overline{M} = M/xM$ and $\overline{M}^* = \frac{\overline{M}}{0_{\overline{M}} : I^\infty}$. Then $\overline{M}^* \simeq \frac{M}{xM : I^\infty}$. By (i), we have

$$l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}\right) = l_A\left(\frac{J^n I_1^{n_1}\cdots I_s^{n_s}M^*}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}M^*}\right) - l_A\left(\frac{J^n I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{J^{n+1}I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right)$$

for all large $n, n_1, \dots, n_s$. Since $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_j^{[k_j]}, \dots, I_s^{[k_s]}; M) \neq 0$ and $k_j > 0$, it follows that $l_A\left(\dfrac{J^n I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}\right)$ is a polynomial of degree $q-2$ for all large $n, n_1, \dots, n_s$. Thus,

$$\dim\left(\frac{M}{xM : I^\infty}\right) = \dim \overline{M}^* = q - 1\,.$$

Hence x is an (FC)-element of M with respect to U.

The proof of (iii): We will prove (iii) by using induction on $t = k_1 + \cdots + k_s$. For $t = 1$, the result was proved in the part (ii). Assume that the result is true for $t-1 > 0$. As the next step, we claim that the result is true for t. Since $k_1 + \cdots + k_s > 0$, there exists j such that $k_j > 0$ and an (FC)-element $x_1 \in I_j$ with respect to U. By (i), we have

$$\begin{aligned} &e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}; M) \\ &\quad = e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_j^{[k_j-1]}, \dots, I_s^{[k_s]}; M/x_1M) \neq 0\,. \end{aligned} \tag{2.7}$$

Let $\bar{x}_2, \dots, \bar{x}_t$ denote the images of $x_2, \dots, x_t$ in $A/(x_1)$, respectively. Then $\bar{x}_2, \dots, \bar{x}_t$ is a weak-(FC)-sequence of M/x_1M with respect to $(\bar{J}, \bar{I}_1, \dots, \bar{I}_s)$ of $t-1$ elements in $A/(x_1)$ consisting of k_1 elements of $\bar{I}_1, \dots, (k_j - 1)$ elements of $\bar{I}_j, \dots, k_s$ elements of $\bar{I}_s$, where

$$\bar{J} = J.A/(x_1);\ \bar{I}_i = I_i.A/(x_1),\ i = 1, \dots, s\,.$$

By (2.7) and inductive assumption, $\bar{x}_2, \dots, \bar{x}_t$ is an (FC)-sequence of M/x_1M with respect to $(\bar{J}, \bar{I}_1, \dots, \bar{I}_s)$. Consequently, $x_1, \dots, x_t$ is an (FC)-sequence of M with respect to U.

The proof of (iv): Assume that x is a weak-(FC)-element of M with respect to U.

Set $\overline{M} = M/xM$ and $\overline{M}^* = \dfrac{\overline{M}}{0_{\overline{M}} : I^\infty}$. Then $\overline{M}^* \simeq \dfrac{M}{xM : I^\infty}$. By (i), we have

$$l_A\left(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}\right) = l_A\left(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M^*}\right) - l_A\left(\frac{J^n I_1^{n_1} \cdots I_j^{n_j-1} \cdots I_s^{n_s} M^*}{J^{n+1} I_1^{n_1} \cdots I_j^{n_j-1} \cdots I_s^{n_s} M^*}\right)$$

for all large $n, n_1, \dots, n_s$. Thus, $l_A\left(\dfrac{J^n I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}\right)$ is a polynomial of degree $\leq q-2$ for all large $n, n_1, \dots, n_s$. Hence

$$\dim \overline{M}^* = \dim\left(\frac{M}{xM : I^\infty}\right) \leq \dim M^* - 1\,.$$

From the above facts and by using induction, we get (iv).

The proof of (v): By (iv), the length of any maximal weak-(FC)-sequence in I_j with respect to U is finite. Since I is not contained in $\sqrt{\mathrm{Ann} M}$ and Proposition 2.3, there exists a weak-(FC)-element $x_1 \in I_j$ with respect to U.

Set $\overline{M} = M/x_1M$ and $\overline{M}^* = \dfrac{\overline{M}}{0_{\overline{M}} : I^\infty}$. Then $\overline{M}^* \simeq \dfrac{M}{x_1M : I^\infty}$. Assume that $x_1, \ldots, x_p$ is a maximal weak-(FC)-sequence in I_j with respect to U. Then there exists a positive integer u such that

(1) $l_A\left(\dfrac{J^n I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} \overline{M}^*}\right)$; $l_A\left(\dfrac{J^n I_1^{n_1} \cdots I_s^{n_s} M^*}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M^*}\right)$; $l_A\left(\dfrac{J^n I_1^{n_1} \cdots I_s^{n_s} \overline{M}}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} \overline{M}}\right)$

are polynomials for all $n, n_1, \ldots, n_s \geq u$.

(2) For any $n_j \geq u$, we have

$$l_A\left(\frac{J^u I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}^*}{J^{u+1} I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}^*}\right)$$
$$= l_A\left(\frac{J^u I_1^u \cdots I_j^{n_j} \cdots I_s^u M^*}{J^{u+1} I_1^u \cdots I_j^{n_j} \cdots I_s^u M^*}\right) - l_A\left(\frac{J^u I_1^u \cdots I_j^{n_j-1} \cdots I_s^u M^*}{J^{u+1} I_1^u \cdots I_j^{n_j-1} \cdots I_s^u M^*}\right).$$

(3) For any $n_j \geq u$, we have

$$l_A\left(\frac{J^u I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}^*}{J^{u+1} I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}^*}\right) = l_A\left(\frac{J^u I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}}{J^{u+1} I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}}\right).$$

Fix u, let $P(u, n_j)$ and $Q(u, n_j)$ denote the polynomials in an indeterminate n_j of functions $l_A\left(\dfrac{J^u I_1^u \cdots I_j^{n_j} \cdots I_s^u M^*}{J^{u+1} I_1^u \cdots I_j^{n_j} \cdots I_s^u M^*}\right)$ and $l_A\left(\dfrac{J^u I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}^*}{J^{u+1} I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}^*}\right)$, respectively.

Set $t = \deg P(u, n_j)$. Since $Q(u, n_j) = P(u, n_j) - P(u, n_j - 1)$, $\deg Q(u, n_j) = t - 1$. We will show that $p = t + 1$. The proof is by induction on t.
For $t = 0$, by (2) and (3), we have

$$l_A\left(\frac{J^u I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}^*}{J^{u+1} I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}^*}\right) = l_A\left(\frac{J^u I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}}{J^{u+1} I_1^u \cdots I_j^{n_j} \cdots I_s^u \overline{M}}\right) = 0\,.$$

It implies that I is contained in $\sqrt{\operatorname{Ann}\overline{M}}$. Therefore, $p = 1 = t + 1$.

Suppose that the result has been proved for $t - 1 \geq 0$, we will show that the result is true for t. From $\deg Q(u, n_j) = t - 1$ and $\bar{x}_2, \ldots, \bar{x}_p$ is a maximal weak-(FC)-sequence of $\overline{M}$ with respect to $\overline{U} = (\bar{J}, \bar{I}_1, \ldots, \bar{I}_s)$, where $\bar{x}_2, \ldots, \bar{x}_p$ the images of $x_2, \ldots, x_p$ in $A/(x_1)$, respectively, and $\bar{J} = J.A/(x_1)$; $\bar{I}_1 = I_1.A/(x_1), \ldots, \bar{I}_s = I_s.A/(x_1)$. By inductive assumption, we get

$$p - 1 = \deg Q(u, n_j) + 1 = (t - 1) + 1 = t\,.$$

Hence $p = t + 1$.

The proof of (vi): Assume the contrary, that $p < h$. This follows that

$$\mathrm{ht}\left[\frac{I + \mathrm{Ann}(M/(x_1,\ldots,x_p)M)}{\mathrm{Ann}(M/(x_1,\ldots,x_p)M)}\right] > 0\,.$$

Thus, I is not contained in $\sqrt{\mathrm{Ann}\left(\dfrac{M}{(x_1,\ldots,x_p)M}\right)}$. From Proposition 2.3, there exists $x \in \bigcup_{i=1}^{s} I_i$ such that $x_1,\ldots,x_p,x$ is a weak-(FC)-sequence in $\bigcup_{i=1}^{s} I_i$ with respect to U. This contradicts that $x_1,\ldots,x_p$ is a maximal weak-(FC)-sequence of M with respect to U. Therefore, $h \le p$. By (iv), we have

$$\dim\left(\frac{M}{(x_1,\ldots,x_{h-1})M : I^\infty}\right) \le \dim\left(\frac{M}{0_M : I^\infty}\right) - (h-1)\,.$$

Since $\mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M) = h$, $\mathrm{ht}\left[\dfrac{I + \mathrm{Ann}(M/(x_1,\ldots,x_{h-1})M)}{\mathrm{Ann}(M/(x_1,\ldots,x_{h-1})M)}\right] > 0$. Therefore,

$$\dim\left(\frac{M}{(x_1,\ldots,x_{h-1})M}\right) = \dim\left(\frac{M}{(x_1,\ldots,x_{h-1})M : I^\infty}\right).$$

Since $\mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M) = h > 0$, $\dim M = \dim\left(\dfrac{M}{0_M : I^\infty}\right)$.

It is clear that

$$\dim\left(\frac{M}{(x_1,\ldots,x_{h-1})M}\right) \ge \dim M - (h-1)\,.$$

From the above facts, we get

$$\dim\left(\frac{M}{(x_1,\ldots,x_{h-1})M : I^\infty}\right) = \dim\left(\frac{M}{0_M : I^\infty}\right) - (h-1)\,.$$

Hence by (iv), $x_1,\ldots,x_{h-1}$ is an (FC)-sequence of M with respect to U.
The proof of Proposition 3.3 is complete. ∎

By Proposition 3.3, Lemma 3.2 and applying the argument as in [8, Theorem 3.4], we get the main theorem of this section as follows.

THEOREM 3.4. *Let* $(I_1,\ldots,I_s)$ *be a set of ideals such that* $I = I_1\cdots I_s$ *is not contained in* $\sqrt{\mathrm{Ann}M}$ *and* J *be an* $\mathfrak{m}$*-primary ideal. Assume that* $\dim\left(\dfrac{M}{0_M : I^\infty}\right) = q > 0$. *Let* $k_0,\ldots,k_s$ *be non-negative integers such that* $k_0 + k_1 + \cdots + k_s = q - 1$. *Then the following statements hold.*

(i) $e_A(J^{[k_0+1]}, I_1^{[k_1]},\ldots,I_s^{[k_s]}; M) = e_A(J; \overline{M}^*)$ *for any (FC)-sequence* $x_1,\ldots,x_t$ *with respect to* $(J, I_1,\ldots,I_s)$ *of* $t = k_1 + \cdots + k_s$ *elements consisting of* k_1 *elements of* $I_1,\ldots,k_s$ *elements of* I_s, *where* $\overline{M}^* = \dfrac{M}{(x_1,\ldots,x_t)M : I^\infty}$.

(ii) $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_j^{[k_j]}, \ldots, I_s^{[k_s]}; M) \neq 0$ *if and only if there exists an* (FC)-*sequence of* M *with respect to* $(J, I_1, \ldots, I_s)$ *of* $k_1 + \cdots + k_s$ *elements consisting of* k_1 *elements of* $I_1, \ldots, k_s$ *elements of* I_s.

(iii) *The length of maximal* (FC)-*sequences of* M *in* I_j *with respect to* $(J, I_1, \ldots, I_s)$ *is an invariant and this invariant is equal to* $\max\{k_j \mid e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_j^{[k_j]}, \ldots, I_s^{[k_s]}; M) \neq 0\}$.

(iv) *Let* p *be the length of maximal weak-*(FC)*-sequences in* I_j *with respect to* $(J, I_1, \ldots, I_s)$ *and* f *the length of maximal* (FC)-*sequences in* I_j *with respect to* $(J, I_1, \ldots, I_s)$, $j = 1, \ldots, s$. *Then* $f < p \leq q$.

From Proposition 3.3 and Theorem 3.4, we get an interesting result as follows.

PROPOSITION 3.5. *Let* $(I_1, \ldots, I_s)$ *be a set of ideals such that* $I = I_1 \cdots I_s$ *is not contained in* $\sqrt{\mathrm{Ann}M}$ *and* J *be an* $\mathfrak{m}$-*primary ideal. Suppose that* f *is the length of maximal* (FC)-*sequences of* M *in* I_j *with respect to* $U = (J, I_1, \ldots, I_s)$. *When if* $p \leq f$ *and* $y_1, \ldots, y_p$ *is a weak-*(FC)*-sequence of* M *in* I_j *with respect to* U. *Then* $y_1, \ldots, y_p$ *is an* (FC)-*sequence of* M *in* I_j *with respect to* U.

In the case that $\mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M) > 0$, we get the following result.

THEOREM 3.6. *Let* J *be an* $\mathfrak{m}$-*primary ideal and* $(I_1, \ldots, I_s)$ *be a set of ideals such that* $\mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M) > 0$, *where* $I = I_1 \cdots I_s$. *Let* $k_0, \ldots, k_s$ *be non-negative integers such that* $k_0 + k_1 + \cdots + k_s = d - 1$. *Then the following statements hold.*

(i) $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M) = e_A(J; \overline{M}^*)$ *for any* (FC)-*sequence* $x_1, \ldots, x_t$ *with respect to* $(J, I_1, \ldots, I_s)$ *of* $t = k_1 + \cdots + k_s$ *elements consisting of* k_1 *elements of* $I_1, \ldots, k_s$ *elements of* I_s, *where* $\overline{M}^* = \dfrac{M}{(x_1, \ldots, x_t)M : I^\infty}$.

(ii) *If* $t = k_1 + \ldots + k_s < \mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M)$, *then*

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M) = e_A(J; \overline{M})$$

for any (FC)-*sequence* $x_1, \ldots, x_t$ *with respect to* $(J, I_1, \ldots, I_s)$ *consisting of* k_1 *elements of* $I_1, \ldots, k_s$ *elements of* I_s, *where* $\overline{M} = \dfrac{M}{(x_1, \ldots, x_t)M}$.

(iii) $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M) \neq 0$ *if and only if there exists an* (FC)-*sequence of* M *with respect to* $(J, I_1, \ldots, I_s)$ *of* $k_1 + \cdots + k_s$ *elements consisting of* k_1 *elements of* $I_1, \ldots, k_s$ *elements of* I_s.

PROOF. Set $\overline{M}_1^* = \dfrac{\overline{M}}{0_{\overline{M}} : I^\infty}$. Then $\overline{M}_1^* \simeq \overline{M}^*$. From Theorem 3.4, we immediately get (i) and (iii).

The proof of (ii): By Theorem 3.4 (i),

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M) = e(J; \overline{M}^*) = e(J; \overline{M}_1^*)\,.$$

It is clear that $\mathrm{ht}(I+\mathrm{Ann}\overline{M}/\mathrm{Ann}\overline{M})>0$. Thus, applying Lemma 3.2, we have

$$e(J;\overline{M}_1^*)=e(J;\overline{M}).$$

From the above facts, we get

$$e_A(J^{[k_0+1]},I_1^{[k_1]},\ldots,I_s^{[k_s]};M)=e_A(J;\overline{M}). \qquad \blacksquare$$

From Proposition 3.3 (vi), Theorem 3.4 and Theorem 3.6, we have the following proposition.

PROPOSITION 3.7. *Let $(I_1,\ldots,I_s)$ be a set of ideals such that $I=I_1\cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$ and J be an $\mathfrak{m}$-primary ideal. Assume that* $\mathrm{ht}(I+\mathrm{Ann}M/\mathrm{Ann}M)=h>0$. *Set $U=(J,I_1,\ldots,I_s)$. Then the following statements hold.*

(i) *If $k_1+\cdots+k_s\le h-1$, then $e_A(J^{[k_0+1]},I_1^{[k_1]},\ldots,I_s^{[k_s]};M)\ne 0$.*

(ii) *If $x_1,\ldots,x_t$ is a weak-(FC)-sequence of M with respect to U of $t=k_1+\cdots+k_s\le h-1$ elements consisting of k_1 elements of $I_1,\ldots,k_s$ elements of I_s, then*

$$e_A(J^{[k_0+1]},I_1^{[k_1]},\ldots,I_s^{[k_s]};M)=e_A\left(J;\frac{M}{(x_1,\ldots,x_t)M}\right).$$

4. Multiplicity of Rees modules

In this section, we show some results on the multiplicity of Rees modules. Let I be an ideal of A and finitely generated A-module M. Set $R_M(I)=\bigoplus_{n\ge 0}I^nMt^n$ and $R_A(I)=\bigoplus_{n\ge 0}I^nt^n$, where t is an indeterminate. We call $R_M(I)$ the Rees module of I and $R_A(I)$ the Rees algebra of I. Then $R_A(I)$ is a Noetherian graded ring and $R_M(I)$ is a finitely generated graded $R_A(I)$-module. Now, we study multiplicities of Rees modules.

By applying Theorem 3.4, Theorem 3.6 and arguing as in the proof of Theorem 4.1 [8], we get the following theorem.

THEOREM 4.1. *Let J be $\mathfrak{m}$-primary and I be an ideal such that* $\mathrm{ht}(I+\mathrm{Ann}M/\mathrm{Ann}M)=h>0$. *Suppose that $x_1,\ldots,x_f$ is a maximal (FC)-sequence of M in I with respect to (J,I). Then*

(i) $e_A(J^{[d-i]},I^{[i]};M)=e\left(J;\dfrac{M}{(x_1,\ldots,x_i)M}\right)$ *for all* $i\le h-1$.

(ii) $e_A(J^{[d-i]},I^{[i]};M)=e\left(J;\dfrac{M}{(x_1,\ldots,x_i)M:I^\infty}\right)$ *for* $h\le i\le f$.

(iii) $e_A(J^{[d-i]},I^{[i]};M)\ne 0$ *if and only if* $i\le f$.

The following results are generalizations of Section 4 in [8] to modules.
First, from Theorem 4.1 and the results in [5] and [7], we get some multiplicity formulas for Rees modules.

THEOREM 4.2. *Let J be $\mathfrak{m}$-primary and I be an ideal such that* $\mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M) = h > 0$. *Then*

$$e((J, It); R_M(I)) = \sum_{i=0}^{h-1} e\left(J; \frac{M}{(x_1, \ldots, x_i)M}\right) + \sum_{i=h}^{f} e\left(J; \frac{M}{(x_1, \ldots, x_i)M : I^\infty}\right)$$

for any maximal (FC)-sequence $x_1, \ldots, x_f$ in I with respect to (J, I).

PROOF. Using the argument as in [5], we get

$$e((J, It); R_M(I)) = \sum_{i=0}^{d-1} e_A(J^{[d-i]}, I^{[i]}; M)\,.$$

Now, applying Theorem 4.1 we get Theorem 4.2. ∎

In the case that I is $\mathfrak{m}$-primary, we have $h = \mathrm{ht}(I + \mathrm{Ann}M/\mathrm{Ann}M) = d$. From Theorem 4.2, we get the following result.

THEOREM 4.3. *Let I and J be $\mathfrak{m}$-primary ideals. Then there exists an (FC)-sequence $x_1, \ldots, x_{d-1}$ in I with respect to (J, I) and*

$$e((J, It); R_M(I)) = \sum_{i=0}^{d-1} e\left(J; \frac{M}{(x_1, \ldots, x_i)M}\right).$$

PROOF. Since I is $\mathfrak{m}$-primary, by Proposition 2.5, there exists a maximal (FC)-sequence of M in I with respect to (J, I) consisting of $d - 1$ elements. Next, the proof is complete by applying Theorem 4.2. ∎

Let $R_M(I^{[s]})$ denote $\bigoplus_{\alpha_1,\ldots,\alpha_s \geq 0} I_1^{\alpha_1} \ldots I_s^{\alpha_s} M t_1^{\alpha_1} \ldots t_s^{\alpha_s}$, where $I_1 = \ldots = I_s = I$ and $t_1, \ldots, t_s$ indeterminates. We call $R_M(I^{[s]})$ the multi-Rees module of I. In particular, $M = A$ we call $R_A(I^{[s]})$ the multi-Rees algebra of I. It is easy to see that $R_A(I^{[s]})$ is a Noetherian graded ring and $R_M(I^{[s]})$ is a finitely generated graded $R_A(I^{[s]})$-module.

By an argument analogous to that used for the proof of Theorem 4.3 and Theorem 1.4 [7] and applying Theorem 3.6, we get the following theorem.

THEOREM 4.4. *Let I and J be $\mathfrak{m}$-primary ideals. Let $R = R_A(I^{[s]})$ and $R_M(I^{[s]})$ be multi-Rees algebra of I and multi-Rees module of I, respectively. Let R^+ be the positively graded part of R. Suppose that $x_1, \ldots, x_{d-1}$ is an (FC)-sequence in I with respect to (J, I). Then*

$$e((J, R^+); R_M(I^{[s]})) = \sum_{i=0}^{d-1} \frac{(i + s - 1)!}{i!(s - 1)!} e\left(J; \frac{M}{(x_1, \ldots, x_i)M}\right). \quad ∎$$

ACKNOWLEDGEMENT. We would like to thank Prof. Ryuji Tsushima for his help and advices.

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