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# An Example of Set-theoretic Complete Intersection Lattice Ideal

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**Abstract.** We prove that the monomial curve $(t^{17}, t^{19}, t^{25}, t^{27})$ is set-theoretic complete intersection.

## 1. Introduction

It has been questioned whether every curve in affine space is set-theoretic complete intersection or not. We say that the curve in affine $N$-space is set-theoretic complete intersection, if it is defined by $N-1$ polynomials, that is, its defining ideal is generated by $N-1$ polynomials up to radical. In general, Cowsik and Nori proved that it is true, if the characteristic of the base field is positive ([3]). If the characteristic is zero, the question is open now. Even the monomial curve case is open in general. A monomial curve is defined as the curve $\{(t^{n_1}, t^{n_2}, \dots, t^{n_N}) : t \in k\}$ where $n_1, n_2, \dots, n_N$ are natural numbers whose greatest common divisor is one. There are a lot of partial results for this question for monomial curves. See [1, 2, 4, 5, 8, 9, 10, 12]. In most of them, it is affirmatively proved by finding $N-2$ binomials (a binomial is the polynomial of the form a monomial minus a monomial) and one polynomial so that the defining ideal $I$ of a monomial curve is generated by them up to radical, or finding set-theoretic complete intersection subideal $J$ of $I$ and a polynomial $f$ satisfying $I = \sqrt{J + (f)}$. In fact, $I$ is generated by binomials and it is proved in [11] that $I$ is a complete intersection if it is generated by $N-1$ binomials up to radical. Hence, in general case, to prove set-theoretic complete intersection for monomial curves, we have to find more than one polynomials which is not binomials. The case when we only needs one such polynomial is studied in [6]. And it also proved that the monomial curve $C = (t^{17}, t^{19}, t^{25}, t^{27})$ is never defined by two binomials and one polynomial up to radical. In this paper, we prove

THEOREM 1. *The monomial curve $(t^{17}, t^{19}, t^{25}, t^{27})$ is set-theoretic complete intersection.*

In fact, we find one binomial and two polynomials which is not binomials so that $C$ is defined by them, by new and unique method.

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## 2. The monomial curve $(t^{17}, t^{19}, t^{25}, t^{27})$

Let $k$ be a field. In this section, we prove that the monomial curve $C = \{(t^{17}, t^{19}, t^{25}, t^{27}) : t \in k\}$ in affine 4-space is set-theoretic complete intersection.

Let $V = \mathrm{Ker}(17, 19, 25, 27)$ the submodule in $\mathbf{Z}^4$ (note that we regard (17, 19, 25, 27) as the map from $\mathbf{Z}^4$ to $\mathbf{Z}$). For each $v \in V$, put $F(v) = X^{v^-} - X^{v^+}$ in $A = k[X_1, X_2, X_3, X_4]$ where $v^+ = \sum_{i=1}^4 \max\{\sigma_i(v), 0\}e_i$, $v^- = \sum_{i=1}^4 \max\{-\sigma_i(v), 0\}e_i$, and $\sigma_i$ denotes the $i$-th entry of $v$ for each $i$. Then the defining ideal $I$ of $C$ is generated by all $F(v)$ where $v \in V$ (cf. [8]). In general, for given submodule $W$ in $\mathbf{Z}^N$, the ideal in a polynomial ring generated by all $F(v)$ for $v \in W$ is called a lattice ideal.

The defining ideal $I$ of $C$ is weighted homogeneous; i.e. if we put $\deg X_1 = 17$, $\deg X_2 = 19$, $\deg X_3 = 25$ and $\deg X_4 = 27$, then each $F(v)$ is homogeneous, thus $I$ is homogeneous. Throughout this paper, the degree means this weighted degree.

Put $v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -3 \\ 4 \\ -1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 6 \\ 0 \\ -3 \\ -1 \end{pmatrix}$. Then $V = \mathbf{Z}v_1 + \mathbf{Z}v_2 + \mathbf{Z}v_3$ (recall $V = \mathrm{Ker}(17, 19, 25, 27)$). It is certified from the following calculation of the determinant;

$$\begin{vmatrix} 1 & 4 & 0 \\ 1 & -1 & -3 \\ -1 & 0 & -1 \end{vmatrix} = 17\,.$$

Further, $I$ is minimally generated by

$$\begin{gathered} F(v_1),\, F(v_2),\, F(v_2 - v_1),\, F(v_2 - 2v_1),\, F(v_2 - 3v_1) \\ F(v_3),\, F(v_3 + v_1),\, F(v_3 + 2v_1),\, F(v_3 + 3v_1). \end{gathered} \tag{$*$}$$

This follows from the following Gastinger's theorem;

THEOREM 2 ([7]). *Let $A = k[X_1, \dots, X_N]$ be a polynomial ring, $I \subset A$ the defining ideal of a monomial curve defined by natural numbers $n_1, \dots, n_N$ whose greatest common divisor is* 1. *And let $J \subset I$ be a subideal. Then $J = I$ if and only if* $\dim_k A/J + (X_i) = n_i$ *for some $i$.* (*Note that the above conditions are also equivalent to* $\dim_k A/J + (X_i) = n_i$ *for any $i$.*)

Indeed, let $J$ be the ideal generated by the all binomials in $(*)$. Since

$$J + (X_1) = (X_1, X_2X_3, X_2^4, X_2^3X_4, X_2^2X_4^2, X_3^4 - X_2X_4^3, X_3^3X_4, X_3^2X_4^2, X_3X_4^3, X_4^4)\,,$$

$A/J + (X_1)$ is the vector space whose basis consists of the images of monomials

$$\begin{gathered} 1, X_2, X_2^2, X_2^3, X_3, X_3^2, X_3^3, X_4, X_2X_4, X_2^2X_4, X_3X_4, X_3^2X_4, \\ X_4^2, X_2X_4^2, X_3X_4^2, X_4^3, X_2X_4^3\,, \end{gathered}$$

hence its dimension is 17. By Theorem 2, we have $J = I$. Further, we see that it is a minimal generating system again by Theorem 2 (if one binomial in $J$ were removed, the dimension of $A/J + (X_1)$ increases).

Now we start to prove that $I$ is set-theoretic complete intersection. For each $n$, we define the polynomial

$$f_n(Z_0, \dots, Z_n) = \sum_{i=0}^{n} (-1)^n \binom{n}{i} Z_i \,.$$

Put

$$\begin{array}{lll} M_{00} = X_1^{27} & M_{10} = X_1^{21} X_3^3 X_4 & M_{20} = X_1^{15} X_3^6 X_4^2 \\ M_{01} = X_1^{18} X_2^4 X_3^2 X_4 & M_{11} = X_1^{12} X_2^4 X_3^5 X_4^2 & M_{21} = X_1^6 X_2^4 X_3^8 X_4^3 \\ M_{02} = X_1^9 X_2^8 X_3^4 X_4^2 & M_{12} = X_1^3 X_2^8 X_3^7 X_4^3 & M_{22} = X_2^5 X_3^7 X_4^7 \\ M_{03} = X_2^{12} X_3^6 X_4^3 & M_{13} = X_2^6 X_3^3 X_4^{10} & M_{23} = X_4^{17} \end{array}$$

$G_i = f_3(M_{i0}, M_{i1}, M_{i2}, M_{i3})$ for $i = 0, 1, 2$ and $G = f_2(G_0, G_1, G_2)$. Note that the degree of $M_{ij}$ is 459 for each $i, j$, thus $G_i \in I$ for each $i$ and $G \in I$.

Put $w = 50e_2 \in \mathbf{Z}^4$. Then $X^w = X_2^{50}$ and $X^{w+6v_1-14v_2-6v_3} = X_3^{38}$. And put $A_l = \sum_{i=0}^{l} (-1)^i \binom{20}{i}$ for $l = 0, \dots, 20$ and $B_l = \sum_{i=0}^{l} (-1)^i \binom{14}{i}$ for $l = 0, \dots, 14$. We define the polynomial $H$ as follows;

$$\begin{aligned} H = X^w &+ \sum_{i=1}^{12} (A_i - B_{i-1}) X^{w-iv_2} + (-A_1 + B_1) X^{w-3v_1-v_2-v_3} \\ &+ \sum_{i=2}^{11} (-A_i + B_i) X^{w-iv_2-v_3} + \sum_{i=12}^{16} (-A_i + A_{i+1}) X^{w-12v_2-(i-11)v_3} \\ &+ (-A_{17} + B_{12}) X^{w-12v_2-6v_3} + (A_{18} - B_{12}) X^{w+3v_1-13v_2-5v_3} \\ &+ (-A_{18} + B_{13}) X^{w+3v_1-13v_2-6v_3} + X_3^{38} \end{aligned}$$

Clearly, each term in $H$ is indeed a monomial in $A$ and has the same degree 950. Note $B_0 = A_0 = 1$ and $B_{13} = A_{19} = -1$. Substituting 1 for all $X_i \in H$, it becomes 0, thus $H \in I$.

Now we prove $I = \sqrt{(F(v_1), G, H)}$. Let $\mathfrak{p}$ be a prime ideal containing the ideal $(F(v_1), G, H)$. If $\mathfrak{p}$ contains a monomial, then it contains all monomials by the choice of $F(v_1)$, $G$ and $H$, hence it is a maximal ideal. Assume that $\mathfrak{p}$ does not contain any monomial. We claim $I = \mathfrak{p}$. If it were proved, the assertion follows from this. To prove the claim, it is enough to prove it in the Laurent ring $k[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}, X_4^{\pm 1}]$.

Consider $M_{ij}M_{i(j+1)}^{-1}$ for each $i, j$. Then each of them is equal to one of $X_1^9X_2^{-4}X_3^{-2}X_4^{-1}$, $X_1^6X_2^{-1}X_3X_4^{-4}$, $X_1^3X_2^2X_3^4X_4^{-7}$ or $X_2^5X_3^7X_4^{-10}$. And

$$X_1^9X_2^{-4}X_3^{-2}X_4^{-1} \equiv X_1^6X_2^{-1}X_3X_4^{-4} \equiv X_1^3X_2^2X_3^4X_4^{-7} \equiv X_2^5X_3^7X_4^{-10}$$

mod $(F(v_1))$. Thus we have

$$\begin{aligned}
G_0 &= X_1^{27}f_3(1, X_1^{-9}X_2^4X_3^2X_4, (X_1^{-9}X_2^4X_3^2X_4)^2, (X_1^{-9}X_2^4X_3^2X_4)^3) \\
&= X_1^{27}(1 - X_1^{-9}X_2^4X_3^2X_4)^3 \\
G_1 &\equiv X_1^{21}X_3^3X_4(1 - X_1^{-9}X_2^4X_3^2X_4)^3 \quad \text{mod } (F(v_1)) \\
G_2 &\equiv X_1^{15}X_3^6X_4^2(1 - X_1^{-9}X_2^4X_3^2X_4)^3 \quad \text{mod } (F(v_1)).
\end{aligned}$$

Hence

$$\begin{aligned}
G &\equiv f_2(X_1^{27}, X_1^{21}X_3^3X_4, X_1^{15}X_3^6X_4^2)(1 - X_1^{-9}X_2^4X_3^2X_4)^3 \\
&\equiv X_1^{27}(1 - X_1^{-6}X_3^3X_4)^2(1 - X_1^{-9}X_2^4X_3^2X_4)^3 \\
&\equiv X_1^{27}(1 - X^{-v_3})^2(1 - X^{v_2 - v_3})^3 \quad \text{mod } (F(v_1)).
\end{aligned}$$

Since $\mathfrak{p}$ contains $F(v_1)$ and $G$, it also contains $F(-v_3)$ or $F(v_2 - v_3)$. Thus we have to consider two cases.

(Case 1) Assume that $\mathfrak{p}$ contains $F(-v_3)$. Then

$$\begin{aligned}
H &\equiv X^w + \sum_{i=1}^{12}(A_i - B_{i-1})X^{w-iv_2} + (-A_1 + B_1)X^{w-v_2} \\
&\quad + \sum_{i=2}^{11}(-A_i + B_i)X^{w-iv_2} + \sum_{i=12}^{16}(-A_i + A_{i+1})X^{w-12v_2} \\
&\quad + (-A_{17} + B_{12})X^{w-12v_2} + (A_{18} - B_{12})X^{w-13v_2} \\
&\quad + (-A_{18} + B_{13})X^{w-13v_2} + X^{w-14v_2} \\
&\equiv X^w + \sum_{i=1}^{13}(B_i - B_{i-1})X^{w-iv_2} + X^{w-14v_2} \\
&\equiv X^w + \sum_{i=1}^{13}(-1)^i\binom{14}{i}X^{w-iv_2} + X^{w-14v_2} \\
&\equiv X^w(1 - X^{-v_2})^{14} \quad \text{mod } (F(v_1), F(v_3)) .
\end{aligned}$$

Hence $\mathfrak{p}$ contains $F(v_2)$ and $I$, thus $\mathfrak{p} = I$.

(Case 2) Assume that $\mathfrak{p}$ contains $F(v_2 - v_3)$. Then

$$\begin{aligned}
H &\equiv X^w + \sum_{i=1}^{12}(A_i - B_{i-1})X^{w-iv_2} + (-A_1 + B_1)X^{w-2v_2} \\
&\quad + \sum_{i=2}^{11}(-A_i + B_i)X^{w-(i+1)v_2} + \sum_{i=12}^{16}(-A_i + A_{i+1})X^{w-(i+1)v_2} \\
&\quad + (-A_{17} + B_{12})X^{w-18v_2} + (A_{18} - B_{12})X^{w-18v_2} \\
&\quad + (-A_{18} + B_{13})X^{w-19v_2} + X^{w-20v_2} \\
&\equiv X^w + \sum_{i=1}^{19}(A_i - A_{i-1})X^{w-iv_2} + X^{w-20v_2} \\
&\equiv X^w + \sum_{i=1}^{19}(-1)^i \binom{20}{i} X^{w-iv_2} + X^{w-20v_2} \\
&\equiv X^w(1 - X^{-v_2})^{20} \mod (F(v_1), F(v_2 - v_3)).
\end{aligned}$$

Hence $\mathfrak{p}$ contains $F(v_2)$ and $I$, thus $\mathfrak{p} = I$. In any case, we have $\mathfrak{p} = I$ and conclude

$$\sqrt{(F(v_1), G, H)} = I\ .$$

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