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A Remark on the Analyticity of the Solutions for Non-Linear Elliptic Partial Differential Equations

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Abstract. In this note, the real analyticity of the solutions for non-linear elliptic equations will be proved by the method of Friedman [5] and Kato [7] using Faá di Bruno's formula.

1. Introduction

In 1904, the analyticity of the solutions for elliptic equation of two independent variables was shown by Bernstein [2]. In 1939, Petrowskii [12] proved such analytic property for elliptic system of partial differential equations of any order by the method of continuation into the complex domain. After that, Friedman [5] has shown the same result by the method of real functional analysis.

In this paper, we shall show the analyticity of the solutions of the non-linear elliptic equations of the second order by a simple method of cut-off functions introduced by Kato [7] combined with the method used by Friedman [5]. We shall also use Faá di Bruno's formula [3] as a basic tool. We shall describe some definitions and main Theorem in §2. In §3, we shall describe the real analytic property of the composite function by the method of majorant series using Faá di Bruno's formula. §§4 and 5 are devoted to the proof of the main theorem by applying Kato's method of cut-off functions [7] and Friedman's method [5], where classical method of majorant series and Faá di Bruno's formula for the estimation of the composite functions turned out to be very efficient.

2. Notations, definitions and the result

DEFINITION 2.1. A partial differential equation

$$F(x, u, \nabla u, \nabla^2 u) = 0 \tag{2.1}$$

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is said to be elliptic if

$$\sum_{jk} \frac{\partial F}{\partial u_{jk}}(x, u, u_j, u_{jk})\zeta_j\zeta_k \neq 0\,, \quad \zeta \in \mathbf{R}^n \setminus \{0\} \tag{2.2}$$

where we put $u_j = \partial_j u, u_{jk} = \partial_j \partial_k u$ $(j, k = 1, \ldots, n)$.

THEOREM. *If $F(x, u, \nabla u, \nabla^2 u) = 0$ is elliptic in Ω, $F(x, \zeta)$ is real analytic in a domain $D \subset \mathbf{R}^{n+\mu}$ $(\mu = (n+1)(n+2)/2)$ and $u \in C^\infty(\Omega)$, then u is real analytic in Ω.*

DEFINITION 2.2. Let $f(x), F(x)$ be real analytic functions in an interval $I = \{|x - x_0| < a\}$ expressed by the power series:

$$f(x) = \sum_{j=0}^{\infty} f_j (x - x_0)^j\,, \quad F(x) = \sum_{j=0}^{\infty} F_j (x - x_0)^j\,.$$

Then we call $F(x)$ a majorant of $f(x)$ if

$$|f_j| \leq F_j\,, \quad (j = 0, 1, \ldots)\,.$$

PROPOSITION 2.1. *Let $f \in C^\infty(I)$ for some open interval I. The function f is in fact in $C^\omega(I)$ if and only if, for each $x_0 \in I$, there are an open interval J, with $x_0 \in J \subset I$, and constants $C > 0$ and $A > 0$ such that the derivatives of f satisfy*

$$|f^{(j)}(x)| \leq C j! A^j, \quad \forall x \in J\,.$$

Let $x = (x_1, x_2, \ldots, x_m) \in \mathbf{R}^m$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbf{Z}_+$, where $\mathbf{Z}_+$ denotes the set of non-negative integers. Set

$$\begin{aligned}
\mu! &= \mu_1! \mu_2! \cdots \mu_m!\,, \\
|\mu| &= \mu_1 + \mu_2 + \cdots + \mu_m\,, \\
x^\mu &= x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_m}\,, \\
|x|^\mu &= |x_1|^{\mu_1} |x_2|^{\mu_2} \cdots |x_m|^{\mu_m}\,, \\
(x)_\mu &= \prod_{j=1}^{m} (x_j)_{\mu_j} = \prod_{j=1}^{m} [x_j(x_j - 1) \cdots (x_j - \mu_j + 1)]\,, \\
\binom{x}{\mu} &= \frac{(x)_\mu}{\mu!}\,, \\
\frac{\partial^\mu}{\partial x^\mu} &= \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \frac{\partial^{\mu_2}}{\partial x_2^{\mu_2}} \cdots \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}}\,.
\end{aligned}$$

We write $C^\omega(G)$ the set of real analytic functions defined on an open subset G of $\mathbf{R}^n$.

3. Composite functions of Real Analytic Functions

Differentiation of composite functions is known as Faà di Bruno's formula. We refer to [3], [8], [11], [13] etc. The description in this section is due to [§1.3, Krantz-Parks, 8].

LEMMA 3.1 (Formula of Faà di Bruno). *Let I be an open interval in $\mathbf{R}$ and suppose that $f \in C^\infty(I)$. Assume that g takes real values in an open interval J such that $g(J) \subset I$ and that $g \in C^\infty(J)$. Then the derivatives of $h = f \circ g$ are given by*

$$h^{(n)}(x) = \sum \frac{n!}{k_1!k_2!\cdots k_n!} f^{(k)}(g(x)) \left(\frac{g^{(1)}(x)}{1!}\right)^{k_1} \left(\frac{g^{(2)}(x)}{2!}\right)^{k_1} \cdots \left(\frac{g^{(n)}(x)}{n!}\right)^{k_n}$$

where $k = k_1 + k_2 + \cdots + k_n$ and the summation is taken over all $k_1, k_2, \ldots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$.

LEMMA 3.2. *For each positive integer n and real number R,*

$$\sum \frac{k!}{k_1!k_2!\cdots k_n!} R^k = R(1+R)^{n-1}$$

holds, where $k = k_1 + k_2 + \cdots + k_n$ and the sum is taken over all $k_1, k_2, \ldots, k_n$, for which $k_1 + 2k_2 + \cdots + nk_n = n$.

PROOF. We take $f(y) = \frac{1}{1-R(y-1)}$ and $g(x) = \frac{1}{1-x}$. It is immediate that $h(x) = f \circ g(x) = \frac{1-x}{1-(R+1)x}$. Since all these functions are also geometric series, we have for $|y-1| < 1/R$, $|x| < 1/(R+1)$:

$$f(y) = \sum_{j=0}^{\infty} R^j (y-1)^j, \quad g(x) = \sum_{j=0}^{\infty} x^j .$$

So we have:

$$\begin{aligned} h(x) &= \frac{1}{1-(R+1)x} - \frac{x}{1-(R+1)x} = \sum_{j=0}^{\infty} (1+R)^j x^j - \sum_{j=0}^{\infty} (1+R)^j x^{j+1} \\ &= 1 + \sum_{j=1}^{\infty} R(1+R)^{j-1} x^j . \end{aligned}$$

Evaluating f and h at $y = 1$ and $x = 0$, we find that $f^{(j)}(1) = j!R^j$, $g^{(k)}(0) = k!$, and $h^{(n)}(0) = n!R(1+R)^{n-1}$, from which the lemma follows. ∎

We now apply the previous two lemmas together with the proposition of the rate of the growth of derivatives to study compositions of real analytic functions:

PROPOSITION 3.3. *Suppose that* $f(y)$ *is real analytic in an open interval* $y \in I$ *in* $\mathbf{R}$, *that is*

$$\sup_{y\in I} |f^{(j)}(y)| \le CA^j j! \quad (j = 0, 1, 2, \dots)$$

and $g(x)$ *is real analytic in an open interval in* $y_0 = g(x_0)$, $g(J) \subset I$ *in* $\mathbf{R}$,

$$\sup_{x\in J} |g^{(j)}(x)| \le DB^j j! \quad (j = 0, 1, 2, \dots) .$$

Then the composite function $h(x) = f \circ g(x)$ *is also real analytic and the following estimates holds:*

$$\sup_{x\in J} |h^{(j)}(x)| \le \frac{CDA}{1+DA}((1+DA)B)^j j! \quad (j = 0, 1, 2, \dots) .$$

PROOF. By Lemma 3.1 we have

$$h^{(n)}(x) = \sum \frac{n!}{k_1!k_2!\cdots k_n!} f^{(k)}(g(x)) \left(\frac{g^{(1)}(x)}{1!}\right)^{k_1} \left(\frac{g^{(2)}(x)}{2!}\right)^{k_1} \cdots \left(\frac{g^{(n)}(x)}{n!}\right)^{k_n}$$

where $k = k_1 + k_2 + \cdots + k_n$ and the sum is taken over all $k_1, k_2, \dots k_n$, for which $k_1 + 2k_2 + \cdots + nk_n = n$.
So we can estimate

$$\begin{aligned}
\sup_{x\in J} |h^{(n)}(x)| &\le \sum \frac{n!}{k_1!k_2!\cdots k_n!} CA^k k!(DB)^{k_1}(DB^2)^{k_2}\cdots(DB^n)^{k_n} \\
&= CB^n n! \sum \frac{k!}{k_1!k_2!\cdots k_n!} D^k A^k \\
&= CDAB^n(1+DA)^{n-1} n! = \frac{CDA}{1+DA}((1+DA)B)^n n!
\end{aligned}$$

Thus $h(x)$ satisfies the standard estimates that guarantee it to be a real analytic function. ∎

LEMMA 3.4. *Let* $\varphi(x)$ *be real analytic in an open set* $\Omega \subset \mathbf{R}^n$. *Then there is a majorant of* $\varphi(x)$ *in a neighborhood of each* $x^0 \in \Omega$.

PROOF. By the parallel transformation of the coordinate, we assume, without loss of generality, that $x^0 \in \Omega$ is the origin. For small $\varepsilon > 0$ and $(a_1, a_2, \dots, a_n)(a_j > 0)$, there is a rectangle $Q = \{|x_j| \le a_j + \varepsilon (j = 1, \dots, n)\}$ where φ is analytic. $\varphi(x) = \sum c_\alpha x^\alpha$. Putting $M = \max_Q |\varphi(x)|$ and $a_0 = \max\{a_1, \dots, a_n\}$, we have

$$|c_\alpha| \le \frac{M}{a^\alpha} .$$

We have the estimate for the coefficients

$$|c_\alpha| \le \frac{M}{a_0^{|\alpha|}}$$

This shows that the following function is a majorant for φ;

$$\sum \frac{M}{a^\alpha} x^\alpha = M\left(\sum_{\alpha_1}\left(\frac{x_1}{a_1}\right)^{\alpha_1}\right)\cdots\left(\sum_{\alpha_n}\left(\frac{x_n}{a_n}\right)^{\alpha_n}\right) = \frac{M}{\left(1-\frac{x_1}{a_1}\right)\cdots\left(1-\frac{x_n}{a_n}\right)}$$

Using $\frac{|\alpha|!}{\alpha!} \geq 1$ and $a_1^{\alpha_1}\cdots a_n^{\alpha_n} \geq a_0^{|\alpha|}$, we see that the following function Φ is a majorant for φ:

$$\Phi(x) = M\sum_{k=1}^{\infty}\left(\frac{x_1+x_2+\cdots+x_n}{a_0}\right)^k$$
$$= \frac{M}{1-\dfrac{x_1+\cdots+x_n}{a_0}}$$

For $Q_1 = \{|x_1+\cdots+x_n| \leq a_0\}$, we have

$$\sup_{x\in Q_1}|\partial^\alpha\varphi(x)| \leq M\frac{|\alpha|!}{a_0^{|\alpha|}}\,.$$

■

We have the estimates for the composite function of several variables by using Proposition 3.3 and Lemma 3.4.

PROPOSITION 3.5. *$f(y)$ is real analytic in a region $Q \subset \mathbf{R}^\mu$ and $g_1(x), g_2(x), \ldots, g_\mu(x)$ are real analytic in a region $S \subset \mathbf{R}^n$. $(g_1(x),\ldots,g_\mu(x)) \in Q$ for $x \in S$, that is, $g(S) \subset Q$. Furthermore the following estimates hold*

$$\sup_{y\in Q}|\partial^\alpha f(y)| \leq CA^{|\alpha|}|\alpha|!\,,\quad \sup_{x\in S}|\partial^\alpha g_j(x)| \leq DB^{|\alpha|}|\alpha|!\,.$$

Then $h(x) = f(g(x)) = (f\circ g)(x)$ is real analytic in a neighborhood of the origin and we have the following inequalities

$$\sup_{x\in S}|\partial^\alpha h(x)| \leq \frac{CD\mu A}{1+D\mu A}|\alpha|!((1+D\mu A)B)^{|\alpha|}\,.$$

PROOF. By Proposition 3.4, $f(y)$ has a majorant

$$F(y) = \frac{C}{1-A(y_1+\cdots+y_\mu)}$$

and estimated by

$$\sup_{y\in Q}|\partial^\alpha f(y)| \leq C(\mu A)^{|\alpha|}|\alpha|!\,.$$

By Proposition 3.4, we also have a majorant for $g_j(x)$

$$G(x) = \frac{D}{1 - Bt},$$

here $t = x_1 + x_2 + \cdots + x_n$. So we have the estimate

$$\sup_{x \in S} |\partial^\alpha g_i(x)| \le DB^{|\alpha|}|\alpha|!\,.$$

We have a majorant $F(G(x))$ for $f(g_1(x), \ldots, g_\mu(x))$ and we have the estimates for $h(x) = (f \circ g)(x)$

$$\sup_{x \in S} |\partial^\alpha h(x)| \le \frac{CD\mu A}{1 + D\mu A}((1 + D\mu A)B)^{|\alpha|}|\alpha|!\,.$$

■

4. Proof of Main Theorem, I

For simplicity we denote $\partial^\alpha u = \partial^{|\alpha|}u$. We also denote $N + 1$-times derivatives which contain with two different directions including ∂_j by $\partial^N \partial_j u$. Without loss of generality we prove the following inequality in a neighborhood of the origin. Let $\rho(x)$ be a $C_0^\infty(\omega)$-function which is equal to 1 in a neighborhood of 0. We shall prove the real analyticity of u by induction on $|\alpha|$. By assumption F satisfies the analyticity condition *i.e.* $F \in C^\omega(\omega \times \omega')$

$$\|\rho \partial^{N-2}_{(x,\eta)} F(x, \eta)\|_{H^m(\omega \times \omega')} \le DB^N N!\,. \tag{4.1}$$

where the derivative is taken with variables $(x, \eta) = (x, \zeta, \zeta_j, \zeta_{jk})$. We suppose that $u \in C^\infty(\omega)$ and satisfies

$$\|\rho^N \partial^N u\|_{H^m(\omega)} \le CA^N N!\,. \tag{4.2}$$

Then we shall show that the inequality (4.2) holds for the $N+1$ derivatives of the type $\partial^N \partial_j u$.

LEMMA 4.1. *Let $f, g \in H^m(\omega)$ and $m = [\frac{n}{2}] + 1$. Then we have*

$$\|fg\|_m \le C_1 \|f\|_m \|g\|_m \tag{4.3}$$

where C_1 is a constant which does not depend on f, g. (cf. [1]).

LEMMA 4.2. *If $u \in H_0^{m+2}(\Omega)$, then*

$$\|\partial^\beta u\|_m \le \|\Delta u\|_m \tag{4.4}$$

for any multi-index β, $|\beta| = 2$

Differentiating the equation (2.1) with respect to x_k, we have

$$\frac{\partial F}{\partial x_k} + F_u \frac{\partial u}{\partial x_k} + \sum_{j=1}^n F_{u_j} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{i,j=1}^n F_{u_{ij}} \frac{\partial^2}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_k} = 0\,,$$

where we put $\partial_i \partial_j u = u_{ij}$, $\partial_j u = u_j$. Subtracting the first 3 terms from the both sides, we get

$$\sum_{i,j=1}^{n} F_{u_{ij}} \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial u}{\partial x_k}\right) = -\sum_{j=1}^{n} F_{u_j} \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_k}\right) - F_u \frac{\partial u}{\partial x_k} - \frac{\partial F}{\partial x_k}.$$

Differentiating N times, we have the following equality:

$$\sum_{i,j=1}^{n} F_{u_{ij}} \partial_i \partial_j (\partial^{N+1} u) = \partial^{N+1} G,$$

here we denote $\partial^{N+1} G$ in the right-hand-side the terms which contain the derivatives of u of order $N+2$, at most. Putting $F_{u_{ij}} = a_{ij}$, we have

$$\sum_{i,j} a_{ij}(0) \partial_i \partial_j (\partial^{N+1} u) = \partial^{N+1} G - \sum (a_{ij}(x) - a_{ij}(0)) \partial_i \partial_j (\partial^{N+1} u).$$

By the ellipticity condition (2.2), we take a change of variables at the origin and we have

$$\Delta(\partial^{N+1} u) = \partial^{N+1} \tilde{G} - \sum (\tilde{a}_{ij}(x) - \tilde{a}_{ij}(0)) \partial_i \partial_j (\partial^{N+1} u). \tag{4.5}$$

By multiplying ρ^{N+1}, we have

$$\begin{aligned} \Delta(\rho^{N+1} \partial^{N+1} u) = {} & \rho^{N+1}(\partial^{N+1} \tilde{G}) - \sum \rho^{N+1} (\tilde{a}_{ij}(x) - \tilde{a}_{ij}(0)) \partial_i \partial_j (\partial^{N+1} u) \\ & - \sum 2(N+1) \partial_k ((\partial_k \rho) \rho^N \partial^{N+1} u)) \\ & - \sum 2(N+1)(\partial_k^2 \rho) \rho^N \partial^{N+1} u \\ & - \sum 2(N+1) N (\partial_k^2 \rho) \rho^{N-1} \partial^{N+1} u. \end{aligned} \tag{4.6}$$

By multiplying the parametrix of $\triangle$ and integrating in ω, we can estimate $\rho^{N+1} \partial^{N+1} u$ in the following.

We use the notation $r = |x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ and S_n = the area of the surface of the n-dimensional unit sphere which is equal to $\frac{2\pi^{n/2}}{\Gamma(n/2)}$.

LEMMA 4.3. *The fundamental solution for the Laplace operator Δ is given by*

$$\varphi(x, y) = \begin{cases} \dfrac{1}{2\pi} \log r & n = 2, \\ -\dfrac{1}{(n-2) S_n} r^{2-n} & n \geq 3. \end{cases}$$

We then have

$$\partial_i \partial_j \varphi(x, y) = \begin{cases} C_n n r^{-2-n} (x_i - y_i)(x_j - y_j) & i \neq j, \\ C_n \left(n r^{-n-2} (x_i - y_i)^2 + r^{-n} \right) & i = j. \end{cases}$$

Here $C_n = \frac{1}{2\pi}$ *if* $n = 2$ *or* $C_n = \frac{1}{S_n}$ *if* $n \neq 2$. *Since* $\partial_j \partial_k \varphi(x, y)$ *is a homogeneous function of order* $-n$ *and*

$$\int_{|x|=1} \partial_j \partial_k \varphi(x, 0) dx = 0$$

for $i \neq j$, *the operators* $u \to K_{ij} u = \int u(y)(\text{v.p. } \partial_i \partial_j \varphi(x, y)) dy$ *are bounded operators in* L^2. (cf. [9]).

By the assumption of the derivatives given at first of this section, we can write $\partial_j \partial^N = \partial_p \partial_q \partial^{N-1}$. The first term in the right-hand-side of (4.6) can be written by the following form:

$$\begin{aligned} \rho^{N+1} \partial_p \partial_q \partial^{N-1} \tilde{G} = {} & \partial_p \partial_q (\rho^{N+1} \partial^{N-1} \tilde{G}) - (N+1) \partial_p ((\partial_q \rho) \rho^N \partial^{N-1} \tilde{G}) \\ & - (N+1) \partial_q ((\partial_p \rho) \rho^N \partial^{N-1} \tilde{G}) \\ & + (N+1) N (\partial_p \rho)(\partial_q \rho) \rho^{N-1} \partial^{N-1} \tilde{G} \\ & + (N+1)(\partial_p \partial_q \rho) \rho^N \partial^{N-1} \tilde{G} \end{aligned} \tag{4.7}$$

Multiplying φ and integrating on ω and taking H^m norm, we have the following estimate for the first term in the right-hand-side of (4.7),

$$\begin{aligned} \left\| \int K_{pq} \rho^{N+1} \partial^{N-1} \tilde{G} \, dy \right\|_m & = \| \widehat{K_{pq}} (1 + |\xi|)^m \widehat{\rho^{N+1} \partial^{N-1}} \tilde{G} \| \\ & \leq \sup_{|\xi|=1} |\hat{K_{pq}}| \| \widehat{\rho^{N+1} \partial^{N-1}} \tilde{G} \|_m \end{aligned}$$

By using Proposition 3.5 and the assumption of the induction on u, we have

$$\| \rho^{N+1} \partial^{N-1} \tilde{G} \|_m \leq \frac{C D \mu A}{1 + D \mu A} ((1 + D \mu A) B)^N N! \,. \tag{4.8}$$

Since the 2nd term of (4.7) has the derivative of ρ, their supports are contained in ω and $\equiv 0$ near the origin. By multiplying φ and integrating on ω, taking H^m norm, integrating by parts and by the analyticity of φ outside a neighborhood of the origin, we have the same estimate as (4.8):

$$\begin{aligned} \| \langle (N+1) \partial_p ((\partial_q \rho) \rho^N \partial^{N-1} \tilde{G}), \varphi \rangle \|_m & = (N+1) \| \langle (\partial_q \rho) \rho^N \partial^{N-1} \tilde{G}), \partial_p \varphi \rangle \|_m \\ & \leq C (N+1) \| (\partial_q \rho) \rho^N \partial^{N-1} \tilde{G} \|_m \\ & \leq \frac{C D \mu A}{1 + D \mu A} ((1 + D \mu A) B)^{N+1} (N+1)! \end{aligned}$$

We take $(1 + D\mu A)B \geq 1$. By the similar argument as above, H^m norm of the 3rd, the 4th and the 5th term have the estimates:

$$\frac{CD\mu A}{1 + D\mu A}((1 + D\mu A)B)^{N+1}(N + 1)!$$

Combining these inequalities, we have the estimate of the form

$$\|\langle \rho^{N+1}\partial_p\partial_q\partial^{N-1}\tilde{G}, \varphi\rangle\|_m \leq \frac{CD\mu A}{1 + D\mu A}((1 + D\mu A)B)^{N+1}(N + 1)!$$

We estimate the 2nd term on the right-hand-side of (4.6). As in the same calculation as in (4.7), we have

$$\begin{aligned}(\tilde{a}_{ij}(x) - \tilde{a}_{ij}(0))&\partial_i\partial_j(\rho^{N+1}\partial^{N+1}u)\\ &= \partial_i\partial_j((\tilde{a}_{ij}(x) - \tilde{a}_{ij}(0))\rho^{N+1}\partial^{N+1}u)\\ &\quad - \partial_j((\partial_i\tilde{a}_{ij}(x))\rho^{N+1}\partial^{N+1}u) - \partial_i((\partial_j\tilde{a}_{ij}(x))\rho^{N+1}\partial^{N+1}u)\\ &\quad + (\partial_i\partial_j\tilde{a}_{ij}(x))\rho^{N+1}\partial^{N+1}u\end{aligned} \tag{4.9}$$

Multiplying φ and integrating on ω and taking H^m norm, the main term of these terms can be estimated as follows.

$$\begin{aligned}\|\langle\partial_i\partial_j((\tilde{a}_{ij}(x) - \tilde{a}_{ij}(0))&\rho^{N+1}\partial^{N+1}u), \varphi\rangle\|_m\\ &\leq C\|(\tilde{a}_{ij}(x) - \tilde{a}_{ij}(0))(\rho^{N+1}\partial^{N+1}u)\|_m\\ &\leq C\max_{\omega}|\tilde{a}_{ij}(x) - \tilde{a}_{ij}(0)|\|(1 + |D|)^m\rho^{N+1}\partial^{N+1}u\|_0 + \text{l.o.t}\\ &\leq C(\varepsilon)\|\rho^{N+1}\partial^{N+1}u\|_m + \text{l.o.t}\,.\end{aligned}$$

Here we denote the terms l.o.t. the lower order terms to which we can apply the inductive assumption. The last inequality is obtained by choosing the diameter of ω sufficiently small. The other terms on the right-hand-side of (4.9) can be estimated by the same argument as in (4.7).

To estimate the third term and the 4th terms on the right-hand-side of (4.6), we use the method as estimating the 3rd term and the 4th term of (4.7).

Considering these estimates, we then have

$$\|\rho^{N+1}\partial^N\partial_j u\|_m \leq C(\varepsilon)\|\rho^{N+1}\partial^N\partial_j u\|_m + \frac{CD\mu A}{1 + D\mu A}((1 + D\mu A)B)^{N+1}(N + 1)!\,.$$

From which we have

$$(1 - C(\varepsilon))\|\rho^{N+1}\partial^N\partial_j u\|_m \leq \frac{CD\mu A}{1 + D\mu A}((1 + D\mu A)B)^{N+1}(N + 1)!\,.$$

By taking the radius of the support of the cut-of function sufficiently small, we get the estimate for $\partial^N\partial_j u$. We note the proof carefully so that $((1 + D\mu A)B)$ is replaced by $(2n^2 + 5n + 3)((1 + D\mu A)B)$ ■

5. Proof of the Main Theorem, II

We have the following equality by the same method as in §4:

$$\Delta u = G(x, u, \nabla u, \nabla^2 u) \tag{5.1}$$

Differentiating both sides of the equation (5.1) $N-1$ times with respect to x_j, we have

$$\partial_j^{N-1} \Delta u = \partial_j^{N-1} G(x, u, \nabla u, \nabla^2 u)$$

$$\partial_j^{N+1} u = -\partial_j^{N-1} \Delta' u + \partial_j^{N-1} G'(x, u, \nabla u, \nabla^2 u) + (F_{u_{jj}}(x) - F_{u_{jj}}(0))\partial_j^{N+1} u$$

where Δ' denotes $\Delta - \partial_j^2$. Multiplying $\rho(x)^{N+1}$ both sides, we have

$$\begin{aligned}\rho^{N+1}\partial_j^{N+1} u = &-\rho^{N+1}\partial_j^{N-1}\Delta' u + \rho^{N+1}\partial_j^{N-1} G'(x, u, \nabla u, \nabla^2 u)\\ &+ \rho^{N+1}(F_{u_{jj}}(x) - F_{u_{jj}}(0))\partial_j^{N+1} u\end{aligned}$$

where $\partial_j^{N-1} G'$ does not contain $\partial_j^{N+1} u$. The terms in the right-hand-side except the last term have the type estimated in §4. Hence we can use the assumption on the induction. The last term in the right-hand-side has the estimate as in §4. Therefore we have the estimate

$$\|\rho^{N+1}\partial_j^{N+1} u\|_m \le \frac{CD\mu A}{1 + D\mu A}((1 + D\mu A)B)^{N+1}(N+1)! + C(\varepsilon)\|\rho^{N+1}\partial_j^{N+1} u\|_m .$$

Taking the support ω so small that $C(\varepsilon) < \frac{1}{2}$, we have

$$\|\partial_j^{N+1} u\|_m \le 2\frac{CD\mu A}{1 + D\mu A}((1 + D\mu A)B)^{N+1}(N+1)! .$$

■

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