# A Cohomology ($p$+1) Form Canonically Associated with Certain Codimension-$q$ Foliations on a Riemannian Manifold

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**Abstract.** Let $(M^n, g)$ be a closed, connected, oriented, $C^\infty$, Riemannian, $n$-manifold with a transversely oriented foliation **F**. We show that if $\{X, Y\}$ are basic vector fields, the leaf component of $[X, Y]$, $\mathcal{V}[X, Y]$, has vanishing leaf divergence whenever $\kappa \wedge \chi_{\mathbf{F}}$ is a closed (possibly zero) de Rham cohomology $(p+1)$-form. Here $\kappa$ is the mean curvature one-form of the foliation **F** and $\chi_{\mathbf{F}}$ is its characteristic form. In the codimension-2 case, $\kappa \wedge \chi_{\mathbf{F}}$ is closed if and only if $\kappa$ is horizontally closed. In certain restricted cases, we give necessary and sufficient conditions for $\kappa \wedge \chi_{\mathbf{F}}$ to be harmonic. As an application, we give a characterization of when certain closed 3-manifolds are locally Riemannian products. We show that bundle-like foliations with totally umbilical leaves with leaf dimension greater than or equal to two on a constant curvature manifold, with non-integrable transversal distribution, and with Einstein-like transversal geometry are totally geodesic.

## Introduction

Let $(M, g)$ be an oriented, $n$-dimensional Riemannian manifold admitting a transversely oriented foliation, **F**, of leaf dimension $p$ and codimension $q$ so $p + q = n$. Generally, we will assume that both integers $p$ and $q$ are positive. Let $\kappa$ denote the mean curvature one-form associated with the foliation **F** and let $\chi_{\mathbf{F}}$ denote the characteristic form of the leaves of **F**. Following Kamber and Tondeur, we consider the $p + 1$ form $\kappa \wedge \chi_{\mathbf{F}}$. Suppose $X$ and $Y$ are local basic vector fields orthogonal to the leaves of **F**. If $\kappa \wedge \chi_{\mathbf{F}}$ is a closed form, then the leaf divergence of the leaf component of $[X, Y]$, $\text{div}_{\mathbf{F}} \mathcal{V}[X, Y]$, vanishes identically.

In the special case of $q = 2$, $\kappa \wedge \chi_{\mathbf{F}}$ is closed if and only if $\text{div}_{\mathbf{F}} \mathcal{V}[X, Y] \equiv 0$ for any local basic vector fields $X$ and $Y$. This result, Theorem 1.2 below, is a general result for a foliation on an *arbitrary* Riemannian manifold. It illustrates once more the general principle that when a cohomology form arises, some pleasant geometric consequences often follow.

Now suppose that $(M, g)$ above is a closed manifold and that the foliation **F** is a Riemannian foliation. Then a fundamental result of Dominguez asserts that there then exists a metric $g$ on $M$ so that the associated mean curvature one form $\kappa$ of **F** with respect to $g$ is a basic one-form. In this setting a result of Kamber-Tondeur asserts that $\kappa$ is a closed one-form.

Suppose now $q = 2$. Then it is easy to see that $\kappa \wedge \chi_{\mathbf{F}}$ is a closed form. In fact, we establish in Theorem 1.4 that $\kappa \wedge \chi_{\mathbf{F}}$ is co-closed if and only if $\mathrm{div}_{\mathbf{H}} \tau = 0$, where $\tau$ is the mean curvature vector field dual to $\kappa$ and where the divergence is taken with respect to a basic orthonormal frame orthogonal to $\mathbf{F}$. The proof involves lengthy and not entirely routine calculations, using three sets of arguments. The O'Neill tensors $T$ and $A$ play a crucial role.

Applying Theorem 1.4 to a Riemannian flow on a closed 3-manifold, $M^3$, we show that with respect to a Dominguez metric, $M^3$ decomposes as a local Riemannian product if and only if $\mathrm{Ric}^M(V, V) = 0$, where $V$ is the unit length vector field tangent to the flow, and $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic (Corollary 1.5). The proof uses an important result of Ranjan as developed in [T3]. Using computations from [T3], we show additionally in Corollary 1.6 that for a Riemannian flow on closed $M^n$ splits as a local Riemannian product with respect to the Dominguez metric, if and only if $\mathrm{Ric}^M(V, V) = 0$ and $\mathrm{div}_{\mathbf{H}} \tau = 0$. Since the result of Corollary 1.6 is general, the computation does not depend on Theorem 1.4.

Theorem 1.7 establishes a result similar to that of Theorem 1.4 in the less interesting case $q = 1$, while Theorems 1.8, 1.9 and 1.11 wrap things up in the spirit of [T3].

In section §2 we obtain local properties for bundle-like foliations with totally umbilical leaves on a constant curvature manifold. In Proposition 2.3 we obtain an equivalent condition for $\kappa$ to be horizontally closed for a bundle-like foliation with totally umbilical leaves. Now we assume the $n$-dimensional Riemannian manifold $(M, g)$ has constant curvature $c$. In Proposition 2.4 we show that $\kappa$ is a basic one-form. As a consequence of Theorems 1.4 and 1.7, we get that $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic if and only if $g(\tau, \tau) = -pqc$ provided that the transversal distribution is integrable, the dimension of the leaves $p$ is greater than one and the codimension $q$ is either one or two. Assuming that the transversal distribution is non-integrable (at any point), we obtain a sufficient condition for $\mathbf{F}$ to be totally geodesic. In an important paper, Walschap showed that a bundle-like foliation with totally umbilical leaves and with leaf dimension $p > 1$ on a complete simply connected space of constant curvature $c \geq 0$ is totally geodesic (see [Wa]). Using a different approach, Theorem 2.8 provides a similar result to Theorem 3.1 in [Wa], under no global assumptions and under some additional local ones. A remarkable fact is that Propositions 2.3, 2.4, 2.5, Theorem 2.8 can be extended to the pseudo-Riemannian case with definite induced metrics on leaves, and, additionally only for Theorem 2.8, with induced positive definite transversal metrics.

## 1. The $(p + 1)$ form $\kappa \wedge \chi_{\mathbf{F}}$

Throughout this paper all maps, functions and morphisms are assumed to be at least of class $C^\infty$. On a closed connected oriented $C^\infty$ Riemannian manifold $(M^n, g)$, let $\mathbf{F}$ be a transversely oriented foliation of leaf dimension $p$ and codimension $q = n - p$. Let $\mathbf{V}$ denote the distribution tangent to the foliation $\mathbf{F}$, and $\mathbf{H}$ the distribution orthogonal to $\mathbf{V}$ in $TM$ determined by the metric $g$. If $E$ is a vector field on $M$, $\mathcal{V}E$ and $\mathcal{H}E$ will denote the projections of $E$ onto the distributions $\mathbf{V}$ and $\mathbf{H}$ respectively. Call the vector field $E$ *vertical* if $\mathcal{V}E = E$. Call $E$ *horizontal* if $\mathcal{H}E = E$.

In general a $C^\infty$ *foliation* of *codimension-q* on an *n*-dimensional manifold $M$ can be defined is a maximal family of $C^\infty$ submersions $f_\alpha : U_\alpha \to f_\alpha(U_\alpha) \subset \mathbf{R}^q$ where $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of $M$ and where for each $\alpha, \beta \in \Lambda$ and each $x \in U_\alpha \cap U_\beta$, there exists a local diffeomorphism $\phi_{\beta\alpha}^x$ of $R^q$ so $f_\beta = \phi_{\beta,\alpha}^x \circ f_\alpha$ in some neighborhood $U_x$ of $x$ (see [L], 2,3).

A horizontal vector field $Z$ defined on some open set $U$ where $U \subset U_\alpha$ is called $f_\alpha$-*basic* provided $f_{\alpha*}Z$ is a well defined vector field on $f_\alpha(U)$. As pointed out in [E1] (for *any* metric $g$), if $U \subset U_\beta$, then $Z$ is also $f_\beta$- *basic*, so one can speak of $Z$ as a *local basic vector field*. We sometimes drop the word "local." Let $i(W)$ and $\theta(W)$ denote the interior product and the Lie derivative with respect to a vector field $W$. A differential form $\phi$ is called *basic* provided $i(W)\phi = 0$ and $\theta(W)\phi = 0$ for all vertical vector fields $W$ ([T1], 118). We follow the conventions of [AMR] for the formalism of differential forms and their exterior derivatives.

$D$ will denote the Levi-Civita connection on $M$ and, following [EP], we introduce the tensors $T$ and $A$ as follows. For vector fields $E$ and $F$ on $M$,

$$(1.1) \qquad T_E F = \mathcal{V}D_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}D_{\mathcal{V}E}\mathcal{V}F, \text{ and}$$

$$(1.2) \qquad A_E F = \mathcal{V}D_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}D_{\mathcal{H}E}\mathcal{V}F.$$

Then $T$ and $A$ are tensors of type $(1, 2)$. These tensors satisfy the usual properties outlined in [EP]. We note that if $X$ and $Y$ are horizontal,

$$(1.3) \qquad A_X Y \neq -A_Y X, \text{ in general},$$

unless the foliation $\mathbf{F}$ is $bundle - like$ with respect to the metric $g$ (see [JW], Lemma (1.2)) that is, if $X$ is a basic vector field, $Wg(X, X) = 0$ for every vertical vector field $W$. If $\{V_1, V_2, V_3, \cdots V_p\}$ is a local orthonormal frame tangent to the foliation, we define the *mean curvature one-form $\kappa$* as follows:

$$(1.4) \qquad \kappa(E) = \sum_{i=1}^{p} g(E, T_{V_i}V_i).$$

Call $\kappa$ *horizontally closed* if $d\kappa(Z_1, Z_2) = 0$ for any horizontal vector fields $Z_1, Z_2$. Using the usual properties of the tensor $T$, one sees easily that if $X$ is basic,

$$(1.5) \qquad \kappa(X) = \sum_{i=1}^{p} g([X, V_i], V_i).$$

Following [T-1], page 65–66, let $\chi_\mathbf{F}$ denote the characteristic form for the foliation $\mathbf{F}$. Then with $\{V_1, \cdots, V_p\}$ as above and for vector fields $\{E_1, \cdots, E_p\}$ on $M^n$, we have:

$$(1.6) \qquad \chi_\mathbf{F}(E_1, E_2 \cdots, E_p) = \det(g(E_i, V_j)).$$

This characteristic differential form (see [T3], page 37) is independent of the local orthonormal frame $\{V_1, \cdots V_p\}$. If any one of the arguments $E_i$ is horizontal, then the left hand side of (1.6) vanishes. This fact will be used repeatedly in the computations below.

We say a **F** is a *Riemannian* foliation of leaf dimension $p$ and codimension-$q$, provided that there is some Riemannian metric $g$ on $M^n$ with respect to which **F** is bundle-like in the sense above. If **F** is a Riemannian foliation on a compact manifold $M^n$, then a fundamental result of Dominguez, [D], shows that there always exists a metric $g$ for which **F** is bundle-like and for which the associated mean curvature one-form, $\kappa$, is basic. We call this metric, a *Dominguez metric*.

Part (a) of the following result is proven in the appendix of [EP].

LEMMA 1.1.   (a)   *Let $(M^n, g)$ be a connected, oriented, $C^\infty$ Riemannian n-manifold with a transversely oriented codimension-q foliation* **F**, *with $q \geq 2$. Suppose X and Y are basic vector fields. Then $\mathcal{V}[X, Y]$ has vanishing leaf divergence if and only if $\kappa$ is horizontally closed.*

(b)   *Let* **F** *be a transversely oriented Riemannian foliation on a closed, oriented Riemannian manifold $(M, g')$. Then there exists a Dominguez metric $g$ on $M$ so that if X and Y are basic with respect to $g$, then $\mathrm{div}_{\mathbf{F}} \mathcal{V}[X, Y] = 0$ and indeed $\mathrm{div}_M \mathcal{V}[X, Y] = 0$.*

PROOF.   (a)   This follows immediately from formula (3) of [EP] which can be expressed this way:

$$(1.7) \qquad\qquad d\kappa(X, Y) = -\mathrm{div}_{\mathbf{F}} \mathcal{V}[X, Y],$$

where the right hand side denotes the divergence of $\mathcal{V}[X, Y]$ along a leaf of **F**. A more succinct proof of (1.7) appears in [CE 1].

(b)   Let $g$ be such a metric for $M$. Then **F** is bundle-like with respect to $g$ and the associated mean curvature one form $\kappa$ is basic by Dominguez's Theorem. Then $\kappa$ is closed by a result of Kamber-Tondeur [T3, p. 82], and so in particular, $\kappa$ is horizontally closed. Thus, by Lemma 1.1(a) and the appendix to [EP], $\mathrm{div}_{\mathbf{F}} \mathcal{V}[X, Y] = \mathrm{div}_M \mathcal{V}[X, Y] = 0$.

The form $\kappa \wedge \chi_{\mathbf{F}}$ arises in the important role in the work of Kamber and Tondeur on foliations, especially Riemannian foliations ([T1], pages 121 and 152, [T3], page 82). It turns out that when this form is closed, the following pleasant property obtains for *arbitrary* foliations on Riemannian manifolds of codimension $q \geq 2$ (actually $q \geq 1$). The result illustrates once more the tie between cohomology and geometry.

THEOREM 1.2.   *Let $(M^n, g)$ be a closed, connected, oriented, $C^\infty$ Riemannian n-manifold with a transversely oriented codimension-q foliation* **F**. *Suppose X and Y are basic vector fields. Then $\mathcal{V}[X, Y]$ has vanishing leaf divergence (equivalently $\kappa$ is horizontally closed) whenever $\kappa \wedge \chi_{\mathbf{F}}$ is a closed (possibly zero) de Rham cohomology $p + 1$ form. In fact, if the codimension of* **F**, $q = 2$, *then $\kappa$ is horizontally closed if and only if $\kappa \wedge \chi_{\mathbf{F}}$ is closed. Finally, if* **H** *is also integrable, then $\kappa \wedge \chi_{\mathbf{F}}$ is always closed.*

PROOF.   Note, if $q = 1$, $\kappa \wedge \chi_{\mathbf{F}}$ is an $n$-form and hence closed. If $X$ and $Y$ are basic vector fields, then in the codimension-one case, $X = fZ$ and $Y = hZ$ where $f$ and $h$ are functions defined on an appropriate open set and $Z$ is a unit length horizontal vector field on that set. Then, $\mathcal{V}[X, Y] = 0$ and the theorem always holds in this trivial case.

To establish this for $q \geq 2$, we will use the local frame, $\{V_1, V_2, V_3, \cdots V_p, X, Y\}$ where $X$ and $Y$ are basic and span a subspace of $\mathbf{H}$ at each $x \in U$ where $X_x$ and $Y_x$ are defined. Note, we make no requirements that $X$ and $Y$ form a basic *orthonormal* frame for a subspace of $\mathbf{H}$, since we do not yet assume the metric $g$ on $M$ is bundle-like. A fundamental result of Rummler [Ru], yields:

$$(1.8) \qquad d\chi_{\mathbf{F}}(V_1, \cdots, V_p, X) = (-1)^{p+1}\kappa(X)\chi_{\mathbf{F}}(V_1, \cdots, V_p).$$

At this point it is worth pointing out that for any $(p+2)$-form $\gamma$,

$$\gamma(V_1, V_2, V_3, \cdots V_p, X, Y) = \gamma(X, Y, V_1, V_2, V_3, \cdots V_p).$$

Using the formulas in [AMR] page 394 and the remarks above, we have,

$$(1.9) \qquad \begin{aligned} d(\kappa \wedge \chi_{\mathbf{F}})(V_1, \cdots, V_p, X, Y) &= (d\kappa \wedge \chi_{\mathbf{F}})(V_1, \cdots, V_p, X, Y) \\ &\quad - (\kappa \wedge d\chi_{\mathbf{F}})(V_1, \cdots, V_p, X, Y) \\ &= (d\kappa \wedge \chi_{\mathbf{F}})(X, Y, V_1, \cdots, V_p) \\ &\quad - (\kappa \wedge d\chi_{\mathbf{F}})(V_1, \cdots, V_p, X, Y), \end{aligned}$$

which becomes,

$$(1.10) \qquad \begin{aligned} &= d\kappa(X, Y)\chi_{\mathbf{F}}(V_1, \cdots, V_p) - (-1)^p\kappa(X)d\chi_{\mathbf{F}}(V_1, \cdots, V_p, Y) \\ &\quad -(-1)^{p+1}\kappa(Y)d\chi_{\mathbf{F}}(V_1, \cdots, V_p, X) \\ &= d\kappa(X, Y)\chi_{\mathbf{F}}(V_1, \cdots, V_p) + (-1)^{p+1}\kappa(X)d\chi_{\mathbf{F}}(V_1, \cdots, V_p, Y) \\ &\quad +(-1)^{p+2}\kappa(Y)d\chi_{\mathbf{F}}(V_1, \cdots, V_p, X), \end{aligned}$$

which then becomes by (1.8),

$$(1.11) \qquad \begin{aligned} &d\kappa(X, Y)\chi_{\mathbf{F}}(V_1, \cdots, V_p) + (-1)^{p+1}\kappa(X)(-1)^{p+1}\kappa(Y)\chi_{\mathbf{F}}(V_1, \cdots, V_p) \\ &+(-1)^{p+2}\kappa(Y)(-1)^{p+1}\kappa(X)\chi_{\mathbf{F}}(V_1, \cdots, V_p) \end{aligned}$$

which becomes,

$$(1.12) \qquad\qquad\qquad\qquad\qquad d\kappa(X, Y),$$

since $\chi_{\mathbf{F}}(V_1, \cdots, V_p) = 1$. Thus,

$$(1.13) \qquad\qquad d(\kappa \wedge \chi_{\mathbf{F}})(V_1, \cdots, V_p, X, Y) = d\kappa(X, Y).$$

Hence, if $\kappa \wedge \chi_{\mathbf{F}}$ is a closed $(p+1)$-form, then in particular, the left hand side of (1.13) vanishes, so $\kappa$ is horizontally closed and the result follows by Lemma 1.1. In the codimension-2 case, $\kappa \wedge \chi_{\mathbf{F}}$ is closed if and only if the left hand side of (1.13) vanishes.

That $\kappa \wedge \chi_{\mathbf{F}}$ is closed when $\mathbf{H}$ is integrable follows from the above computations together with the observation $d(\kappa \wedge \chi_{\mathbf{F}})(E_1, E_2, \cdots, E_{p+2})$ vanishes identically if three or more of the $E_j$ are basic. We omit the details.

We offer the following improvement of a result that appeared in [E2]. It should be noted that in Theorem 1.3 below we do *not* require that the flow **F** is *Riemannian* in the sense above.

THEOREM 1.3. *Let $(M^3, g)$ be a closed, connected, oriented $C^\infty$ Riemannian manifold of dimension 3 with a transversely oriented flow **F**. Suppose the following conditions obtain.*

(a) **F** *admits a basic transverse volume form $\mu$.*

(b) $\kappa \wedge \chi_{\mathbf{F}}$ *is a closed 2-form.*

(c) *Let $X$ and $Y$ denote local basic vector fields so that $\mu(X, Y) = 1$. Assume the globally defined vector field, $\mathcal{V}[X, Y]$ satisfies the following*:

$$[Z, \mathcal{V}[X, Y]] = f_Z \mathcal{V}[X, Y]$$

*for any basic vector field $Z$ and for some function $f_Z$ depending on $Z$.*

*Then either*:

i) $\mathcal{V}[X, Y]$ *vanishes identically on $M$, so **H** is integrable and the leaves of **H** are minimal surfaces in $M$, or,*

ii) $\mathcal{V}[X, Y]$ *never vanishes and so **H** is always a contact structure.*

PROOF. Condition (b) replaces the condition in [E2] that $\kappa$ is horizontally closed. These are equivalent, since when $n = 3$, $p = 1$ and so $q = 2$. In this case the last part of Theorem 1.2, (in particular (1.13)), applies so $\kappa$ is horizontally closed if and only if $\kappa \wedge \chi_{\mathbf{F}}$ is closed. Then the argument given in [E2] carries over and **H** is a foliation of M by minimal surfaces, whenever $\mathcal{V}[X, Y]$ vanishes at one point. The only other possibility is that $\mathcal{V}[X, Y]$ never vanishes, and in this case, **H** is a contact structure.

Now assume that **F** is a *Riemannian* foliation of leaf dimension p and codimension $q = 2$. Although far more restrictive, this assumption allows us to give a definitive answer to the question: when is $\kappa \wedge \chi_{\mathbf{F}}$ co-closed? To address this problem, we need the following preparation. Because **F** is bundle-like with respect to $g$, the local submersions defining **F** are Riemannian submersions in the sense of [O'N] (see [T1], [T2], [E] and [EP]), and so we can choose a local orthonormal frame. $\{X_1, X_2, V_1, V_2, \cdots, V_p\}$ so $X_1, X_2$ are basic vector fields, and so $\{V_1, V_2, \cdots, V_p\}$ is a local orthonormal frame for **V**. Indeed, at a fixed $x \in M$, we can choose $X_1, X_2$ so $(\mathcal{H}D_{X_i} X_j)_x = 0$ for $1 \leq i, j \leq 2$. Because, **F** is bundle-like with respect to $g$, $A_{X_i} X_i = 0$, for $1 \leq i \leq 2$. Let $\{V_1, V_2, \cdots, V_p\}$ be a local orthonormal frame for **V**. At a given $x \in M$, we can choose this frame so that $(\mathcal{V}D_{V_i} V_j)_x = 0$. Note, for *any* vector field $E$, $Eg(V_i, V_i) = 2g(D_E V_i, V_i) = 0$ and $0 = Eg(V_i, V_j) = g(D_E V_i, V_j) + g(V_i, D_E V_j)$. We will exploit these well known facts extensively in the computations below. When the orthonormal frame $\{X_1, X_2, V_1, V_2, \cdots, V_p\}$ enjoys these additional properties at $x \in M$, we'll call the frame *a preferred orthonormal frame at x*.

Set $\tau = \sum_{i=1}^{p} \mathcal{H}D_{V_i} V_i$. (We follow the conventions in foliations and suppress the usual constant.) Note, when $g$ is a Dominguez metric for **F**, $\tau$ is basic, because it is dual to the one-form $\kappa$ with respect to $g$ by (1.4). Set $\mathrm{div}_{\mathbf{H}} \tau = g(X_1, D_{X_1}\tau) + g(X_2, D_{X_2}\tau)$. Then, it

is well known ([P], page 151) that $\delta(\kappa \wedge \chi_{\mathbf{F}}) = -\operatorname{div}(\kappa \wedge \chi_{\mathbf{F}})$. We have the following result. Note, the operator $\delta$ below is on $M$ itself.

THEOREM 1.4. *Let $M^n$ be a closed, oriented, $C^\infty$, Riemannian manifold with a transversely oriented codimension-2 Riemannian foliation $\mathbf{F}$. Let $g$ be a Dominguez metric (with respect to which $\mathbf{F}$ is bundle-like and $\kappa$ is basic). Then, $\kappa \wedge \chi_{\mathbf{F}}$ is closed. In fact, $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic if and only if $\operatorname{div}_{\mathbf{H}} \tau = 0$, where $\tau$ is the mean curvature vector field dual to $\kappa$.*

PROOF.    To show $\kappa \wedge \chi_{\mathbf{F}}$ is closed, it suffices to observe that in the case under consideration, $d(\kappa \wedge \chi_{\mathbf{F}})$ is a closed $n$-form. Evaluating, $d(\kappa \wedge \chi_{\mathbf{F}})$ on a preferred orthonormal frame $\{X_1, X_2, V_1, V_2, \cdots, V_p\}$, we see that (1.13) becomes

$$(1.14) \qquad d(\kappa \wedge \chi_{\mathbf{F}})(V_1, \cdots, V_p, X_1, X_2) = d\kappa(X_1, X_2).$$

But, since $\kappa$ is basic, it is closed by a fundamental result of Kamber-Tondeur for bundle-like foliations with $\kappa$ basic (see [T3], page 82). This means that the left hand side of (1.14) vanishes and so $\kappa \wedge \chi_{\mathbf{F}}$ is closed, as claimed.

We now will show that under the stated hypotheses, $\kappa \wedge \chi_{\mathbf{F}}$ is co-closed on $M$. To do this we will use exclusively the preferred orthonormal frame above. The result follows from lengthy computations of $\delta(\kappa \wedge \chi_{\mathbf{F}})$ on three sets of arguments: $(V_1, V_2, \cdots, V_p)$, $(X_1, V_1, \cdots, \hat{V}_j, \cdots, V_p)$ and $(X_1, X_2, V_1 \cdots, V_{p-2})$. It should be mentioned (that up to sign) it suffices to use $X_1$ in the second set of arguments.

In the first evaluation, the reader should keep in mind the following principles: $g(\mathcal{V}D_{X_i}V_j, V_j) = 0$, mentioned before. Secondly, terms with *repeated* vertical vector fields vanish, thirdly, the sum

$\sum_{i=1}^p (\kappa \wedge \chi_{\mathbf{F}})(D_{V_i}V_i, V_1, V_2, \cdots, V_p)$ and the terms $(\kappa \wedge \chi_{\mathbf{F}})(V_1, D_{V_1}V_1, V_2, \cdots, V_p)$, $(\kappa \wedge \chi_{\mathbf{F}})(V_2, V_1, D_{V_2}V_2, \cdots, V_p), \cdots, (\kappa \wedge \chi_{\mathbf{F}})(V_p, V_1, \cdots, D_{V_p}V_p)$ sum to zero. Also, $(\kappa \wedge \chi_{\mathbf{F}})$ vanishes identically on $(E_1, E_2, E_3, \cdots, E_{p+1})$ if two or more of the arguments are horizontal vector fields. Finally, $\chi_{\mathbf{F}}(E_1, \cdots, E_p)$ will vanish identically even if all the $E_j$ are vertical but linearly dependent. In the expansion below, we have rearranged some of the terms in the expansion, but they are all there. We carry out each of these computations at $x \in M$ above.

$$(1.15) \qquad \delta(\kappa \wedge \chi_{\mathbf{F}})(V_1, V_2, \cdots, V_p)$$

$$= -\sum_{i=1}^p (D_{V_i}(\kappa \wedge \chi_{\mathbf{F}}))(V_i, V_1, V_2, \cdots, V_p) - \sum_{a=1}^2 (D_{X_a}(\kappa \wedge \chi_{\mathbf{F}}))(X_a, V_1, V_2, \cdots, V_p)$$

$$= -\sum_{i=1}^p V_i(\kappa \wedge \chi_{\mathbf{F}})(V_i, V_1, V_2, \cdots, V_p) + \sum_{i=1}^p (\kappa \wedge \chi_{\mathbf{F}})(D_{V_i}V_i, V_1, V_2, \cdots, V_p)$$

$$+ \sum_{i=1}^p (\kappa \wedge \chi_{\mathbf{F}})(V_i, D_{V_i}V_1, V_2, \cdots, V_p) + \sum_{i=1}^p (\kappa \wedge \chi_{\mathbf{F}})(V_i, V_1, D_{V_i}V_2, \cdots, V_p)$$

$$+ \cdots + \sum_{i=1}^{p} (\kappa \wedge \chi_{\mathbf{F}})(V_i, V_1, \cdots, D_{V_i} V_p)$$

$$- \sum_{a=1}^{2} X_a (\kappa \wedge \chi_{\mathbf{F}})(X_a, V_1, V_2, \cdots, V_p) + \sum_{a=1}^{2} (\kappa \wedge \chi_{\mathbf{F}})(D_{X_a} X_a, V_1, V_2, \cdots, V_p)$$

$$+ \sum_{a=1}^{2} (\kappa \wedge \chi_{\mathbf{F}})(X_a, D_{X_a} V_1, V_2, \cdots, V_p) + \sum_{a=1}^{2} (\kappa \wedge \chi_{\mathbf{F}})(X_a, V_1, D_{X_a} V_2, \cdots, V_p)$$

$$+ \cdots + \sum_{a=1}^{2} (\kappa \wedge \chi_{\mathbf{F}})(X_a, V_1, V_2, \cdots, D_{X_a} V_p) .$$

Now $(\mathcal{H}D_{X_i} X_i)_x = 0$, and $\mathcal{V}D_{X_i} X_i = 0$ where defined, since the metric $g$ is assumed bundle-like. Since $\kappa$ annihilates vertical vector fields, $\chi_{\mathbf{F}}$ annihilates horizontal fields, the above becomes,

$$(1.16) \qquad \begin{aligned} &= -X_1 \kappa(X_1) - X_2 \kappa(X_2) = -X_1 g(X_1, \tau) - X_2 g(X_2, \tau) \\ &= -g(X_1, D_{X_1} \tau) - g(X_2, D_{X_2} \tau) = -\mathrm{div}_{\mathbf{H}} \tau . \end{aligned}$$

In the next expansion, note that $X_1$ is basic and that $\hat{V}_j$ means that $V_j$ is omitted. We can use $X_1$ as our basic vector field essentially without loss of generality.

$$(1.17) \qquad \delta(\kappa \wedge \chi_{\mathbf{F}})(X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$= - \sum_{a=1}^{2} (D_{X_a}(\kappa \wedge \chi_{\mathbf{F}}))(X_a, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p))$$

$$- \sum_{i=1, i \neq j}^{p} (D_{V_i}(\kappa \wedge \chi_{\mathbf{F}}))(V_i, X_1, V_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$- (D_{V_j}(\kappa \wedge \chi_{\mathbf{F}})(V_j, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)) .$$

Expanding (1.17), we have the following expression.

$$(1.18) \qquad - \sum_{a=1}^{2} X_a (\kappa \wedge \chi_{\mathbf{F}})(X_a, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \sum_{a=1}^{2} (\kappa \wedge \chi_{\mathbf{F}})(D_{X_a} X_a, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \sum_{a=1}^{2} (\kappa \wedge \chi_{\mathbf{F}})(X_a, D_{X_a} X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \sum_{a=1}^{2} (\kappa \wedge \chi_{\mathbf{F}})(X_a, X_1, D_{X_a} V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p) + \cdots$$

$$+ \sum_{a=1}^{2} (\kappa \wedge \chi_{\mathbf{F}})(X_a, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, D_{X_a} V_p)$$

$$- \sum_{i=1, i \neq j}^{p} V_i (\kappa \wedge \chi_{\mathbf{F}})(V_i, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \sum_{i=1, i \neq j}^{p} (\kappa \wedge \chi_{\mathbf{F}})(D_{V_i} V_i, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \sum_{i=1, i \neq j}^{p} (\kappa \wedge \chi_{\mathbf{F}})(V_i, D_{V_i} X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \sum_{i=1, i \neq j}^{p} (\kappa \wedge \chi_{\mathbf{F}})(V_i, X_1, D_{V_i} V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \sum_{i=1, i \neq j}^{p} (\kappa \wedge \chi_{\mathbf{F}})(V_i, X_1, V_1, D_{V_i} V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \cdots + \sum_{i=1, i \neq j}^{p} (\kappa \wedge \chi_{\mathbf{F}})(V_i, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, D_{V_i} V_p)$$

$$- V_j (\kappa \wedge \chi_{\mathbf{F}})(V_j, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ (\kappa \wedge \chi_{\mathbf{F}})(D_{V_j} V_j, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_j, D_{V_j} X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_j, X_1, D_{V_j} V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_j, X_1, V_1, D_{V_j} V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ \cdots + (\kappa \wedge \chi_{\mathbf{F}})(V_j, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, D_{V_j} V_p).$$

Most of the terms in (1.18) vanish for one of the following reasons: two of the arguments are horizontal; two repeated arguments. Note, at x, $D_{V_i} V_k$ is purely horizontal. The only non-zero summands in (1.18) are:

$$(\kappa \wedge \chi_{\mathbf{F}})(X_2, D_{X_2} X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$- V_j (\kappa \wedge \chi_{\mathbf{F}})(V_j, X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_j, D_{V_j} X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p).$$

Since $\mathcal{H}D_{V_j}X_1 = A_{X_1}V_j$, this becomes:

$$(1.19) \qquad \kappa(X_2)\chi_{\mathbf{F}}(A_{X_2}X_1, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$+V_j(\kappa \wedge \chi_{\mathbf{F}})(X_1, V_j, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$-(\kappa \wedge \chi_{\mathbf{F}})(A_{X_1}V_j, V_j, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)\,.$$

Recall, $g(A_{X_2}X_1, V_j)V_j = g(X_2, A_{X_1}V_j)V_j$. Now $\tau$ is basic, because $\kappa$ is basic and $g$ is bundle-like. Hence $\tau = a_1 X_1 + a_2 X_2$. Then (1.19) becomes:

$$(1.20) \qquad \kappa(X_2)g(A_{X_1}V_j, X_2)\chi_{\mathbf{F}}(V_j, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$\pm V_j\kappa(X_1)\chi(V_1, V_2, \cdots, V_j, \cdots, V_p)$$

$$-\kappa(A_{X_1}V_j)\chi_{\mathbf{F}}(V_j, V_1, V_2, \cdots, \hat{V}_j, \cdots V_p)$$

which becomes

$$(1.21) \quad a_2 g(A_{X_1}V_j, X_2)\chi_{\mathbf{F}}(V_j, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p)$$

$$\pm d\kappa(V_j, X_1) - a_2 g(A_{X_1}V_j, X_2)\chi_{\mathbf{F}}(V_j, V_1, V_2, \cdots, \hat{V}_j, \cdots, V_p) = 0\,,$$

because $\kappa$ is closed for the Dominguez metric.

　　　Our final computation will involve evaluating $\delta(\kappa \wedge \chi_{\mathbf{F}})$ on $(X_1, X_2, V_1, \cdots, V_{p-2})$. Again, we can make this evaluation on our two basic fields and excluding $V_{p-1}$ and $V_p$ as arguments, essentially without loss of generality.

$$(1.22) \qquad \delta(\kappa \wedge \chi_{\mathbf{F}}))(X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$

$$= -\sum_{i=1}^{2}(D_{X_i}(\kappa \wedge \chi_{\mathbf{F}}))(X_i, X_1, X_2, V_1, \cdots, V_{p-2})$$

$$-\sum_{i=1}^{p}(D_{V_i}((\kappa \wedge \chi_{\mathbf{F}}))(V_i, X_1, X_2, V_1, V_2, \cdots, V_{p-2})\,.$$

　　　This expands to:

$$(1.23) \qquad -\sum_{a=1}^{2} X_a(\kappa \wedge \chi_{\mathbf{F}})(X_a, X_1, X_2, V_1, \cdots, V_{p-2})$$

$$+\sum_{a=1}^{2}(\kappa \wedge \chi_{\mathbf{F}})(D_{X_a}X_a, X_1, X_2, V_1, \cdots, V_{p-2})$$

$$+\sum_{a=1}^{2}(\kappa \wedge \chi_{\mathbf{F}})(X_a, D_{X_a}X_1, X_2, V_1, \cdots, V_{p-2})$$

$$+\sum_{a=1}^{2}(\kappa \wedge \chi_{\mathbf{F}})(X_a, X_1, D_{X_a}X_2, V_1, \cdots, V_{p-2})$$

$$+\sum_{a=1}^{2}(\kappa \wedge \chi_{\mathbf{F}})(X_a, X_1, X_2, D_{X_a}V_1, \cdots, V_{p-2})$$

$$+\cdots+\sum_{a=1}^{2}(\kappa \wedge \chi_{\mathbf{F}})(X_a, X_1, X_2, V_1, \cdots, D_{X_a}V_{p-2})$$

$$-\sum_{i=1}^{p-2} V_i(\kappa \wedge \chi_{\mathbf{F}})(V_i, X_1, X_2, V_1, \cdots, V_{p-2})$$

$$+\sum_{i=1}^{p-2}(\kappa \wedge \chi_{\mathbf{F}})(D_{V_i}V_i, X_1, X_2, V_1, \cdots, V_{p-2})$$

$$+\sum_{i=1}^{p-2}(\kappa \wedge \chi_{\mathbf{F}})(V_i, D_{V_i}X_1, X_2, V_1, \cdots, V_{p-2})$$

$$+\sum_{i=1}^{p-2}(\kappa \wedge \chi_{\mathbf{F}})(V_i, X_1, D_{V_i}X_2, V_1, \cdots, V_{p-2})$$

$$+\sum_{i=1}^{p-2}(\kappa \wedge \chi_{\mathbf{F}})(V_i, X_1, X_2, D_{V_i}V_1, \cdots, V_{p-2})$$

$$+\cdots+\sum_{i=1}^{p-2}(\kappa \wedge \chi_{\mathbf{F}})(V_i, X_1, X_2, V_1, \cdots, D_{V_i}V_{p-2})$$

$$-V_{p-1}(\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$

$$+(\kappa \wedge \chi_{\mathbf{F}})(D_{V_{p-1}}V_{p-1}, X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$

$$+(\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, D_{V_{p-1}}X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$

$$+(\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, X_1, D_{V_{p-1}}X_2, V_1, V_2, \cdots, V_{p-2})$$

$$+(\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, X_1, X_2, D_{V_{p-1}}V_1, V_2, \cdots, V_{p-2})$$

$$+\cdots+(\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, X_1, X_2, V_1, V_2, \cdots, D_{V_{p-1}}V_{p-2})$$

$$-V_p(\kappa \wedge \chi_{\mathbf{F}})(V_p, X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$

$$+(\kappa \wedge \chi_{\mathbf{F}})(D_{V_p}V_p, X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$

$$+(\kappa \wedge \chi_{\mathbf{F}})(V_p, D_{V_p}X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$

$$+(\kappa \wedge \chi_{\mathbf{F}})(V_p, X_1, D_{V_p}X_2, V_1, V_2, \cdots, V_{p-2})$$

$$+(\kappa \wedge \chi_{\mathbf{F}})(V_p, X_1, X_2, D_{V_p}V_1, V_2, \cdots, V_{p-2})$$

$$+\cdots+(\kappa \wedge \chi_{\mathbf{F}})(V_p, X_1, X_2, V_1, V_2, \cdots, D_{V_p}V_{p-2}).$$

All terms above with two horizontal vector field arguments vanish. Terms in

$\sum_{i=1}^{p-2} (\kappa \wedge \chi_{\mathbf{F}})(V_i, D_{V_i} X_1, X_2, V_1, \cdots, V_{p-2})$ vanish individually because the arguments $V_i$ repeat when $1 \le i \le p - 2$. Accordingly, the only non-zero terms are:

(1.24)
$$(\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, D_{V_{p-1}} X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, X_1, D_{V_{p-1}} X_2, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_p, D_{V_p} X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_p, X_1, D_{V_p} X_2, V_1, V_2, \cdots, V_{p-2}).$$

Only the vertical components of $D_{V_i} X_j$ matter in the above calculations because when $\kappa \wedge \chi_{\mathbf{F}}$ is evaluated on $p + 1$ arguments with two or more horizontal the result is zero. Recall, $\mathcal{V} D_{V_j} X_i = T_{V_j} X_i$. Hence, we have,

(1.25)
$$(\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, T_{V_{p-1}} X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, X_1, T_{V_{p-1}} X_2, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_p, T_{V_p} X_1, X_2, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_p, X_1, T_{V_p} X_2, V_1, V_2, \cdots, V_{p-2}).$$

A routine argument using the properties of the tensor $T$ introduced in the beginning shows that $g(T_{V_{p-1}} X_j, V_p) = g(T_{V_p} X_j, V_{p-1})$, where $j = 1$ or $j = 2$. This means if the $V_p$-component of $T_{V_{p-1}} X_1$ is $a$, then the $V_{p-1}$-component of $T_{V_p} X_1$ is also $a$. Likewise, if $c$ is the $V_p$-component of $T_{V_{p-1}} X_2$, then c is also the $V_{p-1}$-component of $T_{V_p} X_2$. Hence, (1.25) becomes:

(1.26)
$$(\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, a V_p, X_2, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_{p-1}, X_1, c V_p, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_p, a V_{p-1}, X_2, V_1, V_2, \cdots, V_{p-2})$$
$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_p, X_1, c V_{p-1}, V_1, V_2, \cdots, V_{p-2}) = 0.$$

The proof of Theorem 1.4 is now complete, provided we observe that in the very special case that $\mathbf{F}$ is a flow on $M^3$, the third computation is superfluous.

As an application of Theorem 1.4 we establish the following result which also uses a result of Ranjan (see [Ra]). We will follow the exposition of Ranjan's Theorem as given in [T3], pages 76 and 77 (see also the Corollary on the top of page 89 in [Ra]). Essentially, our result says gives necessary and sufficient conditions for a closed 3-manifold with a Dominguez metric to admit a non-trivial local de-Rham decomposition. $\mathrm{Ric}^M(E, E)$ denotes the Ricci tensor with respect to the Levi-Civita connection on $M$ evaluated on a vector field $E$.

COROLLARY 1.5. *Let $M^3$ be a closed, oriented, $C^\infty$, 3-manifold, with a transversely oriented Riemannian flow, $\mathbf{F}$. Suppose $g$ is a Dominguez metric for the flow , $\mathbf{F}$, and let $V$ be a unit length vector field tangent to this flow.*

*If* $\mathrm{Ric}^M(V, V) \equiv 0$ *on* $M^3$ *and* $\kappa \wedge \chi_{\mathbf{F}}$ *is harmonic, then* $\mathbf{H}$ *is integrable,* $\mathbf{F}$ *is totally geodesic,* $\kappa \wedge \chi_{\mathbf{F}} \equiv 0$, *and locally* $M^3$ *is isometric to a product of the plaques of the leaves of* $\mathbf{H}$ *and* $\mathbf{F}$.

*Conversely, if* $\mathbf{H}$ *is integrable and* $\mathbf{F}$ *is totally geodesic, then on* $M^3$, $\mathrm{Ric}^M(V, V) \equiv 0$ *and* $\kappa \wedge \chi_{\mathbf{F}} \equiv 0$. *In particular,* $\kappa \wedge \chi_{\mathbf{F}}$ *is harmonic.*

PROOF.    In the proof we will let $\{X_1, \cdots, X_q\}$ denote a local basic orthonormal frame for $\mathbf{H}$, with $q = 2$. We use this seemingly cumbersome notation because the same work will yield another somewhat more general result essentially at no extra cost. First note, $\kappa \wedge \chi_{\mathbf{F}}$ is closed because of Theorem 1.4 and the theorem of Kamber-Tondeur ([T3] page 82) which applies in the case of a Dominguez metric. Note all the calculations are independent of the local orthonormal frame for $\mathbf{H}$. The idea is to exploit equations 6.22 and 6.21 of [T3] in that order. Equation 6.22 of [T3] yields $\mathrm{Ric}^M(V, V) = \mathrm{div}_M \tau + \sum_{i=1}^{q} g(A_{X_i} V, A_{X_i} V)$. If $\mathrm{Ric}^M(V, V) = 0$ on $M^3$, then integration yields $\int_M \sum_{i=1}^{q} g(A_{X_i} V, A_{X_i} V) = 0$, so each $A_{X_i} V = 0$ so $A \equiv 0$. Since $A_X Y = (1/2)\mathcal{V}[X, Y]$, we see $\mathbf{H}$ is integrable. Since $A_X Y = \mathcal{V} D_X Y$, $\mathbf{H}$ is totally geodesic.

Using the fact that for our flow $\mathcal{V} D_X V = 0$ for any basic $X$, equation 6.21 of [T3] yields $\mathrm{Ric}^M(V, V) = \mathrm{div}_{\mathbf{H}} \tau - \sum_{i=1}^{q} g(T_V X_i, T_V X_i) + \sum_{i=1}^{q} g(A_{X_i} V, A_{X_i} V)$. Now $\mathrm{Ric}^M(V, V) \equiv 0$ and $A \equiv 0$ and, when $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic, $\mathrm{div}_{\mathbf{H}} \tau = 0$, when $q = 2$. Hence, $\sum_{i=1}^{q} g(T_V X_i, T_V X_i) = 0$. This means $T \equiv 0$ or $\mathbf{F}$ is totally geodesic and so locally $M^3$ is isometric to a product of the plaques of the foliations $\mathbf{H}$ and $\mathbf{F}$. The proof of the converse follows by observing that under the stated hypotheses, our version of 6.22 of [T3] yields that $\mathrm{Ric}^M(V, V) \equiv 0$. Then our version of 6.21 of [T3] yields $\mathrm{div}_{\mathbf{H}} \tau = 0$ which when $q = 2$ means, $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic. If $T = 0$, $\tau \equiv 0$ and so $\kappa \wedge \chi_{\mathbf{F}} \equiv 0$.

The second result using the proof above works for a Riemannian flow of arbitrary codimension on a closed, connected manifold.

COROLLARY 1.6.    *Let $M$ be a closed, oriented, $C^\infty$, n-manifold, with a transversely oriented Riemannian flow, $\mathbf{F}$. Suppose $g$ is a Dominguez metric for the flow , $\mathbf{F}$, and let $V$ be a unit length vector field tangent to this flow.*

*If $\mathrm{Ric}^M(V, V) \equiv 0$ on $M$ and $\mathrm{div}_{\mathbf{H}} \tau = 0$, then $\mathbf{H}$ is integrable, $\mathbf{F}$ is totally geodesic, $\tau = 0$, and locally $M$ is isometric to a product of the plaques of the leaves of $\mathbf{H}$ and $\mathbf{F}$.*

*Conversely, if $\mathbf{H}$ is integrable and $\mathbf{F}$ is totally geodesic, then on $M$, $\mathrm{Ric}^M(V, V) \equiv 0$ and $\mathrm{div}_{\mathbf{H}} \tau = 0$. In particular, $\tau = 0$.*

PROOF.    As noted, the result follows from the proof of 1.5 with minor modifications.

Now suppose $M^n$ is a closed, connected, oriented, Riemannian manifold admitting a codimension-one Riemannian foliation $\mathbf{F}$. Let $g$ be a Dominguez metric for $\mathbf{F}$. Then $\tau = \sum_{i=1}^{n-1} \mathcal{H} D_{V_i} V_i$. (As above, we follow the conventions in foliations and suppress the usual constant.) And $\mathrm{div}_{\mathbf{H}} \tau = g(X, D_X \tau)$, where $X$ is a unit length basic vector field. We have the following result.

REMARK.    For a general codimension-one, transversely oriented foliation on a closed, oriented, Riemannian manifold, Kamber and Tondeur have shown that the leaves of the foliation are minimal submanifolds with respect to the given metric if and only if $d\chi_{\mathbf{F}} = 0$ as shown in Theorem 7.35 of [T1], page 92. But for foliations of codimension one, it is also the case (see [T1], page 80) that $d\chi_{\mathbf{F}} = -\kappa \wedge \chi_{\mathbf{F}}$. Hence, the leaves of the foliation are minimal in this setting if and only if $\kappa \wedge \chi_{\mathbf{F}} = 0$, or equivalently in this setting, $\kappa \wedge \chi_{\mathbf{F}}$ is a harmonic $n$-form, by the Hodge Theorem. The next result gives a sufficient explicit condition for $\kappa \wedge \chi_{\mathbf{F}}$ to be harmonic in the very special case that the codimension-one foliation is bundle-like with respect to a Dominguez metric. We include it because the key condition is essentially the same as that for the codimension $q = 2$ case in Theorem 1.4.

THEOREM 1.7.   *Let $M^n$ be a closed, connected, $C^\infty$, oriented Riemannian manifold admitting a transversely oriented, codimension-one, Riemannian foliation $\mathbf{F}$. Let $g$ be a Dominguez metric for the foliation $\mathbf{F}$ (with respect to which $\mathbf{F}$ is bundle-like and $\kappa$ is basic). Then $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic (and hence by the above remark identically 0 in this case) if and only if $\operatorname{div}_{\mathbf{H}} \tau = 0$, where $\tau$ is the mean curvature one-form dual to $\kappa$.*

PROOF.    $\kappa \wedge \chi_{\mathbf{F}}$ is an $n$-form and hence is closed. Because the chosen metric, $g$, is a Dominguez metric, the mean curvature one-form $\kappa$ is basic. Just as before, it is closed by the Kamber-Tondeur Theorem. We will show under the stated hypotheses, $\delta(\kappa \wedge \chi_{\mathbf{F}}) = 0$. We choose a preferred orthonormal frame at $x \in M$. That is, we chose $\{X, V_1, V_2, \cdots, V_{n-1}\}$, so $X$ is basic, with $(\mathcal{H}D_X X)_x = 0$ and so $(\mathcal{V}D_{V_i}V_j)_x = 0$, where $\{V_1, V_2, \cdots, V_{n-1}\}$ is an orthonormal frame for $\mathbf{V}$. Then at $x \in M$, we have,

$$(1.27)\qquad \delta(\kappa \wedge \chi_{\mathbf{F}})(V_1, V_2, \cdots, V_{n-1})$$

$$= -(D_X(\kappa \wedge \chi_{\mathbf{F}}))(X, V_1, V_2, \cdots, V_{n-1}) - \sum_{j=1}^{n-1}(D_{V_j}(\kappa \wedge \chi_{\mathbf{F}}))(V_j, V_1, V_2, \cdots, V_{n-1})$$

$$= -X(\kappa \wedge \chi_{\mathbf{F}})(X, V_1, V_2, \cdots, V_{n-1}) + (\kappa \wedge \chi_{\mathbf{F}})(D_X X, V_1, V_2, \cdots, V_{n-1})$$

$$+ (\kappa \wedge \chi_{\mathbf{F}})(X, D_X V_1, V_2, \cdots, V_{n-1}) + \cdots + (\kappa \wedge \chi_{\mathbf{F}})(X, V_1, V_2, \cdots, D_X V_{n-1})$$

$$- \sum_{j=1}^{n-1} V_j(\kappa \wedge \chi_{\mathbf{F}})(V_j, V_1, V_2, \cdots, V_{n-1}) + \sum_{j=1}^{n-1}(\kappa \wedge \chi_{\mathbf{F}})(D_{V_j}V_j, V_1, V_2, \cdots, V_{n-1})$$

$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_1, D_{V_1}V_1, V_2, \cdots, V_{n-1}) + (\kappa \wedge \chi_{\mathbf{F}})(V_1, V_1, D_{V1}V_2, \cdots, V_{n-1})$$

$$+ \cdots + (\kappa \wedge \chi_{\mathbf{F}})(V_1, V_1, V_2, \cdots, D_{V_1}V_{n-1}) + (\kappa \wedge \chi_{\mathbf{F}})(V_2, D_{V2}V_1, V_2, \cdots, V_{n-1})$$

$$+ (\kappa \wedge \chi_{\mathbf{F}})(V_2, V_1, D_{V2}V_2, \cdots, V_{n-1})$$

$$+ \cdots + (\kappa \wedge \chi_{\mathbf{F}})(V_2, V_1, V_2, \cdots, D_{V_2}V_{n-1})$$

$$+ \cdots + (\kappa \wedge \chi_{\mathbf{F}})(V_{n-1}, D_{V_{n-1}}V_1, V_2, \cdots, V_{n-1})$$

$$+ \cdots + (\kappa \wedge \chi_{\mathbf{F}})(V_{n-1}, V_1, V_2, \cdots, D_{V_{n-1}}V_{n-1}).$$

Just as in the proof of Theorem 1.4, the expressions $\sum_{j=1}^{n-1}(\kappa \wedge \chi_{\mathbf{F}})(D_{V_j}V_j, V_1,$ $V_2, \cdots, V_{n-1}), (\kappa \wedge \chi_{\mathbf{F}})(V_1, D_{V_1}V_1, V_2, \cdots, V_{n-1}), (\kappa \wedge \chi_{\mathbf{F}})(V_2, V_1, D_{V2}V_2, \cdots, V_{n-1}), \cdots,$ and, $(\kappa \wedge \chi_{\mathbf{F}})(V_{n-1}, V_1, V_2, \cdots, D_{V_{n-1}}V_{n-1})$ sum to zero. Except for the first term, the remaining terms in (1.27) vanish because of repeated arguments, the fact that $D_X V_j$ has no non-zero $V_j$ component and so these expressions are evaluated with two purely horizontal arguments and hence vanish as well.

Then (1.27) becomes,

$$(1.28) \qquad -X(\kappa \wedge \chi_{\mathbf{F}})(X, V_1, V_2, \cdots, V_{n-1}) = -X\kappa(X) = -\operatorname{div}_{\mathbf{H}}\tau\,,$$

which must vanish identically if $\kappa \wedge \chi_{\mathbf{F}}$ is co-closed. Our theorem will be proven if we can show $\delta(\kappa \wedge \chi_{\mathbf{F}})(X, V_1, V_2, \cdots, \hat{V}_l, \cdots, V_{n-1}) = 0$. Essentially without loss of generality, we will show $\delta(\kappa \wedge \chi_{\mathbf{F}})(X, V_1, V_2, \cdots, V_{n-2}) = 0$, since by renumbering the vertical vectors, up to sign, the computation will always evaluate to zero.

$$(1.29) \quad \delta(\kappa \wedge \chi_{\mathbf{F}})(X, V_1, V_2, \cdots, V_{n-2})$$

$$= -(D_X(\kappa \wedge \chi_{\mathbf{F}}))(X, X, V_1, V_2, \cdots, V_{n-2}) - \sum_{j=1}^{n-2}(D_{V_j}(\kappa \wedge \chi_{\mathbf{F}}))(V_j, X, V_1, V_2, \cdots, V_{n-2})$$

$$\quad - (D_{V_{n-1}}(\kappa \wedge \chi_{\mathbf{F}}))(V_{n-1}, X, V_1, V_2, \cdots, V_{n-2})$$

$$= -X(\kappa \wedge \chi_{\mathbf{F}})(X, X, V_1, V_2, \cdots, V_{n-2}) + (\kappa \wedge \chi_{\mathbf{F}})(D_X X, X, V_1, V_2, \cdots, V_{n-2})$$

$$\quad + (\kappa \wedge \chi_{\mathbf{F}})(X, D_X X, V_1, V_2, \cdots, V_{n-2}) + (\kappa \wedge \chi_{\mathbf{F}})(X, X, D_X V_1, V_2, \cdots, V_{n-2})$$

$$\quad + \cdots + (\kappa \wedge \chi_{\mathbf{F}})(X, X, V_1, V_2, \cdots, D_X V_{n-2})$$

$$\quad - \sum_{j=1}^{n-2} V_j(\kappa \wedge \chi_{\mathbf{F}})(V_j, X, V_1, V_2, \cdots, V_{n-2}) + \sum_{j=1}^{n-2}(\kappa \wedge \chi_{\mathbf{F}})(D_{V_j}V_j, X, V_1, V_2, \cdots, V_{n-2})$$

$$\quad + \sum_{j=1}^{n-2}(\kappa \wedge \chi_{\mathbf{F}})(V_j, D_{V_j}X, V_1, \cdots, V_j, \cdots, V_{n-2})$$

$$\quad + \cdots + \sum_{j=1}^{n-2}(\sum_{i=1}^{n-2}(\kappa \wedge \chi_{\mathbf{F}})(V_j, X, \cdots, D_{V_j}V_i, \cdots, V_{n-2}))$$

$$\quad - V_{n-1}(\kappa \wedge \chi_{\mathbf{F}})(V_{n-1}, X, V_1, \cdots, V_{n-2}) + (\kappa \wedge \chi_{\mathbf{F}})(D_{V_{n-1}}V_{n-1}, X, V_1, \cdots, V_{n-2})$$

$$\quad + (\kappa \wedge \chi_{\mathbf{F}})(V_{n-1}, D_{V_{n-1}}X, V_1, \cdots, V_{n-2})$$

$$\quad + (\kappa \wedge \chi_{\mathbf{F}})(V_{n-1}, X, D_{V_{n-1}}V_1, \cdots, V_{n-2})$$

$$\quad + \cdots + (\kappa \wedge \chi_{\mathbf{F}})(V_{n-1}, X, V_1, \cdots, D_{V_{n-1}}V_{n-2})\,.$$

Then (1.29) becomes,

$$(1.30) \quad -V_{n-1}(\kappa \wedge \chi_{\mathbf{F}})(V_{n-1}, X, V_1, \cdots, V_{n-2}) = \pm V_{n-1}\kappa(X) = \pm d\kappa(V_{n-1}, X) = 0\,,$$

because $\kappa$ is a basic form when $g$ is a Dominguez metric. This completes the proof of Theorem 1.7.

Following [T3], page 99, we define the following connection, $\tilde{D}$. For vector fields $E$ and $F$ on $M$, we set:

$$(1.31) \qquad \tilde{D}_E F = \mathcal{V} D_E \mathcal{V} F + \mathcal{H} D_E \mathcal{H} F,$$

where $D$ is the Levi-Civita connection on $M$. Again following [T3] (page 102) or [Mi-Ri-To], let $\omega$ be a basic $r$-form. Let $\{E_2, \cdots, E_r\}$ be vector fields on $M$. Let $\{V_1, \cdots, V_p, X_1, \cdots, X_q\}$ be an orthonormal frame for a bundle-like foliation of leaf dimension $p$ and codimension $q$. Set,

$$(1.32) \quad \tilde{\delta}\omega(E_2, \cdots, E_r)$$

$$= -\sum_{j=1}^{p} V_j(\omega(V_j, E_2, \cdots, E_r)) + \sum_{j=1}^{p} \omega(\tilde{D}_{V_j} V_j, E_2, \cdots, E_r)$$

$$+ \sum_{j=1}^{p}\sum_{i=2}^{r} \omega(V_j, E_2, \cdots, \tilde{D}_{V_j} E_i, \cdots, E_r) - \sum_{k=1}^{q} X_k(\omega(X_k, E_2, \cdots, E_r))$$

$$+ \sum_{k=1}^{q} \omega(\tilde{D}_{X_k} X_k, E_2, \cdots, E_r) + \sum_{k=1}^{q}\sum_{i=2}^{r} \omega(X_k, E_2, \cdots, \tilde{D}_{X_k} E_i, \cdots, E_r).$$

Then by [T3], page 102, if $\omega$ is a basic $r$-form, $\tilde{\delta}\omega$ is a basic $(r-1)$-form. In particular, if $\mathbf{F}$ is a transversely oriented, Riemannian foliation on a closed, oriented Riemannian manifold $(M, g)$, where $g$ is a Dominguez metric for $\mathbf{F}$, a straightforward calculation yields the following:

$$(1.33) \qquad\qquad \tilde{\delta}\kappa = -\operatorname{div}_{\mathbf{H}} \tau.$$

We have the following theorem which combines Theorems 1.4 and 1.7.

THEOREM 1.8.   *Let $(M^n, g)$ be a closed, oriented, $C^\infty$, Riemannian manifold, with a transversely oriented, codimension-$q$, Riemannian foliation $\mathbf{F}$, with $q = 1$ or $q = 2$. Suppose $g$ is a Dominguez metric for $\mathbf{F}$. Then $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic on $M$ if and only if $\tilde{\delta}\kappa = 0$. Under the stated hypotheses when $q = 1$, $\kappa \wedge \chi_{\mathbf{F}} = 0$.*

PROOF.   The proof follows immediately from Theorems 1.4 and 1.7 and the above remarks.

It might be useful to rephrase Theorem 1.8 in the following way. It should be noted however, that thanks to the fundamental result of Dominguez, we always know there exists a bundle-like metric $g$ for $\mathbf{F}$ so $\kappa$ is basic, so in a sense the reformulation is redundant.

THEOREM 1.9. *Let $(M^n, g)$ be a closed, oriented, $C^\infty$, Riemannian manifold, with a transversely oriented, codimension-$q$, foliation* **F**, *with $q = 1$ or $q = 2$. Suppose* **F** *is bundle-like with respect to $g$. Then $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic on $M$ if and only if $\kappa$ is basic and $\tilde{\delta}\kappa = 0$. Under the stated hypotheses when $q = 1, \kappa \wedge \chi_{\mathbf{F}} = 0$.*

PROOF. $\kappa \wedge \chi_{\mathbf{F}}$ is always closed if $q = 1$. If $q = 2$, then $\kappa \wedge \chi_{\mathbf{F}}$ is closed if and only if $\kappa$ is horizontally closed by (1.14). But if $\kappa$ is basic, $\kappa$ is closed by the already mentioned result of Kamber-Tondeur. $\delta(\kappa \wedge \chi_{\mathbf{F}}) = 0$ iff $\tilde{\delta}\kappa = -\operatorname{div}_{\mathbf{H}} \tau = 0$.

REMARK. A straightforward calculation shows $\delta\kappa = \kappa(\tau) - \operatorname{div}_{\mathbf{H}} \tau$. If additionally, $\delta\kappa = 0$, then $\kappa$ would be closed and co-closed and hence harmonic on $M$ itself, a situation not necessary to our work here. If we set $\tilde{\triangle} = d\tilde{\delta} + \tilde{\delta}d$ as in [T3], page 102, then $d\kappa = 0$ and $\tilde{\delta}\kappa = 0$ implies $\tilde{\triangle}\kappa = 0$. However, $\tilde{\triangle}$ is not self-adjoint.

Now let $M^n$ be any closed, oriented Riemannian manifold admitting a transversely oriented foliation **F** of leaf dimension $p$ and codimension $q$. Let $\tau$ be the mean curvature vector field of the foliation **F**.

Then,

$$(1.34) \qquad \operatorname{div}_M \tau = \sum_{\alpha=1}^{n-p} g(D_{X_\alpha}\tau, X_\alpha) + \sum_{i=1}^{p} g(D_{V_i}\tau, V_i).$$

Using the standard properties of the tensor $T$, this becomes

$$(1.35) \qquad \operatorname{div}_M \tau + g(\tau, \tau) = \operatorname{div}_{\mathbf{H}} \tau.$$

Integrating we get,

$$(1.36) \qquad \int_M g(\tau, \tau)dV = \int_M \operatorname{div}_{\mathbf{H}} \tau dV.$$

We have the following lemma, which applies to an arbitrary transversely oriented foliation on a closed oriented Riemannian manifold $M^n$, not just Riemannian foliations.

LEMMA 1.10. *Let* **F** *be any transversely oriented foliation of leaf dimension $p$ on a closed, oriented, Riemannian manifold $M^n$.*

   (1)  *If $\int_M \operatorname{div}_{\mathbf{H}} \tau dV = 0$, then $\tau = 0$ and the leaves of* **F** *are minimal.*
   (2)  *Conversely, if $\tau = 0$ on such an $M^n$, then $\operatorname{div}_{\mathbf{H}} \tau = 0$ so $\int_M \operatorname{div}_{\mathbf{H}} \tau dV = 0$.*

THEOREM 1.11. *Let $M^n$ be a closed, oriented Riemannian manifold admitting a transversely oriented Riemannian foliation* **F** *of codimension $q$ with $q = 1$ or $2$. Then $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic with respect to a Dominguez metric for* **F** *if and only if the mean curvature one-form $\kappa = 0$ and so the leaves of* **F** *are minimal submanifolds of $M^n$.*

PROOF. For $q = 2$ or $1$, Theorems 1.4 and 1.7 respectively guarantee $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic if and only if $\operatorname{div}_{\mathbf{H}} \tau = 0$. The result now follows directly from Lemma 1.10.

## 2.   Bundle-like foliations with totally umbilical leaves

We begin by recalling some basic local properties of Riemannian submersions and of bundle-like foliations. The convention for the Riemannian tensor on a Riemannian manifold $(M, g)$ is:

$$R(E, F)G = D_E D_F G - D_F D_E G - D_{[E,F]}G \quad \text{and} \quad R(E, F, G, G') = -g(R(E, F)G, G').$$

If $F$ is a $p$-dimensional leaf of foliation $\mathbf{F}$ then $T_U V$ is the second fundamental form of the leaf and the mean curvature vector field $\tau$ is given by:

$$\tau = \sum_{i=1}^{p} T_{V_i} V_i,$$

where $\{V_i\}_{1 \leq i \leq p}$ is a local orthonormal frame of vector fields tangent to leaves.

A $p$-dimensional submanifold $F$ of a Riemannian manifold $(M, g)$ is said to be totally umbilical if the second fundamental form $T$ is given by, $T(U, V) = (1/p)g(U, V)\tau$ for any vectors $U$, $V$ tangent to $\mathbf{F}$.

The following equations, usually called *O'Neill's equations*, characterize the geometry of a bundle-like foliation $\mathbf{F}$ on $(M, g)$ (see [T3] page 51, or the known results for Riemannian submersion [O'N, Gr]).

PROPOSITION 2.1.   *For every vertical vector fields $U$, $V$, $W$, $W'$ and for every horizontal vector fields $X$, $Y$, $Z$, $Z'$, we have the following formulas*:

i)   $R(U, V, W, W') = \hat{R}(U, V, W, W') - g(T_U W, T_V W') + g(T_V W, T_U W')$,

ii)  $R(U, V, W, X) = g((D_V T)_U W, X) - g((D_U T)_V W, X)$,

iii) $R(X, U, Y, V) = g((D_X T)_U V, Y) - g(T_U X, T_V Y) + g((D_U A)_X Y, V)$
$\qquad\qquad\qquad + g(A_X U, A_Y V)$,

iv)  $R(X, Y, Z, U) = g((D_Z A)_X Y, U) + g(A_X Y, T_U Z) - g(A_Y Z, T_U X)$
$\qquad\qquad\qquad - g(A_Z X, T_U Y)$,

v)   $R(X, Y, Z, Z') = R^*(X, Y, Z, Z') - 2g(A_X Y, A_Z Z') + g(A_Y Z, A_X Z')$
$\qquad\qquad\qquad - g(A_X Z, A_Y Z')$,

*where we denote by $R$, $\hat{R}$ and $R^*$ the Riemannian tensors for the connections $D$ of $M$, $\hat{D}$ of $\mathbf{F}$, and $D^*$ on the transversal distribution $\mathbf{H}$, respectively.*

Using O'Neill's equations, we get the following lemma.

LEMMA 2.2.   *If $\mathbf{F}$ is a bundle-like foliation on $(M, g)$ with totally umbilical leaves then:*

a)   $R(U, V, U, V) = \hat{R}(U, V, U, V) + [g(U, V)^2 - g(U, U)g(V, V)]g(\frac{\tau}{p}, \frac{\tau}{p})$;

b)   $R(X, U, X, U) = g(U, U)[g(D_X \frac{\tau}{p}, X) - g(X, \frac{\tau}{p})^2] + g(A_X U, A_X U)$;

c)   $R(X, Y, X, Y) = R^*(X, Y, X, Y) - 3g(A_X Y, A_X Y)$.

PROPOSITION 2.3.   *Let $(M, g)$ be a Riemannian manifold with a bundle-like foliation* **F**. *We assume that* **F** *has totally umbilical leaves and $X$, $Y$ are basic vector fields. Then $A_X Y$ is a Killing vector field along leaves if and only if $g(D_X \tau, Y) = g(D_Y \tau, X)$.*

PROOF.   Using Proposition 2.1 from [EP] we have:

$$(2.1) \qquad g(D_U (A_X Y), V) + g(D_V (A_X Y), U) = g(U, V) d\kappa(X, Y).$$

On the other hand,

$$d\kappa(X, Y) = X g(\tau, Y) - g(\tau, D_X Y) - Y g(\tau, X) + g(\tau, D_Y X)$$
$$= g(D_X \tau, Y) - g(D_Y \tau, X).$$

REMARK.   (i)   The affirmation of Proposition 2.3 holds if $\tau$ is parallel in the transversal distribution along leaves.

(ii)   $\kappa$ is horizontally closed if and only if $g(D_X \tau, Y) = g(D_Y \tau, X)$ for any horizontal vector fields $X$, $Y$.

In the next proposition we establish under some certain conditions that $\kappa$ is a basic one-form.

PROPOSITION 2.4.   *Let $(M, g)$ be a Riemannian manifold with constant curvature. If* **F** *is a bund -like foliation with totally umbilical leaves and the dimension of the leaves is $p \geq 2$ then*:

a)   *$\tau$ is parallel in the transversal distribution along leaves. (i.e $\mathcal{H} D_V \tau = 0$ for any vertical vector field $V$).*
b)   *$\tau$ is basic, which implies that $\kappa$ is basic.*
c)   *$A_\tau = 0$.*

PROOF.   Let $\{V_i\}_{1 \leq i \leq p}$ be a local orthonormal basis of vertical vector fields. Let $V$ be a vertical vector field and $X$ an horizontal one. By Proposition 2.1, we have:

$$(*) \qquad \sum_{i=1}^{p} R(V_i, V, V_i, X) = \sum_{i=1}^{p} g((D_V T)_{V_i} V_i, X) - \sum_{i=1}^{p} g((D_{V_i} T)_V V_i, X).$$

Since $(M, g)$ is of constant curvature we have: $\sum_{i=1}^{p} R(V_i, V, V_i, X) = 0$. We compute the first term of the right hand side,

$$\sum_{i=1}^{p} g((D_V T)_{V_i} V_i, X) = \sum_{i=1}^{p} g(D_V (T_{V_i} V_i), X) - \sum_{i=1}^{p} g(T_{D_V V_i} V_i), X) - \sum_{i=1}^{p} g(T_{V_i} (D_V V_i), X)$$

$$= g(D_V \tau, X) - 2(1/p) \sum_{i=1}^{p} g(D_V V_i, V_i) g(\tau, X)$$

$$= g(D_V \tau, X) - (1/p) \sum_{i=1}^{p} V g(V_i, V_i) g(\tau, X)$$

$$= g(D_V \tau, X).$$

Then for the second term of the right hand side of $(*)$ we get:

$$\sum_{i=1}^{p} g((D_{V_i} T)_V V_i, X) = \sum_{i=1}^{p} g(D_{V_i} T_V V_i, X) - \sum_{i=1}^{p} g(T_{D_{V_i} V} V_i, X) - \sum_{i=1}^{p} g(T_V D_{V_i} V_i, X)$$

$$= (1/p)[\sum_{i=1}^{p} g(D_{V_i}(g(V, V_i)\tau), X) - g(\tau, X)g(D_{V_i} V, V_i)$$

$$- g(\tau, X)g(V, D_{V_i} V_i)]$$

$$= (1/p)[\sum_{i=1}^{p} V_i(g(V, V_i)g(\tau, X)) - g(V, V_i)g(\tau, D_{V_i} X))$$

$$- g(\tau, X)V_i(g(V, V_i))]$$

$$= (1/p)g(D_V \tau, X).$$

Therefore,

$$(1 - (1/p))g(D_V \tau, X) = 0$$

for every vertical vector field $V$ and for every horizontal vector field $X$, which implies that $\tau$ is parallel in the transversal distribution along leaves.

b)  It is sufficient to show that, for any basic vector field $X$, $g(\tau, X)$ is constant along leaves (i.e. $V g(\tau, X) = 0$ for any vertical vector field $V$).

First we shall establish that $g(A_X Y, A_X Y)$ is constant along leaves for any basic vector fields $X$, $Y$.  Since $\mathbf{F}$ is a bundle-like foliation we can consider a local model $B$ in distinguished chart $U$ on $M$.  Then the restriction of the foliation $\mathbf{F}$ to $U$ gives a Riemannian submersion $\pi : \mathbf{F}/U \to B$ and we have:

$$R(X, Y, X, Y) = R^*(\pi X, \pi Y, \pi X, \pi Y) - 3g(A_X Y, A_X Y).$$

Since $(M, g)$ is of constant curvature we get $g(A_X Y, A_X Y)$ is constant along leaves for any basic vector fields $X$, $Y$.  By polarization, it follows $g(A_X Y, A_X Z)$ is constant along leaves for any basic vector fields $X$, $Y$, $Z$.  Therefore, $A_X A_X Z$ is a basic vector field and then again by polarization, $A_X A_Y Z + A_Y A_X Z$ is a basic vector field for any basic vector fields $X$, $Y$, $Z$.

By O'Neill equation iv) in Proposition 2.1 we get:

$$R(X, Y, X, A_X Y) = g((D_X A)_X Y, A_X Y) + 2g(A_X Y, T_{A_X Y} X)$$

$$= g(D_X A_X Y, A_X Y) - g(A_{D_X X} Y, A_X Y)$$

$$- g(A_X D_X Y, A_X Y) + 2g(A_X Y, T_{A_X Y} X)$$

$$(2.2) \qquad = (1/2)X(g(A_X Y, A_X Y)) - g(A_Y \mathcal{H} D_X X, A_Y X) - g(A_X \mathcal{H} D_X Y, A_X Y)$$

$$+2g(A_X Y, T_{A_X Y} X).$$

Since $M$ has constant curvature and $A_X Y$ is a vertical vector, we see that $R(X, Y, X, A_X Y) = 0$. The first three terms of (2.2) are constant along leaves and so should be the last one, that means $V(g(A_X Y, T_{A_X Y} X)) = 0$. Since the leaves are totally umbilical, it follows:

$$(2.3) \qquad\qquad V(g(X, \tau))g(A_X Y, A_X Y) = 0.$$

Let $x \in M$, let $X, Y$ be basic vector fields such that $Y(x) = \tau(x)$. If $(A_X \tau)(x) \neq 0$ then, by (2.3), $V(g(X, \tau)) = 0$. Now we shall consider the case when $(A_X \tau)(x) = 0$. Then:

$$V g(\tau, X)_x = (g(D_V \tau, X) + g(\tau, D_V X))_x = g(\tau, A_X V)_x = -g(A_X \tau, V)_x = 0.$$

In the second equality we have used a), $D_V \tau = 0$.

Therefore, $\tau$ is a basic vector field.

c)  The basicity of $\tau$ implies $[V, \tau]$ is a vertical vector field, so $0 = \mathcal{H}[V, \tau] = \mathcal{H}D_V \tau - \mathcal{H}D_\tau V$. Therefore,

$$g(A_\tau X, V) = g(D_\tau X, V) = -g(X, D_\tau V) = -g(X, D_V \tau) = 0,$$

since $\mathcal{H}D_V \tau = 0$. It follows, $A_\tau X = 0$ for any horizontal vector $X$. For a vertical vector $V$, we have, $g(A_\tau V, X) = -g(V, A_\tau X) = 0$ which implies $A_\tau V = 0$.

Using part (b) of Lemma 2.2 we get the following proposition.

PROPOSITION 2.5.  *Let $(M, g)$ be an n-dimensional Riemannian manifold with constant curvature c. Let $\mathbf{F}$ be a bundle-like foliation with totally umbilical leaves of dimension $p = n - q$ on $(M, g)$. Then*

$$(2.4) \qquad \mathrm{div}_{\mathbf{H}} \tau = cpq + \frac{1}{p}g(\tau, \tau) - p\frac{1}{g(U, U)}\sum_{a=1}^{q} g(A_{X_a} U, A_{X_a} U),$$

*for any non-zero vertical vector $U$ and for any orthonormal frame $\{X_a\}$ of $\mathbf{H}$.*

PROOF.  By Lemma 2.2 (b) we get

$$R(X_a, U, X_a, U) = g(U, U)(g(D_{X_a}\frac{\tau}{p}, X_a) - g(X_a, \frac{\tau}{p})g(X_a, \frac{\tau}{p})) + g(A_{X_a} U, A_{X_a} U)$$

$$= \frac{1}{p}g(U, U)(g(D_{X_a}\tau, X_a) - \frac{1}{p}g(\tau, \tau)) + g(A_{X_a} U, A_{X_a} U).$$

But, since $M$ has constant curvature $c$, we have

$$\sum_{a=1}^{q} R(X_a, U, X_a, U) = qcg(U, U),$$

which implies (2.4).

Now if we assume $A \equiv 0$ in Proposition 2.5 we obtain the following result.

COROLLARY 2.6. *Let $(M, g)$ be an n-dimensional Riemannian manifold with constant curvature c, and let $\mathbf{F}$ be a bundle-like foliation with totally umbilical leaves and with horizontal integrable distribution of dimension $p = n - q$ on $(M, g)$. If $\mathrm{div}_{\mathbf{H}} \, \tau = 0$ then*

(i)  *$c \leq 0$ and*
(ii)  *$g(\tau, \tau) = -pqc$.*

Since Proposition 2.4 ensures us that $\kappa$ is a basic one-form, as a consequence of Theorems 1.4 and 1.7, we get the following result from Corollary 2.6.

THEOREM 2.7. *Let $(M, g)$ be an oriented closed n-dimensional Riemannian manifold with constant curvature c, with g a bundle-like metric for a transversally oriented foliation F of codimension q, with totally umbilical leaves of dimension p. If the horizontal distribution is integrable and $p > 1$ and $q \in \{1, 2\}$ then $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic if and only if $g(\tau, \tau) = -pqc$.*

In Theorem 2.8, we shall consider the case when the horizontal distribution $\mathbf{H}$ is non-integrable (i.e. $A \neq 0$ at every point).

THEOREM 2.8. *Let $(M, g)$ be a Riemannian manifold of constant curvature c and let $\mathbf{F}$ be a bundle-like foliation with totally umbilical leaves on M with the leaf dimension $p \geq 2$. If the transversal distribution $\mathbf{H}$ is non-integrable and the transversal curvature operator $R^*$ is Einstein-like, then $\mathbf{F}$ is totally geodesic.*

PROOF.    From O'Neill equations we obtain:

$$(2.5) \qquad \sum_{a=1}^{q} R^*(Y, X_a, Y, X_a) = (q - 1)cg(Y, Y) + 3 \sum_{a=1}^{q} g(A_Y X_a, A_Y X_a),$$

where $\{X_a\}_{1 \leq a \leq q}$ is a local orthonormal frame of $\mathbf{H} = \mathbf{V}^{\perp}$. By hypothesis, there exists a basic function $\lambda$ on $M$ such that:

$$(2.6) \qquad \mathrm{Ric}^*(Y, Y) = \sum_{a=1}^{q} R^*(Y, X_a, Y, X_a) = \lambda g(Y, Y)$$

and $\lambda \neq (q - 1)c$ since $\mathbf{H}$ is a non-integrable distribution, i.e. $\sum_{a=1}^{q} g(A_Y X_a, A_Y X_a) > 0$ for some $Y$.

Now taking $Y = \tau$ in (2.5), from Proposition 2.4, $A_{\tau} = 0$, and from (2.5) and (2.6), we get:

$$(\lambda - (q - 1)c)g(\tau, \tau) = 0,$$

which implies $\tau = 0$.

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