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A New Identity Relating Mock Theta Functions with Distinct Orders

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1. Introduction

S. Ramanujan listed 17 mock theta functions of orders 3, 5 and 7 in his last letter to G. H. Hardy. A mock theta function is a function $f(q)$ for $|q| < 1$ satisfying the following two conditions:

(i) For every root of unity ζ, there is a theta function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.

(ii) There is no single theta function which works as in (i) for all ζ: i.e., for every theta function $\theta(q)$ there is some root of unity ζ for which $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

G. N. Watson found in [5] three more mock theta functions of order 3. Ramanujan gave more mock theta functions in his lost notebook. B. Gordon and R. J. McIntosh also found in [4] eight mock theta functions of order 8. A formal definition of order is unknown until now. It is, however, known that mock theta functions with the same order are related to each other except for order 7. In this paper we show an interesting new relation between mock theta functions with distinct orders in Theorem 1, and we further prove two new series representations of some 8th order mock theta functions in Theorem 2.

In section 2, we give the relation mentioned above in Theorem 1, which connects two mock theta functions with distinct orders, 3rd and 6th , to a generalized Lambert series.

In section 3, we introduce a function $F(q,t)$ defined by G. E. Andrews in [1]. There he consider three specializations of t; here we add to them the fourth specialization of t, and we will further show that there are relations, called half-shift in [4], among these four functions (see Definition 4 and below). We next give three examples of the function $F(q,t)$ in Propositions 1-3. The last one is particularly interesting, because we again see two mock theta functions appeared in section 2. Finally we give new series representations of two 8th order mock theta functions.

We close this section by introducing some notation.

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DEFINITION 1. Suppose q and a are complex numbers and n is an integer. For $n \geq 1$, we define

$$(a)_n = (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)\,,$$

for $n = 0$,

$$(a)_0 = 1\,,$$

and for $n < 0$, when a does not equal $q, q^2, \cdots, q^{-n}$,

$$(a)_n = (a;q)_n = \prod_{i=1}^{-n}(1-aq^{-i})^{-1}\,.$$

If $|q| < 1$, we define

$$(a)_\infty = (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)\,,$$

and more generally,

$$(x_1, \cdots, x_r; q)_\infty = (x_1)_\infty \cdots (x_r)_\infty\,.$$

Throughout this paper, q will denote a fixed complex number of absolute value less than 1. Then, for all integer n we have

$$(x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty}\,, \tag{1.1}$$

and for other real n, we take this as the definition of $(x;q)_n$.

We introduce some mock theta functions which will be used in this paper.

DEFINITION 2. 3rd order mock theta functions [5]:

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\omega q)_n(-\omega^2 q)_n}\,,$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(\omega q;q^2)_{n+1}(\omega^2 q;q^2)_{n+1}}\,,$$

where, and throughout this paper, ω denotes a primitive cube root of 1.

5th order mock theta functions [1]:

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n}\,,$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}.$$

6th order mock theta function [2]:

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q)_n}{(q^3;q^3)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q)_n(\omega^2 q)_n}.$$

8th order mock theta functions [4]:

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n},$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}.$$

DEFINITION 3. The basic hypergeometric function:

$$ {}_m\phi_n \left(\begin{array}{ccccc} a_1, & \cdots & a_m; & q; & z \\ b_1, & \cdots & b_n & & \end{array} \right) := \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_m)_j z^j}{(q)_j (b_1)_j \cdots (b_n)_j},$$

where $|z| < 1$ and $b_i \neq q^{-k}$ for any non-negative integer k.

2. Main result

Untill now, it is not known that mock theta functions with distinct orders are related to each other. Here we derive a new relation between 3rd order $\chi(q)$ and 6th order $\gamma(q)$.

THEOREM 1.

$$(q)_\infty \{3\chi(q) - \gamma(q)\} = 6 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 + q^{2n} + q^{4n}}.$$

PROOF. We begin with the Watson-Whipple transformation[3, Ch16]:

$$ {}_8\phi_7 \left(\begin{array}{cccccccccc} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f; & q; & a^2q^2/bcdef \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f & & & \end{array} \right)$$

$$= \frac{(aq)_\infty (aq/de)_\infty (aq/df)_\infty (aq/ef)_\infty}{(aq/d)_\infty (aq/e)_\infty (aq/f)_\infty (aq/def)_\infty} {}_4\phi_3\left(\begin{matrix} aq/bc, & d, & e, & f; & q\ ; q \\ aq/b, & aq/c, & def/a, & & \end{matrix} \right),$$

provided that each of d, e or f is of the form q^{-N} , where N is a positive integer.

Now

$$\frac{(q\sqrt{a})_n(-q\sqrt{a})_n}{(\sqrt{a})_n(-\sqrt{a})_n} = \frac{1-\sqrt{a}q^n}{1-\sqrt{a}} \cdot \frac{1+\sqrt{a}q^n}{1+\sqrt{a}} = \frac{1-aq^{2n}}{1-a} .$$

Also, when x tends to ∞,

$$\begin{aligned} (x)_n &= (1-x)(1-xq)\cdots(1-xq^{n-1}) , \\ &= (-x)^n\left(-\frac{1}{x}+1\right)\left(-\frac{1}{x}+q\right)\cdots(-\frac{1}{x}+q^{n-1}) , \\ &\sim (-x)^n q^{n(n-1)/2} . \end{aligned}$$

Let $a \to 1$, $d \to \infty$, $e \to \infty$, $f \to \infty$, $b = e^{i\theta}$ and $c = e^{-i\theta}$; we find that

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)(2-2\cos\theta)q^{n(3n+1)/2}}{1-2q^n\cos\theta+q^{2n}} = (q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(qe^{i\theta})_n(qe^{-i\theta})_n} .$$

In this relation, G. N. Watson in [5] took $\theta = \pi/3$ to get

$$(q)_\infty \chi(q) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)q^{n(3n+1)/2}}{1-q^n+q^{2n}} .$$

Similarly, we take $\theta = 2\pi/3$ to get

$$(q)_\infty \gamma(q) = 1 + 3\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)q^{n(3n+1)/2}}{1+q^n+q^{2n}} .$$

Hence,

$$\begin{aligned} (q)_\infty \{3\chi(q) - \gamma(q)\} &= 3 + 3\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)q^{n(3n+1)/2}}{1-q^n+q^{2n}} \\ &\quad -1 - 3\sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)q^{n(3n+1)/2}}{1+q^n+q^{2n}} \\ &= 2 + 3\sum_{n=1}^{\infty}(-1)^n(1+q^n)q^{n(3n+1)/2}\left(\frac{1}{1-q^n+q^{2n}} - \frac{1}{1+q^n+q^{2n}}\right) \\ &= 2 + 6\sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}(1+q^n)}{1+q^{2n}+q^{4n}} \end{aligned}$$

$$
\begin{aligned}
&= 2 + 6\left(\sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{2n}+q^{4n}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(n+1)/2+n}}{1+q^{2n}+q^{4n}}\right) \\
&= 2 + 6\left(\sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{2n}+q^{4n}} + \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{3n(n-1)/2-n}}{1+q^{-2n}+q^{-4n}}\right) \\
&= 2 + 6\left(\sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{2n}+q^{4n}} + \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{3n(n-1)/2-n+4n}}{1+q^{2n}+q^{4n}}\right) \\
&= 6 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{2n}+q^{4n}} .
\end{aligned}
$$

This completes the proof of Theorem 1.

3. Another relation between $\chi(q)$ and $\gamma(q)$

G. E. Andrews defined the following function $F(q,t)$ in [1], while the notation here is slightly changed from his original one.

DEFINITION 4. Suppose $a \in \mathbf{N}, z, t, a_l, b_m \in \mathbf{C}(1 \le l \le s, 1 \le m \le j), |z| < 1$, then

$$
\begin{aligned}
F(q,t) &:= F\left(\begin{matrix} a_1, & \cdots & a_s\,; & a, & z\,; & q\,, & t \\ b_1, & \cdots & b_j & & & & \end{matrix}\right) \\
&:= \sum_{n=0}^{\infty} \frac{(a_1 t)_n \cdots (a_s t)_n z^n t^{an} q^{an(n-1)/2}}{(t)_{n+1}(b_1 t)_n \cdots (b_j t)_n} .
\end{aligned}
$$

There he considered three functions of q:

$$
H_1(q) := \lim_{t\to -1+0}(1-t)F(q,t) = \sum_{n=0}^{\infty} \frac{(-a_1)_n \cdots (-a_s)_n (-1)^{an} z^n q^{an(n-1)/2}}{(-q)_n(-b_1)_n \cdots (-b_j)_n} , \tag{1}
$$

$$
H_2(q) := F(q, q^{1/2}) = \sum_{n=0}^{\infty} \frac{(a_1 q^{1/2})_n \cdots (a_s q^{1/2})_n z^n q^{an^2/2}}{(q^{1/2})_{n+1}(b_1 q^{1/2})_n \cdots (b_j q^{1/2})_n} \tag{2}
$$

and

$$
H_3(q) := \lim_{t\to 1-0}(1-t)F(q,t) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n z^n q^{an(n-1)/2}}{(q)_n (b_1)_n \cdots (b_j)_n} . \tag{3}
$$

Now we define the function:

$$
H_4(q) := F(q, -q^{1/2}) = \sum_{n=0}^{\infty} \frac{(-a_1 q^{1/2})_n \cdots (-a_s q^{1/2})_n (-1)^{an} z^n q^{an^2/2}}{(-q^{1/2})_{n+1}(-b_1 q^{1/2})_n \cdots (-b_j q^{1/2})_n} . \tag{4}
$$

We will observe the relation among these four functions. Consider the two series

$$\begin{aligned}\sum_{n=0}^{\infty} a_n &:= \sum_{n=0}^{\infty} \frac{(-q^{n+1}, -b_1 q^n, \cdots, -b_j q^n \ ; q)_\infty}{(-a_1 q^n, \cdots, -a_s q^n \ ; q)_\infty} (-1)^{an} z^n q^{an(n-1)/2} \\ &= \frac{(-q, -b_1, \cdots, -b_j \ ; q)_\infty}{(-a_1, \cdots, -a_s \ ; q)_\infty} H_1(q)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} b_n &:= \sum_{n=0}^{\infty} \frac{(-q^{n+\frac{1}{2}+1}, -b_1 q^{n+\frac{1}{2}}, \cdots, -b_j q^{n+\frac{1}{2}}; q)_\infty}{(-a_1 q^{n+\frac{1}{2}}, \cdots, -a_s q^{n+\frac{1}{2}}; q)_\infty} (-1)^{a(n+\frac{1}{2})} z^{n+\frac{1}{2}} q^{a(n+\frac{1}{2})(n-\frac{1}{2})/2} \\ &= (-1)^{\frac{a}{2}} z^{\frac{1}{2}} q^{-\frac{a}{8}} \frac{(-q^{1/2}, -b_1 q^{1/2}, \cdots, -b_j q^{1/2}; q)_\infty}{(-a_1 q^{1/2}, \cdots, -a_s q^{1/2}; q)_\infty} H_4(q)\,.\end{aligned}$$

In these formulas, we obtain last equalities by using (1.1). These two series are related by a shift $b_n = a_{n+\frac{1}{2}}$. That is to say, the function $H_4(q)$ is obtained from the function $H_1(q)$ when we replace n by $n + \frac{1}{2}$ in the sum of a_n. We say that $H_4(q)$ is obtained from $H_1(q)$ by the *half-shift*. Similarly $H_2(q)$ can be obtained from $H_3(q)$ by the half-shift.

Now we give three typical examples of $F(q,t)$.

PROPOSITION 1 (G. E. Andrews [1]). *Let*

$$M\vartheta_{5,1}(q,t) := F\left(\begin{matrix} ; & 2,\ q\ ; q, t \\ & \end{matrix} \right) = \frac{1}{1-t} \sum_{n=0}^{\infty} \frac{q^{n^2} t^{2n}}{(tq)_n}\,.$$

Then we have

$$\begin{aligned}&(1) \quad \lim_{t \to -1+0} (1-t) M\vartheta_{5,1}(q,t) = f_0(q)\,, \\ &(2) \quad M\vartheta_{5,1}(q, q^{1/2}) = F_1(q^{1/2})\,, \\ &(3) \quad \lim_{t \to 1} (1-t) M\vartheta_{5,1}(q,t) = G(q)\,, \\ &(4) \quad M\vartheta_{5,1}(q, -q^{1/2}) = F_1(-q^{1/2})\,,\end{aligned}$$

where

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}\,.$$

PROPOSITION 2. *Let*

$$M\vartheta_{8,1}(q^2,t) := F\left(\begin{matrix} -q; & 1,\ q;\ q^2, t \\ & \end{matrix} \right) = \frac{1}{1-t} \sum_{n=0}^{\infty} \frac{(-tq;q^2)_n q^{n^2} t^n}{(tq^2;q^2)_n}\,.$$

Then we have

$$\text{(1)}\quad \lim_{t\to -1+0}(1-t)M\vartheta_{8,1}(q^2,t) = S_0(-q)\,,$$

$$\text{(2)}\quad M\vartheta_{8,1}(q^2,q) = T_1(-q)\,,$$

$$\text{(3)}\quad \lim_{t\to 1-0}(1-t)M\vartheta_{8,1}(q^2,t) = A(q)\,,$$

$$\text{(4)}\quad M\vartheta_{8,1}(q^2,-q) = \sum_{n=0}^{\infty}\frac{(q^2;q^2)_n(-1)^n q^{n^2+n}}{(-q;q^2)_{n+1}}\,,$$

where

$$A(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}\,.$$

NOTE: This proposition is similar to Proposition 1, but it is still unknown whether $M\vartheta_{8,1}(q^2,-q)$ is a mock theta function.

The following Proposition 3 is particularly interesting, because all four specializations are mock theta functions and only one of them has a distinct order from others.

PROPOSITION 3. *Let*

$$M\vartheta_{3,3}(q,t) := F\begin{pmatrix} q & & ;2,\ q\ ;q,t \\ \omega q, & \omega^2 q & \end{pmatrix} = \frac{1}{1-t}\sum_{n=0}^{\infty}\frac{q^{n^2}t^{2n}}{(\omega q t)_n(\omega^2 q t)_n}\,.$$

Then we have

$$\text{(1)}\quad \lim_{t\to -1+0}(1-t)M\vartheta_{3,3}(q,t) = \chi(q)\,,$$

$$\text{(2)}\quad M\vartheta_{3,3}(q,q^{1/2}) = \frac{1+q^{1/2}+q}{1-q^{1/2}}\,\rho(q^{1/2})\,,$$

$$\text{(3)}\quad \lim_{t\to 1-0}(1-t)M\vartheta_{3,3}(q,t) = \gamma(q)\,,$$

$$\text{(4)}\quad M\vartheta_{3,3}(q,-q^{1/2}) = \frac{1-q^{1/2}+q}{1+q^{1/2}}\,\rho(-q^{1/2})\,.$$

Table I lists the functions in Proposition 1, 2 and 3. Here we attach the order to each mock theta function.

Table I

	(1)	(2)	(3)	(4)
$M\vartheta_{5,1}(q,t)$	$f_0(q)$; $5th$	$F_1(q^{1/2})$; $5th$	$G(q)$	$F_1(-q^{1/2})$; $5th$
$M\vartheta_{8.1}(q^2,t)$	$S_0(-q)$; $8th$	$T_1(-q)$; $8th$	$A(q)$	?
$M\vartheta_{3,3}(q,t)$	$\chi(q)$; $3rd$	$\rho(q^{1/2})$; $3rd$	$\gamma(q)$; $6th$	$\rho(-q^{1/2})$; $3rd$

NOTE: In the bottom row, we see again the mock theta functions $\chi(q)$ and $\gamma(q)$ in Theorem 1.

As an appendix, we give a new representation for 8th order mock theta functions $U_0(q)$ and $U_1(q)$, by using a transformation of the basic hypergeometric function.

THEOREM 2.

$$U_0(q) = \frac{(iq;q^2)_\infty}{(-iq^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n(i;q^2)_n}{(q^2;q^2)_n(iq^2;q^2)_n}(iq)^n\,, \tag{4.1}$$

$$U_1(q) = q\sum_{n=0}^{\infty}\frac{(-iq^2;q^2)_n(-iq)^n}{(iq;q^2)_{n+1}}\,. \tag{4.2}$$

PROOF. We use the transformation [3, Ch16]:

$$\begin{aligned} &{}_3\phi_2\begin{pmatrix} a, & b, & c; & q; & de/abc \\ d, & e & & & \end{pmatrix} \\ &\quad = \frac{(e/a)_\infty(de/bc)_\infty}{(e)_\infty(de/abc)_\infty}\,{}_3\phi_2\begin{pmatrix} a, & d/b, & d/c; & q; & e/a \\ d, & de/bc & & & \end{pmatrix}, \end{aligned}$$

where $|de/abc| < 1$ and $|e/a| < 1$.

Replace q by q^2 in this equation and then let c tend to ∞. Then we have

$$\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q^2)_n(-1)^n q^{n^2-n}}{(q^2;q^2)_n(d;q^2)_n(e;q^2)_n}\left(\frac{de}{ab}\right)^n = \frac{(e/a;q^2)_\infty}{(e;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n(d/b;q^2)_n}{(q^2;q^2)_n(d;q^2)_n}\left(\frac{e}{a}\right)^n.$$

Now take $a = -q,\ b = q^2,\ d = iq^2,\ e = -iq^2$. This gives (4.1). Similarly take $a = q^2,\ b = -q,\ d = iq^3,\ e = -iq^3$ and multiply the resulting identity by $q/(1+q^2)$. This gives (4.2).

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