TOKYO J. MATH.
VOL. 29, NO. 1, 2006

Principal Functions for High Powers of Operators

Dedicated to Professor Seiji Watanabe on his sixtieth birthday

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(Communicated by K. Taniyama)

Abstract. For an operator T with some trace class condition, let g_{T^n} and $g^P_{T^n}$ be the principal functions related to the Cartesian decomposition $T^n = X_n + iY_n$ and the polar decomposition $T^n = U_n|T^n|$ for a positive integer n, respectively. In this paper, we study properties of g_{T^n} and $g^P_{T^n}$ and invariant subspaces of T^n.

1. Introduction

An operator below means a bounded linear operator on a separable infinite dimensional Hilbert space $\mathcal{H}$. The commutator of two operators A, B is denoted by $[A, B] = AB - BA$. Let $\mathcal{C}_1$ be the set of trace-class operators of $B(\mathcal{H})$. Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$. Pincus introduced the principal function g_T related to the Cartesian decomposition $T = X + iY$. Properties of the principal function g_T have been studied ([3], [6], [7], [9], [10]). Especially, C.A. Berger gave the principal functions g_{T^n} of powers T^n of T in terms of g_T and proved that for a sufficiently high n, T^n has a non-trivial invariant subspace for a hyponormal operator T ([1]). On the other hand, we have another principal function g^P_T related to the polar decomposition $T = U|T|$ such that $[|T|, U] \in \mathcal{C}_1$ ([3], [4], [10]).

In this paper, we study properties of g_{T^n} and $g^P_{T^n}$ and invariant subspaces of T^n.

2. Theorem

$\phi(r, z)$ is called *Laurent polynomial* if there exist a non-negative integer N and polynomials $p_k(r)$ such that $\phi(r, z) = \sum_{k=-N}^{N} p_k(r)z^k$. For differentiable functions P, Q of two variables (x, y), let $J(P, Q)(x, y) = P_x(x, y) \cdot Q_y(x, y) - P_y(x, y) \cdot Q_x(x, y)$.

For an operator $T = X + iY = U|T|$, we consider the following trace formulae:

Received October 4, 2004; revised February 11, 2005
2000 *Mathematics Subject Classification.* 47B20, 47A10.
Key words and phrases. Hilbert space, trace, principal function.
This research is partially supported by Grant-in-Aid Scientific Research No. 17540176.

$$\text{(1)} \qquad \operatorname{Tr}([P(X,Y),Q(X,Y)]) = \frac{1}{2\pi i}\iint J(P,Q)(x,y)g_T(x,y)dxdy\,,$$

for polynomials P and Q.

$$\text{(2)} \qquad \operatorname{Tr}([\phi(|T|,U),\psi(|T|,U)]) = \frac{1}{2\pi}\iint J(\phi,\psi)(r,e^{i\theta})e^{i\theta}g_T^P(e^{i\theta},r)drd\theta$$

for Laurent polynomials ϕ and ψ.

If formula (1) holds, the function g_T is called *the principal function related to the Cartesian decomposition* $T = X + iY$. If formula (2) holds, the function g_T^P is called *the principal function related to the polar decomposition* $T = U|T|$. For invertible operator T such that $[T^*, T] \in \mathcal{C}_1$, there exist both g_T and g_T^P ([3], [5]).

First we start with the following

THEOREM 1. *Let* $T = X + iY = U|T|$ *be an operator satisfying the following trace formula*:

$$\operatorname{Tr}([\phi(|T|,U),\psi(|T|,U)]) = \frac{1}{2\pi}\iint J(\phi,\psi)(r,e^{i\theta})e^{i\theta}g_T^P(e^{i\theta},r)drd\theta$$

for any Laurent polynomials ϕ *and* ψ. *Then the principal function* $g_T(x,y)$ *related to the Cartesian decomposition* $T = X + iY$ *of* T *exists and it is given by* $g_T(x,y) = g_T^P(e^{i\theta},r)$, *where* $x + iy = re^{i\theta}$.

PROOF. Let P and Q be polynomials of two variables (x,y). First note that

$$\operatorname{Tr}([P(X,Y),Q(X,Y)]) = \operatorname{Tr}\left(\left[P\left(\frac{T+T^*}{2},\frac{T-T^*}{2i}\right),Q\left(\frac{T+T^*}{2},\frac{T-T^*}{2i}\right)\right]\right).$$

Put

$$\tilde{P}(r,z) = P\left(\frac{zr+r/z}{2},\frac{rz-r/z}{2i}\right) \quad \text{and} \quad \tilde{Q}(r,z) = Q\left(\frac{zr+r/z}{2},\frac{rz-r/z}{2i}\right).$$

Then both $\tilde{P}$ and $\tilde{Q}$ are Laurent polynomials and also the following equations hold:

$$\begin{aligned}
\tilde{P}_r(r,z) &= P_x(r,z)\frac{z+1/z}{2} + P_y(r,z)\frac{z-1/z}{2i},\\
\tilde{P}_z(r,z) &= \frac{r}{2}P_x(r,z)\left(1-\frac{1}{z^2}\right) + \frac{r}{2i}P_y(r,z)\left(1+\frac{1}{z^2}\right),\\
\tilde{Q}_r(r,z) &= Q_x(r,z)\frac{z+1/z}{2} + Q_y(r,z)\frac{z-1/z}{2i},\\
\tilde{Q}_z(r,z) &= \frac{r}{2}Q_x(r,z)\left(1-\frac{1}{z^2}\right) + \frac{r}{2i}Q_y(r,z)\left(1+\frac{1}{z^2}\right).
\end{aligned}$$

Hence we obtain

$$J(\tilde{P}, \tilde{Q})(r, z) = J(P, Q)(x, y)\,\frac{r}{zi}\,.$$

Therefore, it holds

$$J(\tilde{P}, \tilde{Q})(r, e^{i\theta}) = J(P, Q)(x, y)\,\frac{r}{ie^{i\theta}}\,. \tag{3}$$

Since $P(X, Y) = \tilde{P}(|T|, U)$ and $Q(X, Y) = \tilde{Q}(|T|, U)$, it holds

$$\mathrm{Tr}([P(X, Y), Q(X, Y)]) = \mathrm{Tr}([\tilde{P}(|T|, U), \tilde{Q}(|T|, U)])\,. \tag{4}$$

By (3) and (4), we have

$$\begin{aligned}\mathrm{Tr}([P(X, Y), Q(X, Y)]) &= \mathrm{Tr}([\tilde{P}(|T|, U), \tilde{Q}(|T|, U)])\\ &= \frac{1}{2\pi}\iint J(\tilde{P}, \tilde{Q})(r, e^{i\theta})e^{i\theta} g_T^P(e^{i\theta}, r)\,dr\,d\theta\\ &= \frac{1}{2\pi i}\iint J(P, Q)(x, y) g_T^P(e^{i\theta}, r)\,r\,dr\,d\theta\,.\end{aligned}$$

Put $g_T(x, y) = g_T^P(e^{i\theta}, r)$ for $x + iy = re^{i\theta}$. Using the transformation $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}\frac{1}{2\pi i}\iint J(P, Q)(x, y) g_T^P(e^{i\theta}, r)\,r\,dr\,d\theta &= \frac{1}{2\pi i}\iint J(P, Q)(x, y) g_T(x, y)\,dx\,dy\\ &= \mathrm{Tr}([P(X, Y), Q(X, Y)])\,.\end{aligned}$$

Since P and Q are arbitrary, by (1) it completes the proof. □

If an operator $T = U|T|$ is invertible, then U is unitary and $[|T|, U] \in \mathcal{C}_1$ implies $[T^*, T] \in \mathcal{C}_1$, because $[T^*, T] = |T|[|T|, U]U^* + [|T|, U]|T|U^*$. And equation (2) holds by [5, Theorem 4]. So we have the following

COROLLARY 2. *If an invertible operator $T = X + iY = U|T|$ satisfies $[|T|, U] \in \mathcal{C}_1$, then $g_T(x, y) = g_T^P(e^{i\theta}, r)$, where $x + iy = re^{i\theta}$.*

For a relation between g_T^P and $g_{T^n}^P$, we need the following Berger's result:

THEOREM 3 (Berger, Th.4 [1]). *For an operator T, if $[T^*, T] \in \mathcal{C}_1$, then for a positive integer n,*

$$g_{T^n}(x, y) = \sum_{(u+iv)^n = x+iy} g_T(u, v)\,.$$

THEOREM 4. *For an operator T with $[T^*, T] \in \mathcal{C}_1$, if $\iint g_T(x, y)\,dx\,dy \neq 0$, then*

$$\lim_{n\to\infty} \text{ess sup}\,|g_{T^n}| = \infty\,.$$

PROOF. We choose a positive number a such that $\|aT\| < 1$. It holds $g_{aT}(x, y) = g_T(x/a, y/a)$ (cf. [9, p. 242,1)]). Hence we have

$$\text{ess sup}\,|g_{aT}(x, y)| = \text{ess sup}\,|g_T(x, y)|\,.$$

Therefore, we may assume that $\|T\| < 1$. Put $g(x + iy) = g_T(x, y)$ and $g_n(x + iy) = g_{T^n}(x, y)$. Let m_2 be the planar Lebesgue measure. Since $m_2(\{0\}) = 0$, we consider in the set $\mathbf{C} - \{0\}$. Put $S(n, k) = \{z \in \mathbf{C} - \{0\} : \frac{2\pi(k-1)}{n} \leq \arg z < \frac{2\pi k}{n}\}$ $(k = 1, \cdots, n)$. Then by Theorem 3 we have

$$g_n(r^n e^{i\theta}) = \sum_{k=1}^{n} g(re^{i(\theta+2\pi(k-1))/n})\,.$$

Hence it holds

$$\begin{aligned}\iint g_n(r^n e^{i\theta}) r\,d\theta\,dr &= \sum_{k=1}^{n} \iint g(re^{i(\theta+2\pi(k-1))/n}) r\,d\theta\,dr \\ &= \sum_{k=1}^{n} n \iint_{S(n,k)} g(re^{i\theta}) r\,d\theta\,dr = n \iint g(re^{i\theta}) r\,d\theta\,dr \\ &= n \iint g_T(x, y)\,dx\,dy \neq 0\end{aligned}$$

for every n. By [2, Theorem 3.3], for an operator S with $[S^*, S] \in \mathcal{C}_1$, the support of g_S is contained in $[-||S||, ||S||] \times [-||S||, ||S||]$. Let A_n denote the support of g_{T^n}. Since $\|T\| < 1$, it holds $\lim_{n\to\infty} m_2(A_n) = 0$. Hence

$$\lim_{n\to\infty} \text{ess sup}\,|g_{T^n}| = \lim_{n\to\infty} \text{ess sup}\,|g_n| = \infty\,. \qquad \square$$

We remark that a hyponormal operator T with $0 \neq [T^*, T] \in \mathcal{C}_1$ satisfies $\iint g_T(x, y)\,dx\,dy \neq 0$.

Applying Corollary 2 and Theorem 4 to T^n, we have the following.

COROLLARY 5. *For an operator $T = U|T|$, let $T^n = U_n|T^n|$ be the polar decomposition of T^n $(n = 1, 2, \cdots)$. If $[|T^n|, U_n] \in \mathcal{C}_1$ for every non-negative integer n and $\iint g_T^P(e^{i\theta}, r) r\,d\theta\,dr \neq 0$, then*

$$\lim_{n\to\infty} \text{ess sup}\,|g^P_{T^n}| = \infty\,.$$

Let $T = U|T|$ and $T^n = U_n|T^n|$ be the polar decompositions of T and T^n, respectively. Then it holds that if $[T^*, T] \in \mathcal{C}_1$, then $[T^{*n}, T^n] \in \mathcal{C}_1$ for any positive integer n. On the other hand, in the polar decomposition case, it is not clear whether $[|T|, U] \in \mathcal{C}_1$ impilies $[|T^n|, U_n] \in \mathcal{C}_1$ even if $n = 2$. If T is invertible and $[|T|, U] \in \mathcal{C}_1$, then, for every n, it holds $[|T^n|, U_n] \in \mathcal{C}_1$ by [5, Theorem 3].

Next we consider operators with cyclic vectors. First we need the following result. We remark that the proof of [9, Theorem X.4.3] works still for a pair of operators with trace class self-commutator.

THEOREM 6 (Martin and Putinar, Th.X.4.3 [9]). *Let g_T and g_V be the principal functions of operators T and V such that $[T^*, T], [V^*, V] \in \mathcal{C}_1$, respectively. If there exists an operator $A \in \mathcal{C}_1$ such that $AV = TA$ and* $\ker(A) = \ker(A^*) = \{0\}$, *then $g_T \leq g_V$.*

Proof of the following lemma is based on it of [9, Corollary X.4.4].

LEMMA 7. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If T has a cyclic vector, then $g_T \leq 1$.*

PROOF. We may assume that $\|T\| < 1$. Let ξ be a cyclic vector for T. Define an operator $A : \ell^2(\mathbf{N}) \to \mathcal{H}$ by

$$Ae_k = T^k\xi\,, \quad k \geq 0\,,$$

where $\{e_k\}$ denotes the standard basis of ℓ^2. Let V be the unilateral shift on $\ell^2(\mathbf{N})$. Then it holds $AV = TA$. It is easy to see that $\ker(A^*) = \{0\}$. Also we show $\ker(A) = \{0\}$. Assume that $\ker(A) \neq \{0\}$. Since $\ker(A) \neq \{0\}$ if and only if there exists a non-zero analytic function f in the unit disk such that $f(T) = 0$, we have $\{0\} = \sigma(f(T)) = f(\sigma(T))$. Since $\sigma(T)$ has a limit point in the unit disk, we have $f = 0$. It's a contradiction. Hence, we have $\ker(A) = \{0\}$.

By the similar way to Corollary X.4.4 of [9], it follows that A is a trace class operator. Since $g_V \leq 1$, by Theorem 6, we have

$$g_T \leq g_V \leq 1\,. \qquad \square$$

Let S be an operator having the principal function g_S related to the Cartesian decomposition $S = X + iY$. Then $g_{S^*}(x, y) = -g_S(x, -y)$. Hence, as a corollary of Lemma 7, we have the following.

LEMMA 8. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If T^* has a cyclic vector, then $-1 \leq g_T$.*

Finally, we give an invariant subspace result. Using a property of g_{T^n}, Berger showed that $(T - aI)^n$ has a non-trivial invariant subspace for any number a satisfying $g_T(a) \neq 0$ and a sufficiently high n (see also [8]).

We remark that if K is a non-trivial invariant subspace for S^*, then $K^\perp$ is an invariant subspace for S.

THEOREM 9. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. Moreover, if $\iint g_T(x, y)dxdy \neq 0$, then, for a sufficiently high n, T^n has a non-trivial invariant subspace.*

PROOF. For a positive integer n, it holds that $[T^{n*}, T^n] \in \mathcal{C}_1$, $\sigma(T^n)$ is an infinite set and $g_T \neq 0$. If both T^n and T^{*n} have cyclic vectors, then, from Lemmas 7 and 8,

$|g_{T^n}| \le 1$. Hence from Theorem 4, for a sufficiently high n, T^n or T^{*n} has a non-trivial invariant subspace. If T^{*n} has a non-trivial invariant subspace, so does T^n. This completes the proof. □

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