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Surfaces in S^n with Prescribed Gauss Map

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Abstract. Let G be a C^∞-mapping from a connected Riemann surface M into the complex quadric Q_{n-1} in the n-dimensional complex projective space. We give a condition for the existence of a surface in the n-dimensional Euclidean unit sphere S^n such that the Gauss map is G. Under this condition, if M is a torus, there exists a surface in S^n such that the Gauss map is G. We also show that for a connected Riemann surface M there exists an immersion $X : M \to RP^n$ such that a neighborhood of each point of $X(M)$ is covered by a surface in S^n with prescribed Gauss map G where RP^n is the n-dimensional real projective space.

1. Introduction

In this paper by a surface S in an n-dimensional ($n \geq 3$) Riemannian manifold $\hat{M}$ we mean a triple $(M, \hat{M}, X)$ consisting of a connected Riemann surface M, the ambient space $\hat{M}$ and a C^∞-conformal immersion $X : M \to \hat{M}$. Let $S = (M, S^n, X)$ be a surface in the n-dimensional Euclidean unit sphere S^n. We regard it as a surface in the $(n+1)$-dimensional Euclidean space R^{n+1} and consider the (generalized) Gauss map $G : M \to Q_{n-1}$ where Q_{n-1} is the complex quadric in the n-dimensional complex projective space ([1]). It is important to study the property of the Gauss map of surfaces. For a simply-connected Riemann surface M and a C^∞-mapping $G : M \to Q_{n-1}$ with certain conditions, Hoffman and Osserman showed that there exists a surface $S = (M, R^{n+1}, X)$ such that the Gauss map is G and X can be expressed by an integration of C^∞-mappings induced from G ([2]). In this paper we consider the existence of surfaces in S^n with prescribed Gauss map. Since, in case of S^n, the existence of such a surface cannot be showed directly by using the results in [2], we need other method. By using this method, a local existence theorem will be given in Theorem 3.2 of this paper.

Let M be a connected Riemann surface and $G : M \to Q_{n-1}$ a C^∞-mapping. We assume that G satisfies the conditions (1) and (2) in Theorem 3.2 at each point of M. We show in Theorem 5.2 that if M is a torus T^2, there exist a covering space $(\hat{T}^2, T^2, \hat{\pi})$ over T^2 and a surface $S = (\hat{T}^2, S^n, X)$ such that the Gauss map is $G \circ \hat{\pi}$. We also show in Section 6 that there exists a surface $S = (M, RP^n, X)$ in the n-dimensional real projective space RP^n

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with the property that a neighborhood of each point of $X(M)$ is covered by a surface in S^n such that the Gauss map is G.

2. The Gauss map of surfaces in S^n

We assume in this paper that manifolds and apparatus on them are of class C^∞ and that manifolds satisfy the second countability axiom, unless otherwise stated.

Let M be a connected Riemann surface and $(U, z = u_1 + \sqrt{-1}u_2)$ a complex coordinate system of M. For a C^∞-mapping $A : M \to R^k$, we put

$$A_z = \frac{\partial A}{\partial z} = \frac{1}{2}\left(\frac{\partial A}{\partial u_1} - \sqrt{-1}\frac{\partial A}{\partial u_2}\right), \quad A_{\bar{z}} = \frac{\partial A}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial A}{\partial u_1} + \sqrt{-1}\frac{\partial A}{\partial u_2}\right).$$

Let M be a connected Riemann surface and $S = (M, S^n, X)$ a surface in $S^n (n \geq 3)$. We define the Gauss map of surfaces in R^{n+1} following Hoffman and Osserman ([1]). We regard S as a surface in R^{n+1}. $X : M \to S^n$ is said to be conformal if for any complex coordinate system $(U, z = u_1 + \sqrt{-1}u_2)$ of M it satisfies

$$\left|\frac{\partial X}{\partial u_1}\right| = \left|\frac{\partial X}{\partial u_2}\right| \neq 0, \quad \frac{\partial X}{\partial u_1} \cdot \frac{\partial X}{\partial u_2} = 0$$

where $|A|$ denotes the length of a vector A in R^{n+1} and $A \cdot B$ denotes the Euclidean inner product of vectors A and B in R^{n+1}. The conformality condition of X is equivalent to

$$\left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial \bar{z}} \right\rangle = 0$$

where $\langle \ , \ \rangle$ denotes the canonical Hermitian product on C^{n+1}.

Let Q_{n-1} be the complex quadric in the n-dimensional projective space CP^n defined as

$$Q_{n-1} = \{[w] \in CP^n | {w_1}^2 + \cdots + {w_{n+1}}^2 = 0\}.$$

Q_{n-1} is diffeomorphic to the oriented Grassmaniann manifold

$$\tilde{G}(2, n+1) = SO(n+1)/SO(2) \times SO(n-1).$$

For each $u \in U$ we identify the tangent vectors

$$dX_u\left(\left(\frac{\partial}{\partial u_1}\right)_u\right), \quad dX_u\left(\left(\frac{\partial}{\partial u_2}\right)_u\right)$$

with

$$\frac{\partial X}{\partial u_1}(u), \quad \frac{\partial X}{\partial u_2}(u)$$

by parallel translations in R^{n+1} respectively. Then each tangent plane of $X(M)$ corresponds to a unique element of Q_{n-1}. Thus the generalized Gauss map of S can be defined as

$$G : M \to Q_{n-1} \quad \left(u \mapsto \left[\frac{\partial X}{\partial \bar{z}}(u)\right]\right)$$

(see [1]). For simplicity, the generalized Gauss map will be called the Gauss map in this paper.

Let $\mathscr{M}(s,t)$ be the set of all $s \times t$ real matrices. For $K \in \mathscr{M}(s,t)$, let ${}^tK$ stand for the transposed matrix of K. For each $u \in M$ we denote by $\hat{G}(u)$ the element of $\tilde{G}(2, n+1)$ corresponding to $G(u)$. For $u \in M$, we express R^{n+1} as the direct sum

$$R^{n+1} = \hat{G}(u) \oplus \hat{G}^{\perp}(u)$$

where $\hat{G}^{\perp}(u)$ denotes the orthogonal complement to $\hat{G}(u)$ in R^{n+1}. We set

$$P(M, G) = \bigcup_{u \in M} \hat{G}(u)\,.$$

We denote by V the smallest linear subspace in R^{n+1} containing $P(M,G)$. Let $V^{\perp}$ be the orthogonal complement of V in R^{n+1}. In the following we put $k = \dim V$. Then we have $2 \le k \le n+1$.

Let $St(n+1, m)$ denote the Stiefel manifold of m-dimensional frames in R^{n+1}. Let $E = (E^T, E^N) : U \to SO(n+1)$ be a C^{∞}-mapping such that

$$E^T = (E_1, E_2) : U \to St(n+1, 2), \;\; E^N = (E_3, \cdots, E_{n+1}) : U \to St(n+1, n-1)$$

are C^{∞}-mappings and such that $E^T(u) = (E_1(u), E_2(u))$ is an orthonormal frame in $\hat{G}(u)$ and gives the orientation of $\hat{G}(u)$. We also regard E^T and E^N as C^{∞}-mappings $E^T : U \to \mathscr{M}(n+1, 2)$ and $E^N : U \to \mathscr{M}(n+1, n-1)$ respectively. Since X is conformal, we can put

$$\frac{\partial X}{\partial z} = E^T \Psi \tag{2.1}$$

where

$$\Psi : U \to C^2 \setminus \{0\} \quad (u \mapsto {}^t(\psi(u), -\sqrt{-1}\psi(u)))\,.$$

We have on U

$$\left\langle \frac{\partial X}{\partial z}, X \right\rangle = 0\,.$$

Then on U we can put

$$X = \sum_{j=3}^{n+1} a_j E_j = E^N A \tag{2.2}$$

where

$$A : U \to R^{n-1} \setminus \{0\} \quad (u \mapsto {}^t(a_3(u), \cdots, a_{n+1}(u)))\,.$$

By using (2.1) and (2.2) we have

$$\frac{\partial X}{\partial z} = \frac{\partial E^N}{\partial z} A + E^N \frac{\partial A}{\partial z} = E^T \Psi\,. \tag{2.3}$$

Since ${}^t E^N E^T = 0$ and ${}^t E^N E^N = I_{n-1}$, by using (2.3), we get

$$\frac{\partial A}{\partial z} = -{}^t E^N \frac{\partial E^N}{\partial z} A\,. \tag{2.4}$$

Here I_{n-1} is the unit matrix of degree $(n-1)$.

By the Frobenius theorem ([3]), we have the following.

LEMMA 2.1. *Under the notations stated above, a necessary and sufficient condition for the existence of non-zero solutions of the partial differential equation* (2.4) *can be expressed as*

$$\mathrm{Im}\left\{ \frac{\partial\, {}^t E^N}{\partial \bar z} \frac{\partial E^N}{\partial z} + {}^t E^N \frac{\partial E^N}{\partial z} \frac{\partial\, {}^t E^N}{\partial \bar z} E^N \right\} = 0\,. \tag{2.5}$$

3. Existence theorem

Let M be a connected Riemann surface and $G : M \to Q_{n-1}$ $(n \geq 3)$ a C^∞-mapping. For each $u \in M$, let $\hat G(u)$ be as in Section 2. We take a point $m_0 \in M$ and a complex coordinate system $(U, z = u_1 + \sqrt{-1} u_2)$ about m_0 where U is connected. If we take U sufficiently small, there exist C^∞-mappings

$$E_i : U \to R^{n+1} \setminus \{0\} \quad (i = 1, 2)$$

such that for each $u \in U$ $E^T(u) := (E_1(u), E_2(u))$ is an orthonormal frame in $\hat G(u)$ and gives the orientation of $\hat G(u)$. We denote by $E^N(u) := (E_3(u), \cdots, E_{n+1}(u))$ the orthonormal complement of $E^T(u)$ in R^{n+1}. In the following let I denote the unit matrix of degree $(n+1)$ and put

$$B := \frac{\partial\, {}^t E^N}{\partial z} (I - E^N\, {}^t E^N) \frac{\partial E^N}{\partial z}\,. \tag{3.1}$$

For such E^N we consider the following four conditions:

(I)
$$\mathrm{Im}\left\{ \frac{\partial\, {}^t E^N}{\partial \bar z} \frac{\partial E^N}{\partial z} + {}^t E^N \frac{\partial E^N}{\partial z} \frac{\partial\, {}^t E^N}{\partial \bar z} E^N \right\} = 0\,, \tag{3.2}$$

(II) $$(I - E^{N\,t}E^N)\frac{\partial E^N}{\partial u_j} \neq 0 \quad (j = 1, 2)\,, \tag{3.3}$$

(III) there exists $\xi \in R^{n-1} \setminus \{0\}$ such that
$$ {}^t\xi B(m_0)\xi = 0\,, \tag{3.4}$$

(IV) $$\frac{\partial\, {}^tE^N}{\partial z}E^N B + B^tE^N\frac{\partial E^N}{\partial z} - \frac{\partial B}{\partial z} = 0\,. \tag{3.5}$$

We note that (3.2) is equivalent to
$$\frac{\partial\, {}^tE^N}{\partial u_2}\frac{\partial E^N}{\partial u_1} - \frac{\partial\, {}^tE^N}{\partial u_1}\frac{\partial E^N}{\partial u_2} + {}^tE^N\left(\frac{\partial E^N}{\partial u_1}\frac{\partial\, {}^tE^N}{\partial u_2} - \frac{\partial E^N}{\partial u_2}\frac{\partial\, {}^tE^N}{\partial u_1}\right)E^N = 0\,. \tag{3.6}$$

We note that the condition (I) is a necessary and sufficient condition for the existence of a solution of the partial differential equation (2.4). The condition (II) demands that a mapping $X : M \to S^n$ defining a surface $S = (M, S^n, X)$ is an immersion. Moreover the conditions (III) and (IV) ensure that X is conformal. These conditions (I)–(IV) are independent of the choice of complex coordinate systems of M. Moreover we have the following.

LEMMA 3.1. *Under the notations above, the conditions* (I)–(IV) *are independent of the choice of orthonormal frames in* R^{n+1}.

PROOF. Let $SO(m)$ denote the set of all special orthogonal $m \times m$-matrices. Let $E = (E^T, E^N) : U \to SO(n+1)$ and $F = (F^T, F^N) : U \to SO(n+1)$ be different C^∞-mappings such that for each $u \in U$ $E^T(u)$ and $F^T(u)$ are orthonormal frames of $\hat{G}(u)$ and give the same orientation of $\hat{G}(u)$. Then there exists a C^∞-mapping
$$\Omega : U \to SO(n+1)$$
such that
$$F = E\Omega = E\left(\begin{array}{c|c} \Omega_1 & 0 \\ \hline 0 & \Omega_2 \end{array}\right)$$
where $\Omega_1(u) \in SO(2)$ and $\Omega_2(u) \in SO(n-1)$ $(u \in U)$. Hence we have
$$E^N = F^{N\,t}\Omega_2\,. \tag{3.7}$$

We assume that E^N satisfies the conditions (I)–(IV). From (3.7), we have
$$\frac{\partial E^N}{\partial z} = \frac{\partial F^N}{\partial z}{}^t\Omega_2 + F^N\frac{\partial\, {}^t\Omega_2}{\partial z}\,, \quad \frac{\partial\, {}^tE^N}{\partial z} = \frac{\partial \Omega_2}{\partial z}{}^tF^N + \Omega_2\frac{\partial\, {}^tF^N}{\partial z}\,. \tag{3.8}$$

By using (3.2), (3.7) and (3.8), we get
$$\mathrm{Im}\{\Omega_2\frac{\partial\, {}^tF^N}{\partial \bar{z}}\frac{\partial F^N}{\partial z}{}^t\Omega_2 + \Omega_2{}^tF^N\frac{\partial F^N}{\partial z}\frac{\partial\, {}^tF^N}{\partial \bar{z}}F^{N\,t}\Omega_2 + \frac{\partial \Omega_2}{\partial \bar{z}}{}^tF^N\frac{\partial F^N}{\partial z}{}^t\Omega_2$$

$$+\frac{\partial\Omega_2}{\partial z}{}^tF^N\frac{\partial F^N}{\partial\bar z}{}^t\Omega_2-\frac{\partial\Omega_2}{\partial\bar z}{}^t\Omega_2\frac{\partial\Omega_2}{\partial z}{}^t\Omega_2-\frac{\partial\Omega_2}{\partial z}{}^t\Omega_2\frac{\partial\Omega_2}{\partial\bar z}{}^t\Omega_2$$
$$+\Omega_2{}^tF^N\frac{\partial F^N}{\partial\bar z}{}^t\Omega_2\frac{\partial\Omega_2}{\partial z}{}^t\Omega_2+\Omega_2{}^tF^N\frac{\partial F^N}{\partial z}{}^t\Omega_2\frac{\partial\Omega_2}{\partial\bar z}{}^t\Omega_2\}=0$$

where we used

$$\frac{\partial\,{}^t\Omega_2}{\partial z}=-{}^t\Omega_2\frac{\partial\Omega_2}{\partial z}{}^t\Omega_2\,.$$

Hence we have

$$\mathrm{Im}\left\{\Omega_2\left(\frac{\partial\,{}^tF^N}{\partial\bar z}\frac{\partial F^N}{\partial z}+{}^tF^N\frac{\partial F^N}{\partial z}\frac{\partial\,{}^tF^N}{\partial\bar z}F^N\right){}^t\Omega_2\right\}=0\,.$$

Since Ω_2 is real, this shows that F^N satisfies the condition (I).

By using (3.8), we have

$$\begin{aligned}(I-E^{N\,t}E^N)\frac{\partial E^N}{\partial u_j}&=(I-F^{N\,t}F^N)\frac{\partial F^N}{\partial u_j}{}^t\Omega_2+(F^N-F^{N\,t}F^NF^N)\frac{\partial\,{}^t\Omega_2}{\partial u_j}\\&=(I-F^{N\,t}F^N)\frac{\partial F^N}{\partial u_j}{}^t\Omega_2\,,\end{aligned}$$

where we used ${}^tF^NF^N=I_{n-1}$. Hence the assumption of E^N implies

$$(I-F^{N\,t}F^N)\frac{\partial F^N}{\partial u_j}\neq 0$$

showing that F^N satisfies the condition (II).

From (3.8), we compute that

$$\begin{aligned}B&=\left(\frac{\partial\Omega_2}{\partial z}{}^tF^N+\Omega_2\frac{\partial\,{}^tF^N}{\partial z}\right)(I-F^{N\,t}F^N)\left(\frac{\partial F^N}{\partial z}{}^t\Omega_2+F^N\frac{\partial\,{}^t\Omega_2}{\partial z}\right)\\&=\frac{\partial\Omega_2}{\partial z}{}^tF^N(I-F^{N\,t}F^N)\frac{\partial F^N}{\partial z}{}^t\Omega_2+\frac{\partial\Omega_2}{\partial z}{}^tF^N(I-F^{N\,t}F^N)F^N\frac{\partial\,{}^t\Omega_2}{\partial z}\\&\quad+\Omega_2\frac{\partial\,{}^tF^N}{\partial z}(I-F^{N\,t}F^N)\frac{\partial F^N}{\partial z}{}^t\Omega_2+\Omega_2\frac{\partial\,{}^tF^N}{\partial z}(I-F^{N\,t}F^N)F^N\frac{\partial\,{}^t\Omega_2}{\partial z}\\&=\Omega_2\frac{\partial\,{}^tF^N}{\partial z}(I-F^{N\,t}F^N)\frac{\partial F^N}{\partial z}{}^t\Omega_2\\&=\Omega_2\tilde B{}^t\Omega_2\,,\end{aligned}$$

hence we have

$$B=\Omega_2\tilde B{}^t\Omega_2\,. \tag{3.9}$$

It follows from (3.4) that

$$ {}^t\xi B(m_0)\xi = {}^t\xi \Omega_2 \tilde{B}(m_0){}^t\Omega_2\xi = 0\,. $$

Since ${}^t\Omega_2\xi \in R^{n-1} \setminus \{0\}$, F^N satisfies the condition (III).

From (3.8) and (3.9) we have

$$ \frac{\partial\, {}^tE^N}{\partial z} E^N B + B^t E^N \frac{\partial E^N}{\partial z} - \frac{\partial B}{\partial z} = \Omega_2 \left(\frac{\partial\, {}^tF^N}{\partial z} F^N \tilde{B} + \tilde{B}^t F^N \frac{\partial F^N}{\partial z} - \frac{\partial \tilde{B}}{\partial z} \right) {}^t\Omega_2\,. $$

This shows that F^N satisfies the condition (IV). We complete the proof.

We note that the Gauss map of the Clifford torus in S^3 and the one of a sphere embedded totally umbilically in S^3 satisfy the conditions (I)–(IV). The other example will be given in Section 6.

For a C^∞-mapping $G : M \to Q_{n-1}$ satisfying the conditions (I)–(IV), we will show the existence of a surface in S^n whose Gauss map is G. Hoffman and Osserman showed that, under certain conditions (Theorem 2.3 in [2]), there exists a surface in R^n such that the Gauss map is G. However our result does not follow directly from their results.

THEOREM 3.2. *Let M be a connected Riemann surface and $G : M \to Q_{n-1}$ ($n \geq 3$) a C^∞-mapping. Let $(U, z = u_1 + \sqrt{-1}u_2)$ be a complex coordinate system about $m_0 \in M$. Assume that there exists a C^∞-mapping*

$$ E = (E^T, E^N) : U \to SO(n+1)\,, $$

where

$$ E^T = (E_1, E_2) : U \to St(n+1, 2),\ \ E^N = (E_3, \cdots, E_{n+1}) : U \to St(n+1, n-1)\,, $$

with the following properties:

(1) *$E^T(u)$ is an orthonormal frame in $\hat{G}(u)$ and gives the orientation of $\hat{G}(u)$ for any $u \in U$;*

(2) *E^N satisfies the conditions* (I)–(IV).

Then there exists a surface $S = (U_0, S^n, X)$ such that the Gauss map of S is $G|_{U_0}$ where U_0 is a simply connected open neighborhood of m_0.

PROOF. Let U_0 be a simply connected open neighborhood of m_0 such that $U_0 \subset U$. By (3.4), we take $\xi \in R^{n-1} \setminus \{0\}$ such that

$$ {}^t\xi B(m_0)\xi = 0\,. $$

From Lemma 2.1 and (3.2), by taking U_0 sufficiently small, the partial differential equation

$$ \frac{\partial A}{\partial z} = -{}^tE^N \frac{\partial E^N}{\partial z} A \tag{3.10} $$

has a unique C^∞ solution

$$ A : U_0 \to R^{n-1} \setminus \{0\} \quad (z \mapsto {}^t(a_3(z), \cdots, a_{n+1}(z))) $$

with the initial value $A(m_0) = \xi$. We now define a C^∞-mapping $Y : U \to R^{n+1}$ as

$$Y = \sum_{j=3}^{n+1} a_j E_j = E^N A\,. \tag{3.11}$$

It follows from (3.3), (3.10) and (3.11) that

$$\frac{\partial Y}{\partial z} = \frac{\partial E^N}{\partial z} A + E^N \frac{\partial A}{\partial z} = (I - E^N {}^t E^N) \frac{\partial E^N}{\partial z} A \neq 0\,. \tag{3.12}$$

We have

$$\left\langle \frac{\partial Y}{\partial z}, \frac{\partial Y}{\partial \bar{z}} \right\rangle = {}^t A \frac{\partial\, {}^t E^N}{\partial z} (I - E^N {}^t E^N) \frac{\partial E^N}{\partial z} A = {}^t A B A\,. \tag{3.13}$$

Then (3.5) and the initial condition of (3.10) imply ${}^t ABA = 0$. Hence (3.13) shows that Y is conformal. From (3.12), we have

$${}^t E^N \frac{\partial Y}{\partial z} = ({}^t E^N - {}^t E^N E^N {}^t E^N) \frac{\partial E^N}{\partial z} A = 0$$

which shows that $G|_{U_0}$ is the Gauss map of the surface $S_Y = (U_0, R^{n+1}, Y)$. It follows from (3.11) and (3.12) that

$$\left\langle Y, \frac{\partial Y}{\partial \bar{z}} \right\rangle = \left\langle E^N A,\ (I - E^N {}^t E^N) \frac{\partial E^N}{\partial \bar{z}} A \right\rangle = \frac{\partial\, {}^t E^N}{\partial z} (I - E^N {}^t E^N) E^N A = 0\,.$$

Hence the length of Y is constant on U_0 since U_0 is connected. We now define a C^∞- conformal immersion $X : U_0 \to S^n$ as

$$X = \frac{1}{|Y|} Y\,.$$

Since S_Y and $S = (U_0, S^n, X)$ have the same Gauss map, $G|_{U_0}$ is the Gauss map of S. This completes the proof.

4. Surfaces in S^n with the same Gauss map

Hoffman and Osserman showed in [2] the following:
Let $S_1 = (M, R^{n+1}, X)$ and $S_2 = (M, R^{n+1}, Y)$ be distinct surfaces in R^{n+1} with the same Gauss map G. If their mean curvature vectors are different from zero at some point, then $Y = cX + X_0$ where c is a non-zero constant and X_0 is a constant vector (Theorem 2.5 in [2]).

By using this result, we investigate relations between surfaces in S^n with the same Gauss map.

For $r_1 > 0$ and $X_0 \in R^{n+1} \setminus \{0\}$, let $S^{n-1}_{r_1}(X_0)$ denote the $(n-1)$-dimensional Euclidean sphere with radius r_1 and center X_0 such that it is the intersection of S^n and the hyperplane in R^{n+1} which is orthogonal to X_0 and passes through it.

Under the notations stated above, we shall show the following.

THEOREM 4.1. *Let M be a connected Riemann surface and $G : M \to Q_{n-1}$ a C^∞-mapping. Suppose that $S_1 = (M, S^n, X)$ and $S_2 = (M, S^n, Y)$ are distinct surfaces in S^n with the same Gauss map G. Then we have the following*:

(1) *$X \cdot X_0$ and $Y \cdot X_0$ are constant in M where X_0 is a constant vector such that*

$$Y = cX + X_0 .$$

Here c is a non-zero constant such that

$$c = -X \cdot X_0 + \varepsilon\sqrt{(X \cdot X_0)^2 + 1 - |X_0|^2}\,, \quad \varepsilon = \pm 1 .$$

(2) *If $X_0 = 0$, then $Y = -X$.*

(3) *If $X_0 \neq 0$, then*

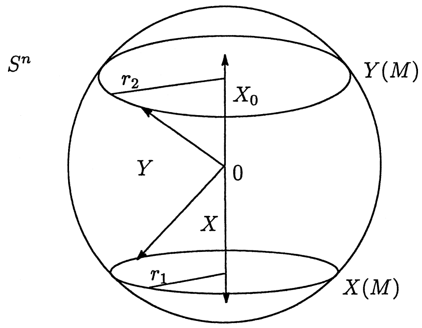
$$X(M) \subset S^{n-1}_{r_1}\left(\frac{\alpha}{|X_0|^2}X_0\right), \quad Y(M) \subset S^{n-1}_{r_2}\left(\frac{\beta}{|X_0|^2}X_0\right)$$

and

$$|c|r_1 = r_2$$

where

$$\alpha = X \cdot X_0\,, \quad \beta = Y \cdot X_0\,, \quad r_1 = \sqrt{1 - \left(\frac{\alpha}{|X_0|}\right)^2}\,, \quad r_2 = \sqrt{1 - \left(\frac{\beta}{|X_0|}\right)^2}\,.$$



PROOF. If we regard $S_1 = (M, S^n, X)$ and $S_2 = (M, S^n, Y)$ as surfaces in R^{n+1}, their mean curvature vector fields in R^{n+1} nowhere vanish. Then, by a theorem due to Hoffman and Osserman (Theorem 2.5 in [2]), X and Y satisfy $Y = cX + X_0$ where c is a constant $(c \neq 0)$ and X_0 is a constant vector in R^{n+1}. In the case where $X_0 = 0$, (1) is evident. To show (1) we may assume $X_0 \neq 0$. We have

$$|Y|^2 = |cX + X_0|^2 = c^2 + 2cX \cdot X_0 + |X_0|^2$$

where we used $|X| = 1$. Since $|Y| = 1$, we have

$$c^2 + 2cX \cdot X_0 + |X_0|^2 - 1 = 0\,. \tag{4.1}$$

Since c is real, we have

$$D := (X \cdot X_0)^2 - |X_0|^2 + 1 \geq 0 \tag{4.2}$$

and get

$$c = -X \cdot X_0 + \varepsilon\sqrt{D} \tag{4.3}$$

where $\varepsilon = \pm 1$. If $|X_0| = 1$, we have

$$X \cdot X_0 = -\frac{c}{2}$$

on M.

From now on we assume that $|X_0| \neq 1$. If the equality holds in (4.2), then we have $X \cdot X_0 = -c$. We now set

$$M_0 = \{\, m \in M \mid X(m) \cdot X_0 \neq -c \,\}\,, \quad M_1 = \{\, m \in M \mid X(m) \cdot X_0 = -c \,\}\,.$$

We shall show that either $M = M_0$ or $M = M_1$ holds. Assume $M_0 \neq \emptyset$. Then M_0 is open in M. Let $m_0 \in M_0$. Let (U, z) be a complex coordinate system about m_0 such that U is connected and $U \subset M_0$. Because of $D > 0$ in M_0, by taking U sufficiently small if necessary, we may assume that

$$\sqrt{D} + \varepsilon(X \cdot X_0) \neq 0$$

holds in U. By differentiating the both sides of the equation (4.3), we get

$$0 = -(X_z \cdot X_0)\left\{\frac{1 - |X_0|^2}{\sqrt{D}(\sqrt{D} + \varepsilon(X \cdot X_0))}\right\}.$$

Since $|X_0| \neq 1$, we have $X_z \cdot X_0 = 0$. Hence $X \cdot X_0$ is constant in U. Thus the function $f := X \cdot X_0$ on M is locally constant in M_0. Take $m_1 \in M_0$ and let $f(m_1) = \alpha$. We set

$$M_2 = \{m \in M_0 \mid f(m) = \alpha\}\,.$$

Since f is locally constant in M_0, we can show that M_2 is closed and open in M. By the connectedness of M it must be $M = M_2$, which implies $M = M_0$. Then we have $X \cdot X_0 = \alpha$ on M where α is constant. Next, we suppose $M_1 \neq \emptyset$. Since f is constant in M_1, we can show that M_1 is closed and open in M. By the connectedness of M, it must be $M = M_1$. Hence we have $X \cdot X_0 = -c$ on M. Then both cases imply that $X \cdot X_0$ is constant on M. Similarly, $Y \cdot X_0$ is constant on M. This proves (1).

If $X_0 = 0$, we have $Y = \varepsilon X$, which proves $Y = -X$, by $X \neq Y$, showing (2).

To show (3), we set

$$r_1 = \sqrt{1 - \cos^2\theta_1}\,, \quad r_2 = \sqrt{1 - \cos^2\theta_2}\,, \tag{4.4}$$

where θ_1 is the angle formed by X and X_0 and θ_2 is the one formed by Y and X_0 respectively. It follows from (1) that $X(M)$ and $Y(M)$ are contained in certain $(n-1)$-dimensional spheres in S^n respectively. We get

$$Y \cdot X_0 = (cX + X_0) \cdot X_0 = cX \cdot X_0 + |X_0|^2 .$$

This is equivalent to

$$|Y||X_0| \cos\theta_2 = c|X||X_0| \cos\theta_1 + |X_0|^2 \tag{4.5}$$

which yields

$$\cos\theta_2 = c\cos\theta_1 + |X_0| . \tag{4.6}$$

From (4.4) and (4.6), we get

$$1 - {r_2}^2 = c^2(1 - {r_1}^2) + 2c|X_0|\cos\theta_1 + |X_0|^2 .$$

By (4.1), we obtain $c^2{r_1}^2 = {r_2}^2$. It follows from (1) that

$$\cos\theta_1 = \frac{\alpha}{|X_0|}$$

where $\alpha = X \cdot X_0$ is constant. Hence the center of the $(n-1)$-dimensional sphere in S^n containing $X(M)$ is given by

$$\cos\theta_1 \frac{X_0}{|X_0|} = \frac{\alpha}{|X_0|^2} X_0 .$$

Similarly the center of the $(n-1)$-dimensional sphere in S^n containing $Y(M)$ is given by

$$\cos\theta_2 \frac{X_0}{|X_0|} = \frac{\beta}{|X_0|^2} X_0$$

where $\beta = Y \cdot X_0$ is constant. Then we see that

$$X(M) \subset S^{n-1}_{r_1}\left(\frac{\alpha}{|X_0|^2} X_0\right), \quad Y(M) \subset S^{n-1}_{r_2}\left(\frac{\beta}{|X_0|^2} X_0\right),$$

where

$$r_1 = \sqrt{1 - \left(\frac{\alpha}{|X_0|}\right)^2}, \quad r_2 = \sqrt{1 - \left(\frac{\beta}{|X_0|}\right)^2} .$$

This completes the proof.

Let M be a connected Riemann surface and $G : M \to Q_{n-1}$ $(n \geq 3)$ a C^∞- mapping. For G, let $\hat{G}$, $P(M, G)$, V, $V^\perp$ and k be as defined in Section 2. For $p \in V^\perp \setminus \{0\}$, let V^n_p be the hyperplane containing the origin in R^{n+1} which is orthogonal to p.

Under the notations stated above, we shall show the following.

LEMMA 4.2. *Let $p \in S^n$. Assume that ${}^tE^T p = 0$ on M. Then we have $p \in V^\perp \cap S^n$.*

PROOF. We set $p = p_1 + p_2$ where $p_1 \in V$ and $p_2 \in V^\perp$. Suppose $p_1 \neq 0$ for contradiction. Since ${}^tE^T p_2 = 0$, we have ${}^tE^T p_1 = 0$. If we express V as the direct sum

$$V = V_1 \oplus \{ap_1 \mid a \in R\},$$

then we have $P(M, G) \subset V_1$. This contradicts the definition of V. Hence it must be $p_1 = 0$, which implies $p \in V^\perp \cap S^n$.

From now on, let (M, S^n, X) be a surface in S^n with the Gauss map G. For $c, t \in R$ and $p \in S^n$, we define a C^∞-conformal immersion $Y_t : M \to R^{n+1}$ by

$$Y_t = cX + tp.$$

Under this notation, we shall show the following lemmas.

LEMMA 4.3. *Let $3 \leq k \leq n$. For any $p \in V^\perp \cap S^n$ the following holds*:

(1) $$X \cdot p = \alpha$$

where α is constant.

(2) *For a constant c such that*

$$c = -\alpha t + \varepsilon\sqrt{(\alpha t)^2 + 1 - t^2}, \quad |t| < 1, \quad \varepsilon = \pm 1,$$

we have $Y_t(M) \subset S^n$.

(3) *For a constant $c = -2\alpha t$ ($|t| = 1$) we have $Y_t(M) \subset S^n$.*

PROOF. Let U be a complex coordinate neighborhood with a coordinate function $z = u_1 + \sqrt{-1}u_2$. We have, by (2.3),

$$\frac{\partial X}{\partial z} = E^T \Psi$$

where

$$\Psi : U \to C^2 \setminus \{0\} \quad (u \mapsto {}^t(\psi(u), -\sqrt{-1}\psi(u))).$$

Since $p \in V^\perp \cap S^n$, we get

$$\frac{\partial}{\partial z}(X \cdot p) = \frac{\partial X}{\partial z} \cdot p = {}^t\Psi {}^tE^T p = 0$$

which implies that $X \cdot p$ is constant on U. By the connectedness of M, (1) holds.

Let

$$c = -\alpha t + \varepsilon\sqrt{(\alpha t)^2 + 1 - t^2}$$

where $\alpha = X \cdot p$ and $|t| < 1$. Then we have $Y_t(M) \subset S^n$, showing (2).

When $|t| = 1$ and $c = -2\alpha t$, we get $|Y_t|^2 = 1$, which shows (3). We complete the proof.

REMARK. We note that the converses of (2), (3) in Lemma 4.3 also hold.

Let $p \in V^{\perp} \cap S^n$. We will denote by S_p^{n-1} the great sphere in S^n which is the intersection of S^n and V_p^n. A C^∞-conformal immersion $Y_t : M \to S^n$ in (2) of Lemma 4.3 gives a surface $S_t = (M, S^n, Y_t)$ whose Gauss map coincides with the one of $S = (M, S^n, X)$. We can choose t_0 ($|t_0| < 1$) such that S_{t_0} is contained in S_p^{n-1}. In the following we denote by $\hat{X}$ instead of Y_{t_0}.

LEMMA 4.4. *If* $3 \leq k \leq n$, *we have*

$$\hat{X}(M) \subset \left(\bigcap_{q \in V^{\perp} \cap S^n} V_q^n \right) \cap S^n .$$

PROOF. Let $q \in V^{\perp} \cap S^n$. We put

$$\hat{Y}_t = c\hat{X} + tq$$

where

$$|t| < 1, \quad c = -t(\hat{X} \cdot q) + \varepsilon \sqrt{t^2(\hat{X} \cdot q)^2 + 1 - t^2}, \quad \varepsilon = \pm 1 .$$

By Theorem 4.1 and Lemma 4.3 each surface $\hat{S}_t = (M, S^n, \hat{Y}_t)$ ($t \neq 0$) is contained in some hyperplane in R^{n+1} which is orthogonal to q. Therefore, we have $\hat{Y}_0(M) \subset V_q^n \cap S^n$, hence $\hat{X}(M) \subset V_q^n \cap S^n$. Since q is an arbitrary point in $V^{\perp} \cap S^n$, we complete the proof.

LEMMA 4.5. *If* $3 \leq k \leq n$, *then the following holds*:

$$V = \bigcap_{q \in V^{\perp} \cap S^n} V_q^n . \tag{1}$$

$$\hat{X}(M) \subset V \cap S^n . \tag{2}$$

PROOF. Let $v \in V$ and $q \in V^{\perp} \cap S^n$. Since $v \cdot q = 0$, we have $v \in V_q^n$. This implies

$$V \subset \bigcap_{q \in V^{\perp} \cap S^n} V_q^n .$$

The other hand let $v \in \bigcap_{q \in V^{\perp} \cap S^n} V_q^n$. For each $q \in V^{\perp} \cap S^n$, we have $v \cdot q = 0$, which implies $v \in V$. This shows (1). (2) follows from Lemma 4.4 and the above. This completes the proof.

Let (M, S^n, X) be a surface whose Gauss map is G. The position of surfaces (M, S^n, Y) with the same Gauss map G depends on k, because $X(M),\ Y(M) \subset S^n$. By using the lemmas showed above, in case of $3 \leq k \leq n$, we will show that there exist many surfaces in S^n with the same Gauss map. Such surfaces are contained in the intersection of S^n and a k-dimensional plane in R^{n+1} which is orthogonal to a vector.

THEOREM 4.6. *Let M be a connected Riemann surface and $G : M \to Q_{n-1}$ a C^∞-mapping. Let V and $V^\perp$ be as defined above. Let $S = (M, S^n, X)$ be a surface in S^n such that the Gauss map is G. If $3 \leq k \leq n$, then the following holds*:

(1) *There exists a surface $\hat{S} = (M, S^n, \hat{X})$ in S^n such that the Gauss map coincides with G and*

$$\hat{X}(M) \subset V \cap S^n .$$

(2) *If the Gauss map of a surface $S_Y = (M, S^n, Y)$ in S^n is G, then Y can be expressed as*

$$Y = c\hat{X} + tq$$

where c and t are constants such that

$$c = \varepsilon\sqrt{1 - t^2}\,, \quad \varepsilon = \pm 1\,, \quad |t| < 1\,,$$

and $q \in V^\perp \cap S^n$.

PROOF. (1) follows from Lemmas 4.3 and 4.4. By (1) and the result of Hoffman and Osserman (Theorem 2.5 in [2]), Y can be expressed as $Y = c\hat{X} + X_0$ ($c \neq 0$) where $X_0 \in R^{n+1}$ and c is a constant such that

$$c = -\alpha + \varepsilon\sqrt{\alpha^2 + 1 - |X_0|^2}\,, \quad \alpha = \hat{X} \cdot X_0\,, \quad \varepsilon = \pm 1\,. \tag{4.7}$$

Here we note that α is a constant by (1) of Theorem 4.1. If $X_0 = 0$, then $Y = \varepsilon\hat{X}$. In the case where $X_0 \neq 0$, we set

$$X_0 = tq\,, \quad q \in S^n\,, \quad t \in R \setminus \{0\}\,.$$

Since $\hat{X}$ is conformal, we have

$$\frac{\partial \hat{X}}{\partial z} = E^T \hat{\Psi}$$

where

$$\hat{\Psi} : U \to C^2 \setminus \{0\} \quad (u \mapsto (\hat{\psi}(u), -\sqrt{-1}\hat{\psi}(u)))\,.$$

We get

$$0 = \frac{\partial \hat{X}}{\partial z} \cdot X_0 = {}^t\hat{\Psi}\,{}^tE^T X_0 = t\,{}^t\hat{\Psi}\,{}^tE^T q\,,$$

because $\hat{X} \cdot X_0$ is constant. This implies that ${}^tE^T q = 0$. From Lemma 4.2 we have $q \in V^\perp \cap S^n$. Since $\alpha = 0$ and $c \neq 0$ in (4.7), we have $c = \varepsilon\sqrt{1 - t^2}$, $|t| < 1$. This completes the proof.

In case of $k = n + 1$, we shall show in the following that there exist at most two surfaces in S^n with the same Gauss map.

THEOREM 4.7. *Let M, G and $S = (M, S^n, X)$ be as in Theorem 4.6. Let $S_Y = (M, S^n, Y)$ be a surface in S^n such that the Gauss map is G. If $k = n+1$, then $Y = \pm X$.*

PROOF. By Theorem 2.5 in [2], Y is expressed as $Y = cX + X_0$ where c is a constant and X_0 is a constant vector in R^{n+1}. If $X_0 \neq 0$, then (3) of Theorem 4.1 holds. This shows $k \leq n$, which contradicts the assumption that $k = n+1$. Then we get $Y = \pm X$.

5. Global existence theorem

The purpose of this section is to show the existence of a surface with prescribed Gauss map. Let M be a connected Riemann surface and $G : M \to Q_{n-1}$ $(n \geq 3)$ a C^∞-mapping. For G, let $\hat{G}$, $P(M, G)$, V, $V^\perp$ and k be as defined in Section 2. If M is the Gaussian plane C or the Riemann sphere S^2, by using the monodromy theorem, we can show the following.

PROPOSITION 5.1. *Let M be the Gaussian plane C or the Riemann sphere S^2. Let $G : M \to Q_{n-1}$ $(n \geq 3)$ be a C^∞-mapping such that $k \geq 3$. Under the notations stated above, assume that for each $z_0 \in M$ there exist a connected open neighborhood U of z_0 and a C^∞-mapping*

$$E = (E^T, E^N) : U \to SO(n+1)\,,$$

where

$$E^T = (E_1, E_2) : U \to St(n+1, 2)\,,\;\; E^N = (E_3, \cdots, E_{n+1}) : U \to St(n+1, n-1)\,,$$

with the following properties:

(1) *$E^T(u)$ is an orthonormal frame in $\hat{G}(u)$ and gives the orientation of $\hat{G}(u)$ for any $u \in U$;*

(2) *E^N satisfies the conditions* (I)–(IV).

Then there exists a surface $S = (M, S^n, X)$ such that the Gauss map is G.

We consider the case where M is a torus T^2. From now on we denote by N and Z the set of all natural numbers and integers respectively. Let $a_1, a_2 > 0$. Let Γ be the transformation group on C generated by translations

$$\varphi_1(z) = u_1 + a_1 + \sqrt{-1}u_2,\;\; \varphi_2(z) = u_1 + \sqrt{-1}(u_2 + a_2)\;(z = u_1 + \sqrt{-1}u_2 \in C)\,.$$

In the following we consider the torus $T^2 = C/\Gamma$. For $k_1,\ k_2 \in N$, let $\Gamma(k_1a_1, k_2a_2)$ denote the subgroup of Γ generating by ${\varphi_1}^{k_1}$ and ${\varphi_2}^{k_2}$ where

$${\varphi_1}^{k_1}(z) = u_1 + k_1a_1 + \sqrt{-1}u_2\,, \quad {\varphi_2}^{k_2}(z) = u_1 + \sqrt{-1}(u_2 + k_2a_2)\,.$$

Here $\Gamma = \Gamma(a_1, a_2)$. We denote by $T^2(k_1a_1, k_2a_2)$ the torus $C/\Gamma(k_1a_1, k_2a_2)$. For each $z \in C$, there exist $l_1,\ l_2 \in Z$ such that

$$z = (t_1 + l_1)k_1a_1 + \sqrt{-1}(t_2 + l_2)k_2a_2$$

where $0 \leq t_1,\ t_2 < 1$. We note that $T^2 = T^2(a_1, a_2)$. Then we have the projection $\pi_{k_1k_2} : C \to T^2(k_1a_1, k_2a_2)$ such that

$$\pi_{k_1k_2}(z) = \pi_{k_1k_2}((t_1+l_1)k_1a_1 + \sqrt{-1}(t_2+l_2)k_2a_2) = \pi_{k_1k_2}(t_1k_1a_1 + \sqrt{-1}t_2k_2a_2)\,.$$

Put $[z] = \pi_{k_1k_2}(z)$ $(z \in C)$.

THEOREM 5.2. *Let $G : T^2 \to Q_{n-1}$ be a C^∞-mapping such that $k \geq 3$. Assume that for each $z_0 \in T^2$ there exist a complex coordinate system (U, z) about z_0 and a C^∞-mapping*

$$E = (E^T, E^N) : U \to SO(n+1)\,,$$

where

$$E^T = (E_1, E_2) : U \to St(n+1, 2)\,,\ \ E^N = (E_3, \cdots, E_{n+1}) : U \to St(n+1, n-1)\,,$$

with the following properties:

(1) *$E^T(u)$ is an orthonormal frame in $\hat{G}(u)$ and gives the orientation of $\hat{G}(u)$ for any $u \in U$*;

(2) *E^N satisfies the conditions* (I)–(IV).

Then there exist a covering space $(\hat{T}^2, T^2, \hat{\pi})$ over T^2 and a surface $S = (\hat{T}^2, S^n, X)$ such that the Gauss map of S is $G \circ \hat{\pi}$.

To prove Theorem 5.2, we shall show some lemmas.

Let $G : T^2 \to Q_{n-1}$ be a C^∞-mapping. For $k_1,\ k_2 \in N$, we define C^∞-mappings

$$\tilde{\pi}_{k_1k_2} : T^2(k_1a_1, k_2a_2) \to T^2, \quad G_{k_1k_2} : T^2(k_1a_1, k_2a_2) \to Q_{n-1}$$

as

$$\tilde{\pi}_{k_1k_2}([z]) = \pi_{11}(z) \quad (z \in C)\,, \quad G_{k_1k_2} = G \circ \tilde{\pi}_{k_1k_2}$$

respectively. Let $\tilde{G} : C \to Q_{n-1}$ be the C^∞-mapping defined by $G \circ \pi_{11}$. This mapping is Γ-invariant. $\tilde{G}$ satisfies the conditions in Proposition 5.1, since $\tilde{\pi}_{k_1k_2}$ is holomorphic. Then there exists a surface $\tilde{S}_1 = (C, S^n, \tilde{X}_1)$ such that the Gauss map is $\tilde{G}$. We define conformal immersions $\tilde{X}_j : C \to S^n$ $(j = 2, 3)$ by $\tilde{X}_2 = \tilde{X}_1 \circ \varphi_1$ and $\tilde{X}_3 = \tilde{X}_1 \circ \varphi_2$ and let $\tilde{S}_j = (C, S^n, \tilde{X}_j)$ be surfaces such that the Gauss map is $\tilde{G}$.

We first consider the case where $k = n+1$. Since $\tilde{S}_1,\ \tilde{S}_2$ and $\tilde{S}_3$ have the same Gauss map $\tilde{G}$, by Theorem 4.6 we have

$$\tilde{X}_1 = \pm\tilde{X}_1 \circ \varphi_1 = \pm\tilde{X}_1 \circ \varphi_2\,.$$

Then the following four cases are possible:

(1) $\tilde{X}_1 = \tilde{X}_1 \circ \varphi_1 = \tilde{X}_1 \circ \varphi_2$, (2) $\tilde{X}_1 = -\tilde{X}_1 \circ \varphi_1 = \tilde{X}_1 \circ \varphi_2$, (3) $\tilde{X}_1 = \tilde{X}_1 \circ \varphi_1 = -\tilde{X}_1 \circ \varphi_2$, (4) $\tilde{X}_1 = -\tilde{X}_1 \circ \varphi_1 = -\tilde{X}_1 \circ \varphi_2$.

For each case we show that Theorem 5.2 holds.

LEMMA 5.3. *If $k = n+1$ and the case* (1) *holds, then there exists a surface $S = (T^2, S^n, X)$ such that the Gauss map is G.*

PROOF. For any $\varphi \in \Gamma$, we have $\tilde{X}_1 = \tilde{X}_1 \circ \varphi$, so define a C^∞-conformal immersion $X : T^2 \to S^n$ as

$$X([z]) = \tilde{X}_1(z) \quad (z \in C)\,.$$

Then we obtain a surface $S = (T^2, S^n, X)$ such that the Gauss map is G. We complete the proof.

LEMMA 5.4. *If $k = n+1$ and the case* (2) *holds, then there exist a covering space $(\hat{T}^2, T^2, \hat{\pi})$ over T^2 and a surface $S = (\hat{T}^2, S^n, X)$ with the Gauss map $G \circ \hat{\pi}$ where $\hat{T}^2$ is $T^2(2a_1, a_2)$.*

PROOF. We define a C^∞-conformal immersion $\tilde{X}_4 : C \to S^n$ by $\tilde{X}_4 = \tilde{X}_1 \circ {\varphi_1}^2$. Since surfaces $\tilde{S}_1$ and $\tilde{S}_4 = (C, S^n, \tilde{X}_4)$ have the same Gauss map $\tilde{G}$, by Theorem 4.6 we have $\tilde{X}_1 = \pm \tilde{X}_1 \circ {\varphi_1}^2$. Then we have the two cases: (i) $\tilde{X}_1 = \tilde{X}_1 \circ {\varphi_1}^2$, (ii) $\tilde{X}_1 = -\tilde{X}_1 \circ {\varphi_1}^2$.

Case (i). We have a C^∞-conformal immersion $X_{21} : T^2(2a_1, a_2) \to S^n$ such that

$$X_{21}([z]) = \tilde{X}_1(z) \quad (z \in C)\,.$$

Then there exists a surface $S_{21} = (T^2(2a_1, a_2), S^n, X_{21})$ with the Gauss map G_{21}.

Case (ii). Since $-\tilde{X}_1 \circ {\varphi_1}^2 = -\tilde{X}_1 \circ \varphi_1$, we have $\tilde{X}_1 = \tilde{X}_1 \circ \varphi_1$. This contradicts the assumption. Then Case (ii) does not occur. We complete the proof.

By using the same argument as in the proof of Lemma 5.4, we have the following lemmas.

LEMMA 5.5. *If $k = n+1$ and the case* (3) *holds, then there exist a covering space $(\hat{T}^2, T^2, \hat{\pi})$ over T^2 and a surface $S = (\hat{T}^2, S^n, X)$ with the Gauss map $G \circ \hat{\pi}$ where $\hat{T}^2$ is $T^2(a_1, 2a_2)$.*

LEMMA 5.6. *If $k = n+1$ and the case* (4) *holds, then there exist a covering space $(\hat{T}^2, T^2, \hat{\pi})$ over T^2 and a surface $S = (\hat{T}^2, S^n, X)$ with the Gauss map $G \circ \hat{\pi}$ where $\hat{T}^2$ is $T^2(2a_1, 2a_2)$.*

LEMMA 5.7. *Under the assumption of Theorem* 5.2, *if $3 \le k \le n$, then there exist a covering space $(\hat{T}^2, T^2, \hat{\pi})$ over T^2 and a surface $S = (\hat{T}^2, V \cap S^n, X)$ such that the Gauss map is $G \circ \hat{\pi}$.*

PROOF. For each $z_0 \in T^2$ it follows from Theorem 3.2 that there exists a surface $S_0 = (U_0, S^n, \Psi_0)$ with the Gauss map $G|_{U_0}$ where U_0 is a simply connected open neighborhood of z_0.

Let $z_0 \in T^2$ and let $S_1 = (U_1, S^n, \Psi_1)$ be a surface as stated above. By the assumption that $3 \le k \le n$ and by (1) of Theorem 4.6, there exists a surface $\hat{S}_1 = (U_1, S^n, \hat{\Psi}_1)$ with the

Gauss map $G|_{U_1}$ and

$$\hat{\Psi}_1(U_1) \subset V \cap S^n .$$

Since dim $V = k$, we can apply Lemmas 5.3, 5.4, 5.5 and 5.6 to the $(k-1)$-dimensional unit sphere $V \cap S^n$. Hence there exist a covering space $(\hat{T}^2, T^2, \hat{\pi})$ over T^2 and a surface $S = (\hat{T}^2, V \cap S^n, X)$ such that the Gauss map is $G \circ \hat{\pi}$. We complete the proof.

By a similar way as in the proof of Theorem 5.2, we have the following.

THEOREM 5.8. *Let M be a cylinder $S^1 \times R$ and $G : M \to Q_{n-1}$ a C^∞-mapping such that $k \geq 3$. Assume that for each $z_0 \in M$ there exist a complex coordinate system (U, z) about z_0 and a C^∞-mapping*

$$E = (E^T, E^N) : U \to SO(n+1),$$

where

$$E^T = (E_1, E_2) : U \to St(n+1, 2), \; E^N = (E_3, \cdots, E_{n+1}) : U \to St(n+1, n-1),$$

with the following properties

(1) *$E^T(u)$ is an orthonormal frame in $\hat{G}(u)$ and gives the orientation of $\hat{G}(u)$ for any $u \in U$;*

(2) *E^N satisfies the conditions* (I)–(IV).

Then there exist a covering space $(\hat{M}, M, \hat{\pi})$ over M and a surface $S = (\hat{M}, S^n, X)$ such that the Gauss map is $G \circ \hat{\pi}$.

6. Surfaces in the real projective space

In the following, let V and k be as in Section 5. We denote by π the natural projection from S^n to the n-dimensional real projective space RP^n.

THEOREM 6.1. *Let M be a connected Riemann surface and $G : M \to Q_{n-1}$ a C^∞-mapping such that $k \geq 3$. Assume that for each $m_0 \in M$ there exist a complex coordinate system (U, z) about m_0 and a C^∞-mapping*

$$E = (E^T, E^N) : U \to SO(n+1),$$

where

$$E^T = (E_1, E_2) : U \to St(n+1, 2), \; E^N = (E_3, \cdots, E_{n+1}) : U \to St(n+1, n-1),$$

with the following properties:

(1) *$E^T(u)$ is an orthonormal frame in $\hat{G}(u)$ and gives the orientation of $\hat{G}(u)$ for any $u \in U$;*

(2) *E^N satisfies the conditions* (I)–(IV).

Then there exists a surface $S = (M, RP^n, X)$ with the property that a neighborhood of each point of $X(M)$ is covered by a surface in S^n with the Gauss map G.

This theorem follows from the lemmas below.

LEMMA 6.2. *Under the assumption of Theorem* 6.1, *if* $k = n + 1$, *then there exists a surface* $S = (M, RP^n, X)$ *with the property that a neighborhood of each point of* $X(M)$ *is covered by a surface in* S^n *with the Gauss map* G.

PROOF. By Theorem 3.2, for each $m_0 \in M$ there exists a surface $\hat{S}_0 = (U_0, S^n, \hat{\Psi}_0)$ with the Gauss map $G|_{U_0}$ where U_0 is a simply connected open neighborhood of m_0. We put $S_0 = (U_0, RP^n, \Psi_0)$ where $\Psi_0 = \pi \circ \hat{\Psi}_0$. Then S_0 has the property as stated in the lemma. Let $\hat{S}_1 = (U_1, S^n, \hat{\Psi}_1)$ and $\hat{S}_2 = (U_2, S^n, \hat{\Psi}_2)$ be such surfaces. Then we have surfaces $S_1 = (U_1, RP^n, \Psi_1)$ and $S_2 = (U_2, RP^n, \Psi_2)$ such that $\Psi_1 = \pi \circ \hat{\Psi}_1$ and $\Psi_2 = \pi \circ \hat{\Psi}_2$. We shall show that if $W = U_1 \cap U_2 \neq \emptyset$, there exists a surface $S_3 = (U_3, RP^n, \Psi_3)$, $U_3 = U_1 \cup U_2$, with the property as stated in the lemma and

$$\Psi_3|_{U_j} = \Psi_j \quad (j = 1, 2)\,.$$

Let $U_3 = U_1 \cup U_2$ and $W = U_1 \cap U_2 \neq \emptyset$. From Theorem 4.7, on each connected component W_0 of W, we have

$$\hat{\Psi}_1|_{W_0} = \pm\hat{\Psi}_2|_{W_0}\,.$$

Then we can define a C^∞-conformal immersion $\Psi_3 : U_3 \to RP^n$ as

$$\Psi_3|_{U_j} = \Psi_j \quad (j = 1, 2)\,.$$

Then $S_3 = (U_3, RP^n, \Psi_3)$ is a desired surface.

We now suppose that M is compact. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of M such that for each $\lambda \in \Lambda$ U_λ is simply connected and there exists a surface $(U_\lambda, RP^n, \Psi_\lambda)$ with the property stated above. Since M is compact, we can choose a finitely many of open sets from $\{U_\lambda\}_{\lambda \in \Lambda}$ so that it is covered by these open sets. By using the argument showed above, we can get a surface $S = (M, RP^n, X)$ with the property stated in the lemma.

In the case where M is non-compact, we choose a sequence $\{K_j\}$ of connected open subsets of M such that $\bar{K}_j$ is compact and

$$M = \bigcup_{j=1}^{\infty} K_j\,, \qquad \bar{K}_j \subset K_{j+1}\,.$$

From what we have shown above, for each j there exists a surface $S_j = (\bar{K}_j, RP^n, \Psi_j)$ with the property stated above and

$$\Psi_{j+1}|_{K_j} = \Psi_j \quad (j = 1, 2, \cdots)\,.$$

Hence we have a desired surface $S = (M, RP^n, X)$ where

$$X|_{K_j} = \Psi_j \quad (j = 1, 2 \cdots)\,.$$

We complete the proof.

By using the same argument as in the proof of Lemma 5.5, Lemma 6.2 implies the following.

LEMMA 6.3. *Under the assumption of Theorem* 6.1, *if* $3 \leq k \leq n$, *then there exists a surface* $S = (M, RP^n, X)$ *with the property that a neighborhood of each point of* $X(M)$ *is covered by a surface in* $V \cap S^n$ *with the Gauss map* G.

EXAMPLE 6.4. We will show an example of a surface such that its Gauss map satisfies the conditions in Theorem 6.1. Let Γ be the transformation group on C generated by translations

$$\varphi_1(z) = u_1 + 2\pi + \sqrt{-1}u_2\,, \quad \varphi_2(z) = u_1 + \sqrt{-1}(u_2 + 2\pi)$$

where $z = u_1 + \sqrt{-1}u_2 \in C$. We define a torus T^2 as C/Γ and a C^∞-conformal immersion $X : T^2 \to S^7$ as

$$X(z) = \frac{1}{2}{}^t(\cos u_1,\ \sin u_1,\ \cos u_2,\ \sin u_2,\ \cos u_1,\ \sin u_1,\ \cos u_2,\ \sin u_2)\,.$$

The Gauss map of the surface $S = (T^2, S^7, X)$ satisfies the conditions in Theorem 6.1.

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