TOKYO J. MATH.
VOL. 29, NO. 1, 2006

The Dirac Operator on Ultrahyperbolic Manifolds

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(Communicated by K. Shinoda)

Abstract. In this paper we consider a projective model for the time- and spacelike ultrahyperbolic unit balls in the orthogonal space $\mathbf{R}^{m,m}$. By means of an associated principal fibre bundle, a Dirac operator on these manifolds is defined and its fundamental solution is constructed (in case $m \in 2\mathbf{N}+1$) with the aid of generalized Riesz distributions. Using the method of descent, we then construct fundamental solutions for the Dirac operator on time- or spacelike ultrahyperbolic unit balls in spaces of signature (m, q) and (p, m) respectively (with $p, q < m$).

1. Introduction

In this paper Clifford analysis techniques are used to construct a fundamental solution for the Dirac operator on specific $\mathrm{SO}(p, q)$-invariant ultrahyperbolic manifolds in $\mathbf{R}^{p,q}$.

Clifford analysis offers a nice and elegant generalization to higher dimension of the theory of holomorphic functions in the complex plane, the Dirac operator being the higher dimensional analogue of the Cauchy-Riemann operator. It is centred around the notion of monogenic functions, i.e. nullsolutions for the Dirac operator. Standard reference books are [1], [7] and [14] and a nice overview of the most basic results can be found in [6].

Whereas these references are mostly concerned with the Dirac operator on flat Euclidean space, a natural generalization consists in studying the Dirac operator on *general* manifolds. We refer e.g. to [14] and the work of Calderbank and Cnops, see [2], [4] and [5]. These latter references study the Dirac operator within the framework of Clifford analysis by embedding the manifolds under consideration into an orthogonal space and by using properties of the Dirac operator on this embedding space. This as opposed to the classical spinor Dirac operator on manifolds, which is studied intrinsically within the framework of differential geometry (see e.g. [17]).

For the more specific choice of a positively curved Riemannian manifold we refer to the work of Sommen and Van Lancker [22] and [23], whereas for the case of a negatively curved Riemannian manifold we refer to the work of Leutwiler and his school [16] and [20] and some of our recent papers [9] and [11]. It should be stressed that the Dirac operator in our approach differs fundamentally from the one considered by Leutwiler: whereas his operator acts on $\mathrm{Spin}(1)$-fields, our operator acts on $\mathrm{Spin}(\frac{1}{2})$-fields, as was already pointed out in [3]. The

Received August 2, 2004
First author supported by F.W.O. Vlaanderen (Fund for Scientific Research Flanders).

hyperbolic Dirac operator studied in our papers is invariant under the automorphism group $\mathrm{SO}(1, m)$ of the hyperbolic unit ball, which is a subgroup of the conformal group $\mathrm{SO}(2, m)$. Therefore, the conformal case is included in the present approach as a special case.

In this paper we consider a projective model for the $\mathrm{SO}(m, m)$-invariant ultrahyperbolic unit balls in $\mathbf{R}^{m,m}$, defined in section 2. An introduction to the Clifford setting will be given in section 3, and this enables us to define the Dirac operator on the ultrahyperbolic manifolds in section 4. In section 5 we reduce the hyperbolic Dirac equation to a scalar equation, which will be solved by means of an explicitely constructed fundamental solution for the ultrahyperbolic wave-operator. In section 6 this solution is then used to construct the fundamental solution for the Dirac operator on ultrahyperbolic unit balls in $\mathbf{R}^{p,q}$, with $p \neq q$.

2. The projective model

Consider the orthogonal space $\mathbf{R}^{p,q}$ with orthonormal basis $(\varepsilon_1, \cdots, \varepsilon_p, e_1, \cdots, e_q)$, endowed with the quadratic form

$$Q_{p,q}(\underline{T}, \underline{X}) = \sum_{i=1}^{p} T_i^2 - \sum_{j=1}^{q} X_j^2 = |\underline{T}|^2 - |\underline{X}|^2 \,.$$

The nullcone $NC_{p,q}$ is defined as the set $\{(\underline{T}, \underline{X}) \in \mathbf{R}^{p,q} : Q_{p,q}(\underline{T}, \underline{X}) = 0\}$ and separates the timelike region $TLR_{p,q}$ from the spacelike region $SLR_{p,q}$, given by

$$\begin{aligned} TLR_{p,q} &= \{(\underline{T}, \underline{X}) \in \mathbf{R}^{p,q} : Q_{p,q}(\underline{T}, \underline{X}) > 0\} \\ SLR_{p,q} &= \{(\underline{T}, \underline{X}) \in \mathbf{R}^{p,q} : Q_{p,q}(\underline{T}, \underline{X}) < 0\} \,. \end{aligned}$$

Both the $TLR_{p,q}$ and the $SLR_{p,q}$ contain a canonical $\mathrm{SO}(p, q)$-invariant subset, with $\mathrm{SO}(p, q)$ the orthogonal group containing the linear transformations of unit determinant on $\mathbf{R}^{p,q}$ leaving the quadratic form $Q_{p,q}(\underline{T}, \underline{X})$ invariant, given by

$$\begin{aligned} M_T(p, q) &= \{(\underline{T}, \underline{X}) \in \mathbf{R}^{p,q} : Q_{p,q}(\underline{T}, \underline{X}) = +1\} \\ M_S(p, q) &= \{(\underline{T}, \underline{X}) \in \mathbf{R}^{p,q} : Q_{p,q}(\underline{T}, \underline{X}) = -1\} \,. \end{aligned}$$

Due to the obvious equivalence between $M_T(p, q)$ and $M_S(q, p)$, we can transpose results for the timelike ultrahyperbolic unit ball in $\mathbf{R}^{p,q}$ to the spacelike ultrahyperbolic unit ball in $\mathbf{R}^{q,p}$ by a simple substitution $\underline{T} \leftrightarrow \underline{X}$.

A projective model for these manifolds $M_T(p, q)$ and $M_S(p, q)$, originated by Gelfand, is then given by the manifolds of halfrays

$$\begin{aligned} \mathrm{Ray}(TLR_{p,q}) &= \{\{\lambda(\underline{T}, \underline{X}) : \lambda \in \mathbf{R}_0^+\} : (\underline{T}, \underline{X}) \in TLR_{p,q}\} \\ \mathrm{Ray}(SLR_{p,q}) &= \{\{\lambda(\underline{T}, \underline{X}) : \lambda \in \mathbf{R}_0^+\} : (\underline{T}, \underline{X}) \in SLR_{p,q}\} \,. \end{aligned}$$

Denoting the multiplicative group $\mathbf{R}_0^+$ by G, these manifolds can be defined in a rigorous geometrical way as principal G-bundles. For that purpose it suffices to consider the action of G on $TLR_{p,q}$ (resp. $SLR_{p,q}$) given by $\lambda(\underline{T}, \underline{X}) = (\lambda\underline{T}, \lambda\underline{X})$. This is a free action and its orbit space is precisely the manifold $\text{Ray}(TLR_{p,q})$ (resp. $\text{Ray}(SLR_{p,q})$), such that in the language of bundles we get

$$\text{Ray}(TLR_{p,q}) = (TLR_{p,q}, \pi, TLR_{p,q}/G)$$
$$\text{Ray}(SLR_{p,q}) = (SLR_{p,q}, \pi, SLR_{p,q}/G)\,,$$

with π the projection on a representant of the fibre, being the orbit under the G-action. These representants can for example be chosen as the intersection of the halfrays with $M_T(p, q)$ (resp. $M_S(p, q)$).

The ultrahyperbolic wave-operator on $\mathbf{R}^{p,q}$ is defined as the differential operator

$$\Box_{p,q} = \sum_{i=1}^{p} \partial^2_{T_i} - \sum_{j=1}^{q} \partial^2_{X_j} = \Delta^{(T)}_p - \Delta^{(X)}_q\,,$$

where $\Delta^{(T)}_p$ (resp. $\Delta^{(X)}_q$) stands for the Laplace operator on $\mathbf{R}^p$ (resp. $\mathbf{R}^q$) in coordinates $\underline{T}$ (resp. $\underline{X}$). In section 4 we will explicitely construct a fundamental solution for the operator $\Box_{m,m}$, in case of an odd dimension m, inspired by the construction of the Riesz potential Z_2 as a fundamental solution for the wave-operator $\Box_{1,m}$ on $\mathbf{R}^{1,m}$ (see [19]).

3. Ultrahyperbolic clifford analysis

In this section we define the Dirac operator on $\mathbf{R}^{p,q}$, a first-order differential operator which factorizes $\Box_{p,q}$ and which is invariant under the group $\text{Spin}(p, q)$, the double covering group for $\text{SO}(p, q)$ (cfr. infra).

For that purpose we define the Clifford algebra $\mathbf{R}_{p,q}$ as the associative algebra generated by the orthonormal basis for $\mathbf{R}^{p,q}$ and the multiplication rules $\varepsilon_r\varepsilon_s + \varepsilon_s\varepsilon_r = 2\delta_{rs}$ for all $1 \le r, s \le p$, $e_ie_j + e_je_i = -2\delta_{ij}$ for all $1 \le i, j \le q$ and $\varepsilon_r e_i + e_i\varepsilon_r = 0$ for all $1 \le r \le p$ and $1 \le i \le q$. Elements of this algebra are so-called Clifford numbers and have the form $a = \sum_{A\subset M} a_A e_A$ with $a_A \in \mathbf{R}$, with $A = \{i_1, \cdots, i_k\} \subset M = \{1, \cdots, p+q\}$, with $e_\phi = 1$ and $e_A = \varepsilon_{i_1} \cdots \varepsilon_{i_j} e_{i_{j+1}} \cdots e_k$ for $1 \le i_1 < \cdots < i_j \le p$ and $p+1 \le i_{j+1} < \cdots < i_k \le p+q$.

Those Clifford numbers for which $|A| = k$ are called k-vectors, and denoting the projection of $a \in \mathbf{R}_{p,q}$ on its k-vector part as $[a]_k$ we thus have $a = \sum_k [a]_k$, with $[a]_k \in \mathbf{R}^{(k)}_{p,q}$. The even subalgebra is then defined as $\mathbf{R}^{(+)}_{p,q} = \sum_{k\in 2\mathbf{N}} \mathbf{R}^{(k)}_{p,q}$ and yields a subalgebra of the algebra $\mathbf{R}_{p,q}$.

The orthogonal space $\mathbf{R}^{p,q}$ can be embedded canonically into the Clifford algebra $\mathbf{R}_{p,q}$ by sending an element $(\underline{T}, \underline{X})$ onto its corresponding 1-vector:

$$(\underline{T}, \underline{X}) \in \mathbf{R}^{p,q} \longrightarrow \underline{T} + \underline{X} \in \mathbf{R}^{(1)}_{p,q}\,.$$

Consider then $(\underline{T}, \underline{X})$ and $(\underline{S}, \underline{Y}) \in \mathbf{R}^{p,q}$. The Clifford product is defined by

$$(\underline{T}, \underline{X})(\underline{S}, \underline{Y}) = (\underline{T}, \underline{X}) \cdot (\underline{S}, \underline{Y}) + (\underline{T}, \underline{X}) \wedge (\underline{S}, \underline{Y})$$

where the *inner* and *outer* product are respectively given by

$$2(\underline{T}, \underline{X}) \cdot (\underline{S}, \underline{Y}) = (\underline{T}, \underline{X})(\underline{S}, \underline{Y}) + (\underline{S}, \underline{Y})(\underline{T}, \underline{X})$$
$$2(\underline{T}, \underline{X}) \wedge (\underline{S}, \underline{Y}) = (\underline{T}, \underline{X})(\underline{S}, \underline{Y}) - (\underline{S}, \underline{Y})(\underline{T}, \underline{X}) .$$

On $\mathbf{R}_{p,q}$ we then define three important involutory (anti)automorphisms: for all $a, b \in \mathbf{R}_{p,q}$ we have

- the *main involution* $a \mapsto \widetilde{a}$ with $\widetilde{\varepsilon}_i = -\varepsilon_i$, $\widetilde{e}_j = -e_j$ and $\widetilde{(ab)} = \widetilde{a}\widetilde{b}$
- the *reversion* $a \mapsto a^*$ with $\varepsilon_i^* = \varepsilon_i$, $e_j^* = e_j$ and $(ab)^* = b^* a^*$
- the *conjugation* $a \mapsto \overline{a}$ with $\overline{\varepsilon}_i = -\varepsilon_i$, $\overline{e}_j = -e_j$ and $\overline{(ab)} = \overline{b}\,\overline{a}$.

The Clifford group $\Gamma(p, q)$ is then defined as the set of all invertible elements $g \in \mathbf{R}_{p,q}$ such that for all $(\underline{T}, \underline{X}) \in \mathbf{R}^{p,q}$ we have $g(\underline{T}, \underline{X})\widetilde{g}^{-1} \in \mathbf{R}^{p,q}$, the group $\mathrm{Pin}(p, q)$ is the quotient group $\Gamma(p, q)/\mathbf{R}^+$ and the group $\mathrm{Spin}(p, q)$ is defined as $\mathrm{Pin}(p, q) \cap \mathbf{R}_{p,q}^{(+)}$. For each $s \in \mathrm{Pin}(p, q)$ the mapping $\chi(s) : \mathbf{R}^{p,q} \mapsto \mathbf{R}^{p,q}$ sending $(\underline{T}, \underline{X}) \mapsto s(\underline{T}, \underline{X})\overline{s}$ induces an orthogonal transformation. In this way $\mathrm{Pin}(p, q)$ defines a double covering of $\mathrm{O}(p, q)$ whereas $\mathrm{Spin}(p, q)$ defines a double covering of $\mathrm{SO}(p, q)$. More information on this can be found in e.g. [18].

The Dirac operator on the orthogonal space $\mathbf{R}^{p,q}$ is then defined as the vector derivative $\partial_{p,q}$ given by

$$\partial_{p,q} = \sum_{i=1}^{p} \varepsilon_i \partial_{T_i} - \sum_{j=1}^{q} e_j \partial_{X_j} .$$

Let $\Omega \subset \mathbf{R}^{p,q}$ be open and let $f \in C^1(\Omega)$ be an $\mathbf{R}_{p,q}$-valued function. If f satisfies $\partial_{p,q} f = 0$ in Ω, f is called *monogenic* with respect to the Dirac operator on $\mathbf{R}^{p,q}$. As the operator $\partial_{p,q}$ factorizes the ultrahyperbolic wave-operator on $\mathbf{R}^{p,q}$, monogenic functions are nullsolutions for $\Box_{p,q}$ but not vice-versa.

Arbitrary vectors belonging to $TLR_{p,q}$ can be decomposed as follows:

$$Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}} \left(\frac{\underline{T}}{Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}}, \frac{\underline{X}}{Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}} \right) = Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}} (\underline{\tau}, \underline{\xi})_T ,$$

where $(\underline{\tau}, \underline{\xi})_T$ belongs to $M_T(p, q)$, and hence is called a *timelike unit vector*. If $(\underline{T}, \underline{X}) \in SLR_{p,q}$ we can replace $Q_{p,q}(\underline{T}, \underline{X})$ by $Q_{q,p}(\underline{X}, \underline{T})$. The corresponding vector $(\underline{\tau}, \underline{\xi})_S$ then belongs to $M_T(p, q)$ and is called a *spacelike unit vector*.

In what follows it will be necessary to let the Dirac operator $\partial_{p,q}$ act on functions defined on $TLR_{p,q}$ (resp. $SLR_{p,q}$). We therefore mention the following decompositions for $\partial_{p,q}$:

$$\partial_{p,q} = \frac{(\underline{\tau}, \underline{\xi})_T}{Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}} (\mathbf{E}_{p,q} + \Gamma_{p,q}) = \frac{(\underline{\tau}, \underline{\xi})_S}{Q_{q,p}(\underline{X}, \underline{T})^{\frac{1}{2}}} (\mathbf{E}_{p,q} + \Gamma_{p,q}) , \qquad (1)$$

where $\mathbf{E}_{p,q}$ denotes the Euler operator measuring the degree of homogeneity with respect to the coordinates $(\underline{T}, \underline{X}) \in \mathbf{R}^{p,q}$ and with $\Gamma_{p,q}$ the ultrahyperbolic angular operator on $\mathbf{R}^{p,q}$ (when acting on functions depending on $Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}$ or $Q_{q,p}(\underline{X}, \underline{T})^{\frac{1}{2}}$ only, this operator vanishes identically). In explicit coordinates these operators are given by

$$\mathbf{E}_{p,q} = \sum_{i=1}^{p} T_i \partial_{T_i} + \sum_{j=1}^{q} X_j \partial_{X_j}$$

and

$$\Gamma_{p,q} = \sum_{i<j} \varepsilon_i \varepsilon_j (T_i \partial_{T_j} - T_j \partial_{T_i}) - \sum_{i<j} e_i e_j (X_i \partial_{X_j} - X_j \partial_{X_i}) - \sum_{i,j} \varepsilon_i e_j (T_i \partial_{X_j} + X_j \partial_{T_i}) .$$

This can easily be verified by calculating the Clifford product $(\underline{T} + \underline{X})\partial_{p,q}$ and by collecting the scalar terms (giving $\mathbf{E}_{p,q}$) and the bivector-terms (giving $\Gamma_{p,q}$). Using these explicit expressions, one can also verify that

$$\Gamma_{p,q}[(\underline{T}, \underline{X}) \cdot (\underline{\sigma}, \underline{\eta})] = (\underline{T}, \underline{X}) \wedge (\underline{\sigma}, \underline{\eta}) ,$$

a relation that will be crucial in what follows.

4. Clifford analysis on spin(p, q)-invariant manifolds

Due to the projective nature of our models for $M_T(p, q)$ and $M_S(p, q)$, the Dirac operator on these manifolds must be defined on Ray$(TLR_{p,q})$ and Ray$(SLR_{p,q})$. For that purpose we define an associated principal G-bundle, for which we refer e.g. to reference [15]. The basic idea is that given a particular principal bundle with structure group G, in our case this is precisely the manifold of rays (where $G = \mathbf{R}_0^+$), one can form a fibre bundle with fibre F for each space F on which G acts as a group of transformations. The main motivation then lies in the fact that sections of this associated bundle will be equivalent to F-valued functions satisfying a certain constraint (cfr. infra). Since monogenic functions are defined as *Clifford algebra-valued* nullsolutions for the Dirac operator, it is thus very natural to construct the principal fibre bundle with fibre $F = \mathbf{R}_{p,q}$ associated to the manifolds of rays.

Consider therefore the following representation Λ of $G = \mathbf{R}_0^+$ on the Clifford algebra $\mathbf{R}_{p,q}$:

$$\Lambda : G \mapsto \text{End}(\mathbf{R}_{p,q}) : \lambda \mapsto \lambda^{-\alpha} , \ \alpha \in \mathbf{C} ,$$

where $\lambda^{-\alpha}$ acts by multiplication. Since we now have a principal G-bundle and the Clifford algebra $\mathbf{R}_{p,q}$ playing the role of a G-space, we may consider the G-product $TLR_{p,q} \times_G \mathbf{R}_{p,q}$ of orbits under the (abelian) action of G on the Carthesian product $TLR_{p,q} \times \mathbf{R}_{p,q}$ given by

$$\lambda((\underline{T}, \underline{X}), a) = (\lambda(\underline{T}, \underline{X}), (\lambda^{-\alpha})^{-1} a) = (\lambda(\underline{T}, \underline{X}), \lambda^{\alpha} a) .$$

By definition, the associated bundle is then given by the bundle

$$(TLR_{p,q} \times_G \mathbf{R}_{p,q}, \pi_G, TLR_{p,q}/G)$$

with fibre $\mathbf{R}_{p,q}$, where the projection π_G is defined as

$$\pi_G : TLR_{p,q} \times_G \mathbf{R}_{p,q} \mapsto TLR_{p,q}/G : [(\underline{T}, \underline{X}), a] \mapsto \pi(\underline{T}, \underline{X}) \,,$$

π being the projection associated to the principal bundle $\mathrm{Ray}(TLR_{p,q})$.

Since the sections of this associated principal fibre bundle are in bijective correspondance with functions $\phi : TLR_{p,q} \mapsto \mathbf{R}_{p,q}$ satisfying

$$\phi(\lambda(\underline{T}, \underline{X})) = (\lambda^{-\alpha})^{-1}\phi(\underline{T}, \underline{X}) = \lambda^{\alpha}\phi(\underline{T}, \underline{X}) \,,$$

we conclude that sections of the associated fibre bundle are *homogeneous* functions on the timelike (resp. spacelike) region in $\mathbf{R}^{p,q}$.

CONCLUSION. *The Dirac operator on the ultrahyperbolic manifolds $M_T(p,q)$ and $M_S(p,q)$ can be defined as the Dirac operator $\partial_{p,q}$ acting on homogeneous functions $F(\underline{T}, \underline{X})$ on either the timelike or spacelike region.*

As the operator $\partial_{p,q}$ is homogeneous of degree (-1) it turns α-homogeneous sections into $(\alpha - 1)$-homogeneous sections, whence this operator is well-defined.

In order to construct the fundamental solution $E_T^{(\alpha)}(\underline{T}, \underline{X})$ (resp. $E_S^{(\alpha)}(\underline{T}, \underline{X})$) for the Dirac operator on $M_T(p,q)$ (resp. $M_S(p,q)$) we first need to derive the corresponding *ultrahyperbolic Dirac equation*. By definition, this is an equation of the following type:

$$\partial_{p,q} E_T^{(\alpha)}(\underline{T}, \underline{X}) = \mathcal{D}_T \text{ (resp. } \partial_{p,q} E_S^{(\alpha)}(\underline{T}, \underline{X}) = \mathcal{D}_S) \,,$$

where $\mathcal{D}_T$ (resp. $\mathcal{D}_S$) stands for a delta distribution on $\mathrm{Ray}(TLR_{p,q})$ (resp. $\mathrm{Ray}(SLR_{p,q})$). This means that $\mathcal{D}_T$ (resp. $\mathcal{D}_S$) is a homogeneous distribution acting on timelike (resp. spacelike) test functions $\varphi(\underline{T}, \underline{X})$ by integration over $TLR_{p,q}$ (resp. $SLR_{p,q}$) and that after multplication with the distribution $\delta(|\underline{T}|^2 - |\underline{X}|^2 \pm 1)$, hence restricting the integration to $M_T(p,q)$ (resp. $M_T(p,q)$) we are left with the point evaluation in a point of this manifold.

Let us for a moment focus on the timelike region. In order to find the explicit form for the distribution $\mathcal{D}_T$ we choose a timelike unit vector $\underline{\tau} \in S^{p-1}$ and we put

$$\mathcal{D}_T = \langle \underline{T}, \underline{\tau} \rangle_+^{\alpha+p+q-2} \delta(\underline{T} \wedge \underline{\tau}) \delta(\underline{X}) \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. The distribution $\mathcal{D}_T$ then clearly represents a delta distribution on the halfray through $\underline{\tau} \in S^{p-1}$, with

$$\langle \delta(|\underline{T}|^2 - |\underline{X}|^2 - 1)\mathcal{D}_T, \varphi(\underline{T}, \underline{X}) \rangle = \varphi(\underline{\tau}, \underline{0}) \,,$$

and has the desired degree of homogeneity $(\alpha - 1)$. Without loosing generality we may choose $\underline{\tau} = \varepsilon_1$ such that the timelike *ultrahyperbolic Dirac equation* on $M_T(p,q)$ with singularity at ε_1 reduces to:

$$\partial_{p,q} E_T^{(\alpha)}(\underline{T}, \underline{X}) = (T_1)_+^{\alpha+p+q-2} \delta(\underline{T}_1) \delta(\underline{X}) \,,$$

where $\underline{T}_1$ stands for the vector $\sum_{i=2}^{p} \varepsilon_i T_i \in \mathbf{R}^{p-1}$. Note that the right-hand side does not exist for $\alpha \in 1-p-q-\mathbf{N}$: these values correspond to the poles of the distribution $(T_1)_+^{\alpha+p+q-2}$ (see reference [13]). In a completely similar way, the spacelike *ultrahyperbolic Dirac equation* on $M_S(p,q)$ with singularity at $\underline{\xi} \in S^{q-1}$ is given by

$$\partial_{p,q} E_S^{(\alpha)}(\underline{T},\underline{X}) = \langle \underline{X}, \underline{\xi} \rangle_+^{\alpha+p+q-2} \delta(\underline{X} \wedge \underline{\xi}) \delta(\underline{T}) .$$

5. The dirac equation on $\mathbf{R}^{m,m}$

In this section we will solve the ultrahyperbolic Dirac equation on $M_T(m,m)$ for $E_T^{(\alpha)}(\underline{T},\underline{X})$, where m is an odd integer. As $\partial_{m,m}^2 = \Box_{m,m}$ it suffices to solve the *scalar* problem

$$\Box_{m,m} \Phi(\underline{T},\underline{X}) = (T_1)_+^{\alpha+2m-2} \delta(\underline{T}_1) \delta(\underline{X}) , \tag{2}$$

and to put $E_T^{(\alpha)}(\underline{T},\underline{X}) = \partial_{m,m} \Phi(\underline{T},\underline{X})$. For that purpose we will first construct a fundamental solution $\mathcal{F}_{m,m}(\underline{T},\underline{X})$ for $\Box_{m,m}$ and then convolute it with the distribution at the right-hand side. In view of the fact that this latter distribution does not exist for $\alpha \in 1-2m-\mathbf{N}$, we expect $\Phi(\underline{T},\underline{X})$ to have poles at the very same values for α.

5.1. The fundamental solution $\mathcal{F}_{m,m}(\underline{T},\underline{X})$.

To construct the fundamental solution for the operator $\Box_{m,m}$ we use a method leading to the ultrahyperbolic analogues of Riesz' distributions. Let us therefore define the function $P_-(\underline{T},\underline{X})$ by

$$P_-(\underline{T},\underline{X}) = \begin{cases} (|\underline{X}|^2 - |\underline{T}|^2)^{\frac{1}{2}} & \text{if} \quad |\underline{T}| < |\underline{X}| \\ 0 & \text{if} \quad |\underline{T}| \geq |\underline{X}| \end{cases}$$

For $\mathrm{Re}(\lambda) > -2$ the function P_-^λ belongs to $L_1^{loc}(\mathbf{R}^{m,m})$ and hence defines a regular distribution. Using the fact that $\Box_{m,m} P_-^\lambda = -\lambda(\lambda+2m-2) P_-^{\lambda-2}$ for $\mathrm{Re}(\lambda) > -2$, the distribution P_-^λ can then for other values of λ be defined by analytic continuation, where the derivatives have to be understood in distributional sense:

$$\langle P_-^\lambda, \varphi \rangle = (-1)^k \frac{\langle \Box_{m,m}^k P_-^{\lambda+2k}, \varphi \rangle}{(\lambda+2)\cdots(\lambda+2k)(\lambda+2m)\cdots(\lambda+2m+2k-2)} .$$

From this expression it is clear that P_-^λ has poles at $\lambda \in -2\mathbf{N}_0$ and at $\lambda \in -2m-2\mathbf{N}$. In the following Lemma it is proved how the pole at $(2-2m)$ can easily be removed by division by an appropriate Gamma function, and how this leads to a fundamental solution for the operator $\Box_{m,m}$ in case of an odd dimension m.

LEMMA 1. *In case* $m \in 2\mathbf{N}+1$, *we have*:

$$\Box_{m,m} \left(\frac{(-1)^{\frac{m+1}{2}}}{4\pi^{m-1}} \frac{P_-^{2-2m}}{\Gamma(2-m)} \right) = \delta(\underline{T}) \delta(\underline{X}) .$$

PROOF. First of all note that the expression above must be understood in the sense

$$\lim_{\lambda\to 2-2m} \frac{(-1)^{\frac{m+1}{2}}}{4\pi^{m-1}} \frac{P_-^\lambda}{\Gamma(1+\frac{\lambda}{2})} .$$

Since $\Box_{m,m} P_-^\lambda = -\lambda(\lambda+2m-2)P_-^{\lambda-2}$ it suffices to prove that

$$\lim_{\mu\to -2m} \frac{(-1)^{\frac{m+1}{2}}}{4\pi^{m-1}} \frac{-2(\mu+2m)\langle P_-^\mu, \varphi\rangle}{\Gamma(1+\frac{\mu}{2})} = \varphi(\underline{0},\underline{0}) .$$

By definition we have

$$\begin{aligned}\langle P_-^\mu, \varphi\rangle &= \iint_{|\underline{X}|>|\underline{T}|} (|\underline{X}|^2 - |\underline{T}|^2)^{\frac{\mu}{2}} \varphi(\underline{T},\underline{X}) d\underline{T} d\underline{X} \\ &= \int_0^\infty |\underline{X}|^{\mu+2m-1} \Phi(|\underline{X}|,\mu) d|\underline{X}| ,\end{aligned}$$

where we have put

$$\Phi(|\underline{X}|,\mu) = \int_0^1 t^{m-1}(1-t^2)^{\frac{\mu}{2}} \psi(t|\underline{X}|, |\underline{X}|) dt ,$$

with

$$\psi(|\underline{T}|,|\underline{X}|) = \int_{S^{m-1}} \int_{S^{m-1}} \varphi(|\underline{T}|\varepsilon, |\underline{X}|e) d\varepsilon de .$$

Recalling the fact that $\mathrm{Res}\{x_+^\lambda, \lambda=-1\} = \delta(x)$ the Lemma easily follows by elementary calculations, hereby invoking the relation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. Note that the result only holds in case $m \in 2\mathbf{N}+1$. □

5.2. Calculation of $\Phi(\underline{T},\underline{X})$. Once this fundamental solution for the ultrahyperbolic wave-operator is found, the solution for equation (2) in case of odd m immediately follows:

$$\Phi(\underline{T},\underline{X}) = \mathcal{F}_{m,m}(\underline{T},\underline{X}) * (T_1)_+^{\alpha+2m-2} \delta(\underline{T}_1)\delta(\underline{X}) .$$

Recalling the explicit expression for the definition of the fundamental solution, the convolution reduces to

$$\begin{aligned}&\frac{(-1)^{\frac{m+1}{2}}}{4\pi^{m-1}} \lim_{\lambda\to 2-2m} \int_{\mathbf{R}^m} \frac{H(|\underline{X}-\underline{Y}| - |\underline{T}-\underline{S}|)(S_1)_+^{\alpha+2m-2}\delta(\underline{S}_1)\delta(\underline{Y})}{\Gamma(1+\frac{\lambda}{2})(|\underline{X}-\underline{Y}|^2 - |\underline{T}-\underline{S}|^2)^{-\frac{\lambda}{2}}} d\underline{S} d\underline{Y} \\ &= \frac{(-1)^{\frac{m+1}{2}}}{4\pi^{m-1}} \lim_{\lambda\to 2-2m} \int_0^\infty \frac{H(|\underline{X}-\underline{Y}| - |\underline{T}-\underline{S}|) S_1^{\alpha+2m-2}}{\Gamma(1+\frac{\lambda}{2})(|\underline{X}|^2 - |\underline{T}-S_1\varepsilon_1|^2)^{-\frac{\lambda}{2}}} dS_1 .\end{aligned}$$

The factor in the denominator is given by $e^{-i\lambda\frac{\pi}{2}}[(S_+ - S_1)(S_1 - S_-)]^{-\frac{\lambda}{2}}$, where we have put

$$S_\pm = |\underline{T}|\theta \pm (|\underline{T}|^2\theta^2 - |\underline{T}|^2 + |\underline{X}|^2)^{\frac{1}{2}} ,$$

with $\theta = \langle \varepsilon, \varepsilon_1 \rangle$ the classical Euclidean inner product. Due to the Heaviside distribution, it is immediately clear that $\text{supp}(\Phi(\underline{T}, \underline{X})) = \{(\underline{T}, \underline{X}) \in TLR_{m,m} : |\underline{X}|^2 \geq (1 - \langle \varepsilon, \varepsilon_1 \rangle^2)|\underline{T}|^2\}$, which is also equal to $(SLR_{m-1,m} \times \mathbf{R}\varepsilon_1) \cap TLR_{m,m}$. This latter expression has a very simple geometrical interpretation, as it is the region in $TLR_{m,m}$ obtained by letting the support of $\mathcal{F}_{m,m}(\underline{T}, \underline{X})$ move along the half ray containing the singularities. We thus get that

$$\Phi(\underline{T}, \underline{X}) = \frac{(-1)^{\frac{m+1}{2}}}{4\pi^{m-1}} \lim_{\lambda \to 2-2m} \int_{S_-}^{S_+} \frac{e^{i(1-m)\pi} S_1^{\alpha+2m-2}}{\Gamma(1+\frac{\lambda}{2})[(S_+ - S_1)(S_1 - S_-)]^{-\frac{\lambda}{2}}} \, dS_1 \, .$$

The evaluation of this integral is the subject of the following Lemma:

LEMMA 2. *For a, b in $\mathbf{R}^+$ and γ in $\mathbf{C}$ with $Re(\gamma) < 1$, one has*

$$\int_a^b \frac{t^\beta}{[(t-a)(b-t)]^\gamma} \, dt = \frac{a^\beta}{(b-a)^{2\gamma-1}} \frac{\Gamma(1-\gamma)^2}{\Gamma(2-2\gamma)} \, F\left(-\beta, 1-\gamma; 2-2\gamma; \frac{a-b}{a} \right).$$

PROOF. The substitution $(b-a)u = t - a$ leads to

$$\int_a^b \frac{t^\beta}{[(t-a)(b-t)]^\gamma} \, dt = a^\beta (b-a)^{1-2\gamma} \int_0^1 \frac{(1 - \frac{a-b}{a} u)^\beta}{u^\gamma (1-u)^\gamma} \, du \, .$$

Using Euler's integral representation formula for the hypergeometric function, we get for all γ such that $\text{Re}(\gamma) < 1$:

$$\int_0^1 \frac{(1 - \frac{a-b}{a} u)^\beta}{u^\gamma (1-u)^\gamma} \, du = \frac{\Gamma(1-\gamma)^2}{\Gamma(2-2\gamma)} \, F\left(-\beta, 1-\gamma; 2-2\gamma; \frac{a-b}{a} \right).$$

This proves the Lemma. □

REMARK. For our purposes, the condition on γ may be ignored. This can easily be seen as follows: in the expression for $\Phi(\underline{T}, \underline{X})$ that needs to be determined, there is an additional factor $\Gamma(1 + \frac{\lambda}{2}) = \Gamma(1 - \gamma)$ in the denominator, such that the last formula of the Lemma reduces to

$$\frac{\Gamma(1-\gamma)}{\Gamma(2-2\gamma)} \, F\left(-\beta, 1-\gamma; 2-2\gamma; \frac{a-b}{a} \right).$$

This expression is defined for all $\gamma \in \mathbf{C}$, whence the expression

$$\frac{1}{\Gamma(1-\gamma)} \int_a^b \frac{t^\beta}{[(t-a)(b-t)]^\gamma} \, dt$$

is defined for all odd dimensions m by analytic continuation.
We therefore find the following expression for $\Phi(\underline{T}, \underline{X})$:

$$\frac{(-1)^{\frac{m-3}{2}}}{2^{5-2m}\pi^{m-\frac{3}{2}}} \frac{(S_-)^{\alpha+2m-2}}{(S_+ - S_-)^{2m-3}} \frac{F(2-2m-\alpha, 2-m; 4-2m; \frac{S_- - S_+}{S_-})}{\Gamma(\frac{5}{2} - m)} \, .$$

By analogy with the hyperbolic case, see e.g. [11], we would like to write the expression for $\Phi(\underline{T}, \underline{X})$ in terms of Gegenbauer functions of the second kind. This is also motivated by the fact that Gegenbauer functions are fundamentally related to the orthogonal group. So, in what follows we will use some basic results from the theory on special functions (in particular on the hypergeometric function and the Gegenbauer function of the second kind, for which we resp. refer to [12] and [8]) to put $\Phi(\underline{T}, \underline{X})$ into another form.

Recalling one of Kummer's relations,

$$F(a, c-b; 1+a-b; 1-z) = z^{-a} F(a, 1+a-c; 1+a-b; 1-z^{-1}),$$

and one of Goursat's quadratic transformations,

$$F\left(a, a+\frac{1}{2}; c; z^2\right) = (1+z)^{-2a} F\left(2a, c-\frac{1}{2}; 2c-1; \frac{2z}{1+z}\right),$$

elementary calculations yield:

$$\Phi(\underline{T}, \underline{X}) = \frac{(-1)^{\frac{m-3}{2}}}{4\pi^{m-\frac{3}{2}}\Gamma(\frac{5}{2}-m)} T_1^{\alpha+1}\left(1-\frac{|\underline{T}|^2-|\underline{X}|^2}{T_1^2}\right)^{\frac{3}{2}-m}$$
$$F\left(\frac{2-2m-\alpha}{2}, \frac{3-2m-\alpha}{2}; \frac{5}{2}-m; 1-\frac{|\underline{T}|^2-|\underline{X}|^2}{T_1^2}\right).$$

Because the singular behaviour of $\Phi(\underline{T}, \underline{X})$ does not change when a nullsolution for $\Box_{m,m}$ is added, we will add the particular solution constructed in the following Lemma:

LEMMA 3. *For all* $(\underline{T}, \underline{X}) \in TLR_{m,m}$ *we have*

$$\Box_{m,m}\left\{T_1^{\alpha+1}F\left(-\frac{\alpha}{2}, -\frac{\alpha+1}{2}; m-\frac{1}{2}; 1-\frac{|\underline{T}|^2-|\underline{X}|^2}{T_1^2}\right)\right\} = 0.$$

PROOF. Putting $\Box_{m,m} = \partial_{T_1}^2 + \Box_{m-1,m}$ and $\rho^2 = |\underline{X}|^2 - (|\underline{T}|^2 - T_1^2)$, it suffices to prove that

$$\left(\partial_{T_1}^2 - \partial_\rho^2 - \frac{2m-1}{\rho}\partial_\rho\right) T_1^{\alpha+1} F\left(-\frac{\alpha}{2}, -\frac{\alpha+1}{2}; m-\frac{1}{2}; \frac{\rho^2}{T_1^2}\right) = 0,$$

because for functions on $\mathbf{R}^{m,m-1}$ depending on ρ only we have that $\Box_{m-1,m} = (\partial_\rho^2 + \frac{2m-1}{\rho}\partial_\rho)$. Putting $T_1 = \lambda$ and $\frac{\rho^2}{T_1^2} = u$, the differential operator between brackets acting on $T_1^{\alpha+1} f(u)$ reduces to

$$\left(u(1-u)\frac{d^2}{du^2} + \left[\left(m-\frac{1}{2}\right) + \left(\alpha-\frac{1}{2}\right)u\right]\frac{d}{du} - \frac{\alpha(\alpha+1)}{4}\right) f(u).$$

The Lemma then follows trivially. □

Adding this regular solution, multiplied by

$$C = \frac{e^{i\pi\frac{m-1}{2}}}{2^{2m-1}\pi^{m-\frac{1}{2}}} \frac{\Gamma(\frac{3}{2}-m)\Gamma(\alpha+m-1)}{\Gamma(\alpha+2)},$$

to the expression for $\Phi(\underline{T}, \underline{X})$ that was found earlier, using the definition of the Gegenbauer function of the second kind in terms of the Legendre function of the second kind

$$D^{\nu}_{\mu}(z) = \frac{e^{i\pi(2\nu-\frac{1}{2})}}{\pi^{\frac{1}{2}}2^{\nu-\frac{1}{2}}} \frac{\Gamma(\mu+2\nu)}{\Gamma(\nu)\Gamma(1+\mu)} (z^2-1)^{\frac{1}{4}-\frac{\nu}{2}} Q^{\frac{1}{2}-\nu}_{\mu+\nu-\frac{1}{2}}(z)$$

and the following expansion for the Legendre function of the second kind,

$$\begin{aligned} Q^{\mu}_{\nu}(z) = {} & \frac{\Gamma(\mu)}{2^{1-\mu}} \frac{e^{i\mu\pi} z^{\nu+\mu}}{(z^2-1)^{\frac{\mu}{2}}} F\left(-\frac{\nu+\mu}{2}, \frac{1-\nu-\mu}{2}; 1-\mu; 1-\frac{1}{z^2}\right) \\ & + \frac{\Gamma(-\mu)\Gamma(1+\mu+\nu)}{2^{\mu+1}\Gamma(1+\nu-\mu)} \frac{e^{i\mu\pi} z^{\nu-\mu}}{(z^2-1)^{-\frac{\mu}{2}}} F\left(\frac{\mu-\nu}{2}, \frac{1+\mu-\nu}{2}; 1+\mu; 1-\frac{1}{z^2}\right), \end{aligned}$$

we eventually get the following solution for equation (2):

$$\Phi(\underline{T}, \underline{X})' = \frac{e^{i\pi\frac{m-1}{2}}}{2\pi^{m-1}} \Gamma(m-1)(|\underline{T}|^2 - |\underline{X}|^2)^{\frac{\alpha+1}{2}} D^{m-1}_{\alpha+1}\left(\frac{T_1}{(|\underline{T}|^2 - |\underline{X}|^2)^{\frac{1}{2}}}\right).$$

REMARK. As the function $D^{\mu}_{\nu}(z)$ has poles at $\nu + 2\mu \in -\mathbf{N}$, this fundamental solution $\Phi(\underline{T}, \underline{X})'$ has indeed poles at the values for which the delta distribution $\mathcal{D}_T$ on $\mathrm{Ray}(TLR_{m,m})$ does not exist.

A last simplification makes use of the fact that the argument of the Gegenbauer function can also be written as $(\underline{\tau}, \underline{\xi})_T \cdot \varepsilon_1$, where $(\underline{\tau}, \underline{\xi})_T$ denotes the timelike unit vector in $M_T(p, q)$ associated to $(\underline{T}, \underline{X}) \in TLR_{m,m}$. Recalling the decomposition (1) of the operator $\partial_{p,q}$ we thus conclude:

$$\begin{aligned} E^{(\alpha)}_T(\underline{T}, \underline{X}) &= \partial_{p,q}\Phi(\underline{T}, \underline{X})' \\ &= \frac{e^{i\pi\frac{m-1}{2}}}{2\pi^{m-1}} \Gamma(m-1)(|\underline{T}|^2 - |\underline{X}|^2)^{\frac{\alpha}{2}} (\underline{\tau}, \underline{\xi})_T \\ &\quad (\Gamma_{p,q} + (\alpha+1)) D^{m-1}_{\alpha+1}((\underline{\tau}, \underline{\xi})_T \cdot \varepsilon_1), \end{aligned}$$

which by means of the fact that $\frac{d}{dz} D^{\mu}_{\nu}(z) = 2\mu D^{\mu+1}_{\nu-1}(z)$ reduces to

$$\begin{aligned} E^{(\alpha)}_T(\underline{T}, \underline{X}) = {} & \frac{e^{i\pi\frac{m-1}{2}}}{2\pi^{m-1}} \Gamma(m-1)(|\underline{T}|^2 - |\underline{X}|^2)^{\frac{\alpha}{2}} (\underline{\tau}, \underline{\xi})_T \\ & ((2m-2)(\underline{\tau}, \underline{\xi})_T \wedge \varepsilon_1 D^{m}_{\alpha}((\underline{\tau}, \underline{\xi})_T \cdot \varepsilon_1) + (\alpha+1) D^{m-1}_{\alpha+1}((\underline{\tau}, \underline{\xi})_T \cdot \varepsilon_1)). \end{aligned}$$

Using the fact that $(\underline{\tau}, \underline{\xi})_T[(\underline{\tau}, \underline{\xi})_T \wedge \varepsilon_1] = \varepsilon_1 - (\underline{\tau}, \underline{\xi})_T\big[(\underline{\tau}, \underline{\xi})_T \cdot \varepsilon_1\big]$ and an elementary recurrence relation for the Gegenbauer functions, we eventually find that

$$E_T^{(\alpha)}(\underline{T}, \underline{X}) = \frac{e^{i\pi\frac{m+1}{2}}}{\pi^{m-1}}\Gamma(m)(|\underline{T}|^2 - |\underline{X}|^2)^{\frac{\alpha}{2}} \\ ((\underline{\tau}, \underline{\xi})_T D^m_{\alpha-1}((\underline{\tau}, \underline{\xi})_T \cdot \varepsilon_1) - \varepsilon_1 D^m_\alpha((\underline{\tau}, \underline{\xi})_T \cdot \varepsilon_1))\,.$$

For the spacelike equation, it suffices to switch $\underline{T} \leftrightarrow \underline{X}$ and to note that the argument of the Gegenbauer function can be written as $-(\underline{\tau}, \underline{\xi})_S \cdot e_1$. A similar calculation then yields:

$$E_S^{(\alpha)}(\underline{T}, \underline{X}) = \frac{e^{i\pi\frac{m+1}{2}}}{\pi^{m-1}}\Gamma(m)(|\underline{X}|^2 - |\underline{T}|^2)^{\frac{\alpha}{2}} \\ ((\underline{\tau}, \underline{\xi})_S D^m_{\alpha-1}(-(\underline{\tau}, \underline{\xi})_S \cdot e_1) + e_1 D^m_\alpha(-(\underline{\tau}, \underline{\xi})_S \cdot e_1))\,.$$

Note that these solutions have indeed the same structure as the fundamental solution for the hyperbolic Dirac operator as obtained in e.g. [11].

6. The Dirac Equation on $\mathbf{R}^{p,q}$

In this section we will use the fundamental solution for the Dirac equation on $M_T(m, m)$ (resp. $M_S(m, m)$) to construct the fundamental solution for the Dirac equation on $M_T(m, q)$ (resp. $M_S(m, q)$). We hereby use the *method of descent*, based on the following argument:

$$\Box_{m,m}\Phi(\underline{T}, \underline{X}) = (T_1)_+^{\alpha+2m-2}\delta(\underline{T}_1)\delta(\underline{X}) \\ \Downarrow \\ \int_{-\rho_m}^{\rho_m} \Box_{m,m}\Phi(\underline{T}, \underline{X})dX_m = (T_1)_+^{\alpha+2m-2}\delta(\underline{T}_1)\delta(\underline{X}_m)\,,$$

where we have put $\underline{X}_m = \sum_{j=1}^{m-1} X_j e_j$ and $\rho_m = (|\underline{T}|^2 - |\underline{X}_m|^2)^{\frac{1}{2}}$. Note that for $(\underline{T}, \underline{X})$ in the spacelike region, the factor ρ_m remains positive if $X_m \in [-\rho_m, \rho_m]$. The integral sign can be switched with the operator $\Box_{m,m-1}$ and for the remaining term we have:

$$\int_{-\rho_m}^{\rho_m} \partial^2_{X_m}\Phi(\underline{T}, \underline{X})dX_m = \partial_{X_m}\Phi(\underline{T}, \underline{X})|_{-\rho_m}^{+\rho_m}\,.$$

Using the following expansion for the Legendre function of the second kind,

$$e^{-i\mu\pi}Q_\nu^\mu(z) = \frac{\pi^{\frac{1}{2}}\Gamma(\mu+\nu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\frac{(z^2-1)^{\frac{1}{2}\mu}}{z^{\mu+\nu+1}}F\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \nu+\frac{3}{2}; \frac{1}{z^2}\right),$$

for which we refer to e.g. [12], it is easily verified that

$$\Phi(\underline{T}, \underline{X}) = \frac{2^{1-\alpha-2m}\Gamma(\alpha+2m-1)}{\Gamma(m-1)\Gamma(\alpha+m+1)}\frac{(|\underline{T}|^2 - |\underline{X}|^2)^{\alpha+m}}{T_1^{\alpha+2m-1}}$$

$$F\left(m+\frac{\alpha-1}{2}, m+\frac{\alpha}{2}; \alpha+m+1; \frac{|\underline{T}|^2-|\underline{X}|^2}{T_1^2}\right). \tag{3}$$

such that $\partial_{X_m}\Phi(\underline{T},\underline{X})|_{-\rho_m}^{+\rho_m}$ vanishes for $\mathrm{Re}(\alpha) > 1-m$. The fundamental solution obtained by means of the method of descent will however be defined for all $\alpha \in \mathbf{C}$ such that $\alpha \notin 1-2m-\mathbf{N}$, these latter values being the ones for which the distribution $(T_1)_+^{\alpha+2m-2}$ does not exist, by means of analytic continuation.

The $(\alpha+2)$-homogeneous solution for the equation

$$\Box_{m,m-1}\Phi(\underline{T},\underline{X}_m) = (T_1)_+^{\alpha+2m-2}\delta(\underline{T}_1)\delta(\underline{X}_m)$$

is thus given by

$$\Phi(\underline{T},\underline{X}_m) = \int_{-\rho_m}^{\rho_m}\Phi(\underline{T},\underline{X})dX_m\,.$$

In order to perform this integration, we recall expression (3). Writing the hypergeometric function as a series

$$\sum_{k=0}^{\infty}\frac{(m+\frac{\alpha-1}{2})_k(m+\frac{\alpha}{2})_k}{k!(\alpha+m+1)_k}\frac{(|\underline{T}|^2-|\underline{X}|^2)^k}{T_1^{2k}} = \sum_{k=0}^{\infty}c_k\frac{(|\underline{T}|^2-|\underline{X}|^2)^k}{T_1^{2k}}\,,$$

the integral reduces, up to a constant, to:

$$\Phi(\underline{T},\underline{X}_m) \sim \sum_{k=0}^{\infty}\frac{c_k}{T_1^{\alpha+2m+2k-1}}\int_{-\rho_m}^{\rho_m}(|\underline{T}|^2-|\underline{X}|^2)^{\alpha+m+k}dX_m\,.$$

Since

$$|\underline{T}|^2-|\underline{X}|^2 = (|\underline{T}|^2-|\underline{X}_m|^2)\left(1-\frac{X_m^2}{|\underline{T}|^2-|\underline{X}_m|^2}\right),$$

we get the following expression for $\Phi(\underline{T},\underline{X}_m)$:

$$2\sum_{k=0}^{\infty}\frac{c_k(|\underline{T}|^2-|\underline{X}_m|^2)^{\alpha+m+k}}{T_1^{\alpha+2m+2k-1}}\int_0^{\rho_m}\left(1-\frac{X_m^2}{|\underline{T}|^2-|\underline{X}_m|^2}\right)^{\alpha+m+k}dX_m\,.$$

Under the conditions on α stated above, the integral can for all $k \in \mathbf{N}$ be reduced to a Beta integral so that we find:

$$\Phi(\underline{T},\underline{X}_m) \sim \sum_{k=0}^{\infty}\frac{c_k(|\underline{T}|^2-|\underline{X}_m|^2)^{\alpha+m+k+\frac{1}{2}}}{T_1^{\alpha+2m+2k-1}}B\left(\frac{1}{2},\alpha+m+k+1\right).$$

Since

$$B\left(\frac{1}{2},\alpha+m+k+1\right)c_k = \pi^{\frac{1}{2}}\frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+m+\frac{3}{2})}\frac{(m+\frac{\alpha-1}{2})_k(m+\frac{\alpha}{2})_k}{k!(\alpha+m+\frac{3}{2})_k}\,,$$

we thus get:

$$\Phi(\underline{T}, \underline{X}_m) = \pi^{\frac{1}{2}} \frac{2^{1-\alpha-2m}\Gamma(\alpha+2m-1)}{\Gamma(m-1)\Gamma(\alpha+m+\frac{3}{2})} \frac{(|\underline{T}|^2 - |\underline{X}_m|^2)^{\alpha+m+\frac{1}{2}}}{T_1^{\alpha+2m-1}}$$
$$F\left(m+\frac{\alpha-1}{2}, m+\frac{\alpha}{2}; \alpha+m+\frac{3}{2}; \frac{|\underline{T}|^2 - |\underline{X}_m|^2}{T_1^2}\right).$$

From this, it immediately follows that the $(\alpha+1)$-homogeneous solution for the ultrahyperbolic Dirac equation

$$\partial_{m,m-1} E_T^{(\alpha+1)}(\underline{T}, \underline{X}_m) = (T_1)_+^{\alpha+2m-2}\delta(\underline{T}_1)\delta(\underline{X}_m)$$

with support in $(SLR_{m-1,m-1} \times \mathbf{R}\varepsilon_1) \cap TLR_{m,m-1}$ is given by

$$E_T^{(\alpha+1)}(\underline{T}, \underline{X}_m) = \frac{e^{i\pi\frac{m}{2}}}{\pi^{m-\frac{3}{2}}}\Gamma\left(m-\frac{1}{2}\right)(|\underline{T}|^2 - |\underline{X}_m|^2)^{\frac{\alpha+1}{2}}$$
$$((\underline{\tau}, \underline{\xi}_m)_T D_\alpha^{m-\frac{1}{2}}((\underline{\tau}, \underline{\xi}_m)_T \cdot \varepsilon_1) - \varepsilon_1 D_{\alpha+1}^{m-\frac{1}{2}}((\underline{\tau}, \underline{\xi}_m)_T \cdot \varepsilon_1)).$$

The very same argument can then be repeated, and in this way an expression is obtained for the fundamental solution of the ultrahyperbolic Dirac equation on the *timelike* manifold $M_T(m,q)$ for $q < m$ or, by substituting $\underline{T} \leftrightarrow \underline{X}$, on the *spacelike* manifold $M_T(p,m)$ for $p < m$. It should however be noted that the method of descent only allows us to lower the amount of spatial (resp. temporal) dimensions in the timelike (resp. spacelike) region. This leads to the following:

- Consider the ultrahyperbolic Dirac equation on $M_T(m,q) \subset \mathbf{R}^{m,q}$ with singularities on the ray through $\underline{\sigma} \in S^{m-1}$:

  $$\partial_{m,q} E_T^{(\alpha)}(\underline{T}, \underline{X}) = (\langle \underline{T}, \underline{\sigma}\rangle)_+^{\alpha+m+q-2}\delta(\underline{T} \wedge \underline{\sigma})\delta(\underline{X}).$$

  The fundamental solution $E_T^{(\alpha)}(\underline{T}, \underline{X})$, whose support is given by

  $$\mathrm{supp}(E_T^{(\alpha)}(\underline{T}, \underline{X})) = (SLR_{m-1,q} \times \mathbf{R}\underline{\sigma}) \cap TLR_{m,q},$$

  is defined as

  $$E_T^{(\alpha)}(\underline{T}, \underline{X}) = \frac{e^{i\pi\frac{q+1}{2}}}{\pi^{\frac{m+q}{2}-1}}\Gamma\left(\frac{m+q}{2}\right)(|\underline{T}|^2 - |\underline{X}_m|^2)^{\frac{\alpha}{2}}$$
  $$((\underline{\tau}, \underline{\xi})_T D_{\alpha-1}^{\frac{m+q}{2}}((\underline{\tau}, \underline{\xi})_T \cdot \underline{\sigma}) - \underline{\sigma} D_\alpha^{\frac{m+q}{2}}((\underline{\tau}, \underline{\xi})_T \cdot \underline{\sigma})),$$

  where $(\underline{\tau}, \underline{\xi})_T$ denotes the vector $(|\underline{T}|^2 - |\underline{X}|^2)^{-\frac{1}{2}}(\underline{T}, \underline{X})$ with $(\underline{T}, \underline{X})$ an element of $TLR_{m,q}$.

- Consider the ultrahyperbolic Dirac equation on $M_S(p, m) \subset \mathbf{R}^{p,m}$ with singularities on the ray through $\underline{\eta} \in S^{m-1}$:

$$\partial_{p,m} E_S^{(\alpha)}(\underline{T}, \underline{X}) = (\langle \underline{X}, \underline{\eta} \rangle)_+^{\alpha+m+p-2} \delta(\underline{X} \wedge \underline{\eta}) \delta(\underline{T}) .$$

  The fundamental solution $E_S^{(\alpha)}(\underline{T}, \underline{X})$, whose support is given by

$$\mathrm{supp}(E_S^{(\alpha)}(\underline{T}, \underline{X})) = (TLR_{p,m-1} \times \mathbf{R}\underline{\eta}) \cap SLR_{p,m} ,$$

  is defined as

$$E_S^{(\alpha)}(\underline{T}, \underline{X}) = \frac{e^{i\pi \frac{p+1}{2}}}{\pi^{\frac{m+p}{2}-1}} \Gamma\left(\frac{m+p}{2}\right) (|\underline{X}|^2 - |\underline{T}|^2)^{\frac{\alpha}{2}}$$
$$((\underline{\tau}, \underline{\xi})_S D_{\alpha-1}^{\frac{m+p}{2}}((\underline{\tau}, \underline{\xi})_S \cdot \underline{\eta}) - \underline{\eta} D_{\alpha}^{\frac{m+p}{2}}((\underline{\tau}, \underline{\xi})_S \cdot \underline{\eta})) ,$$

  where $(\underline{\tau}, \underline{\xi})_S$ denotes the vector $(|\underline{X}|^2 - |\underline{T}|^2)^{-\frac{1}{2}}(\underline{T}, \underline{X})$ with $(\underline{T}, \underline{X})$ an element of $SLR_{p,m}$.

We conclude this section by noting that the fundamental solution for the *conformal* Dirac operator on the timelike (resp. spacelike) ultrahyperbolic manifold can easily be found by putting $\alpha = -\frac{m+q}{2}$ (resp. $\alpha = -\frac{p+m}{2}$). This follows from the fact that the Dirac operator *on* the nullcone in $\mathbf{R}^{p,q}$, which is only defined for these respective values for α, gives rise to the conformally invariant Dirac operator on $\mathbf{R}^{p-1,q-1}$, as was explained in [21].

References

[1] BRACKX, F., DELANGHE, R., SOMMEN, F., *Clifford Analysis*, Research Notes in Mathematics No. 76, Pitman, London (1982).

[2] CALDERBANK, D., Dirac operators and Clifford analysis on manifolds with boundary, Max Planck Institute for Mathematics, Bonn, Preprint number 96–131 (1996).

[3] CEREJEIRAS, P., KÄHLER, U., SOMMEN. F., Clifford Analysis on Projective Hyperbolic Space, J. Nat. Geom. **22** (2002), 19–34.

[4] CNOPS, J., *An Introduction to Dirac Operators on Manifolds*, Progress Notes in Mathematical Physics Vol. 24, Birkhäuser, Boston (2002).

[5] CNOPS, J., Manifolds with and without embeddings, Clifford algebras and their application in mathematical physics (Aachen, 1996), 57–65.

[6] DELANGHE, R., Clifford Analysis: History and Perspective, Comp. Meth. and Func. Theory **1** (2001), 107–153.

[7] DELANGHE, R., SOMMEN, F., SOUČEK, V., *Clifford Algebra and Spinor-Valued Functions*, Kluwer Academic Publishers, Dordrecht (1992).

[8] DURAND, L., FISHBANE, P. M., SIMMONS, L. M. JR., Expansion formulas and addition theorems for Gegenbauer functions, J. Math. Phys. **17** (1976), 1933–1948.

[9] EELBODE, D., Arbitrary complex powers of the Dirac operator on the hyperbolic unit ball, Ann. Acad. Scient. Fenn. **29** (2004), 367–381.

[10] EELBODE, D., SOMMEN, F., The Photogenic Cauchy Transform, Journ. Geom. Phys. **54** (2005), 339–354.

[11] EELBODE, D., SOMMEN, F., Taylor Series on the Hyperbolic Unit Ball, Bull. Belg. Math. Soc.-Simon Stevin **11** (2004), 719–737.

[12] ERDELYI, A., MAGNUS, W., OBERHETTINGER, F., TRICOMI, F. G., *Higher Transcendental Functions*, McGraw-Hill, New York (1953).

[13] GEL'FAND, I. M., SHILOV, G. E., *Generalized Functions: Properties and Operations* (*volume 1*), Academic Press, New York (1964).

[14] GILBERT, J., MURRAY, M. A. M., *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge (1991).

[15] ISHAM, C. J., *Modern Differential Geometry for Physicists*, World Scientific Lecture Notes in Physics, Vol. 61, Singapore (2001).

[16] LEUTWILER, H., Modified quaternionic analysis in $\mathbf{R}^3$, Complex Variables **20** (1991), 19–51.

[17] LAWSON, H. B., MICHELSOHN. M. L., *Spin Geometry*, Princeton University Press, Princeton New Jersey (1989).

[18] PORTEOUS, I., *Topological Geometry* (2nd Edition), Cambridge University Press, New York (1981).

[19] RIESZ, M., L'integrale de Riemann-Liouville et le problème de Cauchy, Acta Math. **81** (1949), 1–223.

[20] ERIKSSON-BIQUE, S-L., LEUTWILER, H., *Hypermonogenic Functions*, *Clifford Algebras and Their Applications in Mathematical Physics*, 287–302, Ryan John et al. (eds) Vol. 2., Birkhäuser, Basel (2002).

[21] SOMMEN, F., Clifford analysis on the Nullcone, submitted.

[22] SOMMEN, F., VAN LANCKER, P., Homogeneous Functions in Euclidean Space, Integr. Transf. Spec. F. **7** No. 3–4 (1998), 285–298.

[23] VAN LANCKER, P., Clifford Analysis on the Unit Sphere, PhD-thesis, Ghent, Belgium (1997).

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