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Splittability of Stellar Singular Fiber with Three Branches

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Abstract. We are concerned with the splittability problem of degenerations with stellar singular fibers. In this paper we give an interesting splitting criterion for such degenerations: if a stellar singular fiber has exactly three branches, and its central component (core) is the projective line, then this degeneration admits a splitting deformation.

1. Introduction

The purpose of the present paper is to show splittability of stellar singular fiber with three branches. If there is a family $\pi_t : M_t \to \Delta$ of degenerations of closed Riemann surfaces such that π_0 has only one normally minimal sigular fiber $\pi_0^{-1}(0)$ and that π_t $(t \neq 0)$ has more than one singular fibers such that they are not obtained as blowing ups of smooth fibers, then the germs of π_0 is called *splittable*. If a singular fiber is not splittable, it is called *atomic*. We are very interested in classification problem of atomic singular fibers, and also splittability problem.

We shall consider splittability problem for stellar singular fiber. The singular fiber X is stellar if it has a core and some branches (Definition 2.1). We assume that the core is a projective line and that the number of branches is 3 or more. In this case, the second author shows that if X has a simple crust then we obtain a splitting family of degeneration X. Here a simple crust is a subdivisor of the singular fiber X satisfying some conditions (Definition 2.5).

In this paper, we show that if the number of branches is exactly three then there exists at least one simple crust of X, and hence the degeneration is not atomic (Theorem 4.1). To prove this theorem, it is sufficient to construct a simple crust of X combinatrially. To show it is a simple crust, we prepare an arithemetic lemma (Lemma 3.2). We call this lemma *sub-continued-fractions lemma*.

In section 5, we consider the case where X has more than 3 branches. We find many examples of stellar singular fibers which don't have a simple crust. We use software *Splitica*, which is developed by the first author, to find such examples.

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2. Preparation

In this section we introduce a stellar singular fiber and a simple crust. These two concepts are presented by Takamura [T3]. Simple crust is the main idea of Takamura's splitting theory.

Let $\pi : M \to \Delta$ be a linear degeneration of complex curve such that the genus of a general fiber is positive. (*A linear degeneration* is defined in p. 273, [T3].) In the sequel we assume that the degeneration $\pi : M \to \Delta$ is normally minimal.

DEFINITION 2.1 (Stellar singular fiber). A singular fiber X of a linear degeneration of complex curve is called *stellar* if the following five conditions (ST1), (ST2), (ST3), (ST4), and (ST5) are satisfied:

(ST1) X is decomposed as follows:

$$X = m_0\Theta_0 + \sum_{j=1}^{b} B^{(j)} ,$$

where b is a positive integer, and $B^{(j)} = \sum_{i=1}^{\lambda_j} m_i^{(j)}\Theta_i^{(j)}$. Here Θ_0 and $\Theta_i^{(j)}$ ($j = 1, 2, \cdots, b, i = 1, 2, \cdots, \lambda_j$) are irreducible components. m_0 and $m_i^{(j)}$ are their multiplicities respectively. We call Θ_0 a core of X. We call $m_0\Theta_0 + B^{(j)}$ a branch of X for $j = 1, 2, \cdots, b$.

(ST2) $\Theta_i^{(j)}$ is biholomorphic to $\mathbf{P}^1$. $\Theta_i^{(j)}$ and $\Theta_{i+1}^{(j)}$ intersect transversally at one point for $j = 1, 2, \cdots, b, i = 1, 2, \cdots, \lambda_j - 1$. Θ_0 and $\Theta_1^{(j)}$ intersect transversally at one point p_j for $j = 1, 2, \cdots, b$, such that p_i's are mutually distinct.

(ST3) For $j = 1, 2, \cdots, b$, $m_i^{(j)}$'s satisfy

$$m_0 > m_1^{(j)} > m_2^{(j)} > \cdots > m_{\lambda_j}^{(j)} > 0 .$$

(ST4) Let r_0 and $r_i^{(j)}$ ($i = 1, 2, \cdots, \lambda_j$) be defined by the followings:

$$r_0 = \frac{m_1^{(1)} + \cdots + m_1^{(b)}}{m_0} ,$$
$$r_i^{(j)} = \begin{cases} \dfrac{m_{i-1}^{(j)} + m_{i+1}^{(j)}}{m_i^{(j)}} & (i = 1, 2, \cdots, \lambda_j - 1) , \\ \dfrac{m_{i-1}^{(j)}}{m_i^{(j)}} & (i = \lambda_j) . \end{cases}$$

r_0 and $r_i^{(j)}$ are integers and $r_i^{(j)} \geq 2$. Here $m_0^{(j)} = m_0$.

(ST5) One of the following four conditions is satisfied.

- (ST5-1) genus$(\Theta_0) = 0$, $b \geq 3$, and $m_0 \geq 2$.
- (ST5-2) genus$(\Theta_0) = 0$, and $b = 2$.
- (ST5-3) genus$(\Theta_0) \geq 1$, $b \geq 2$, and $m_0 \geq 2$.
- (ST5-4) genus$(\Theta_0) \geq 1$, and $b = 0$, and $m_0 \geq 2$.

Next we prepare definition of a simple crust.

Let X be a stellar singular fiber. Suppose that a subdivisor Y of X is represented by the following.

$$Y = n_0\Theta_0 + \sum_{j=1}^{b} S^{(j)}\,,$$

where

$$S^{(j)} = \sum_{i=1}^{e_j} n_i^{(j)}\Theta_i^{(j)} \quad \text{or} \quad S^{(j)} = 0\,.$$

(If $S^{(j)} = 0$ then we regard $e_j = 0$.) We assume that $\mathrm{Supp}(Y)$ is connected and that at least one of $S^{(j)}$ is not zero.

DEFINITION 2.2 (Subbranch). $n_0\Theta_0 + S^{(j)}$ is called *a subbranch of* $m_0\Theta_0 + B^{(j)}$ if the following conditions (SB1), (SB2), (SB3) are satisfied:

(SB1) $0 \le e_j \le \lambda_j$ and if $e_j > 0$ then $n_0 \ge n_1^{(j)} \ge \cdots \ge n_{e_j}^{(j)} > 0$.

(SB2) $m_0 > n_0$, and if $e_j > 0$ then and $m_i^{(j)} \ge n_i^{(j)}$ $(i = 1, \cdots, e_j)$.

(SB3) If $e_j > 1$ then

$$\frac{n_{i-1}^{(j)} + n_{i+1}^{(j)}}{n_i^{(j)}} = r_i^{(j)} \quad (i = 1, \cdots, e_j - 1)\,.$$

Here $n_0^{(j)} = n_0$.

REMARK. In Definition 2.2, the case $e_j = 0$ is allowed. In this case, the subbranch is $n_0\Theta_0 + S^{(j)} = n_0\Theta_0$ and we consider that there is a vacant subbranch, and that $n_1^{(j)} = 0$ for convenience sake.

DEFINITION 2.3 (Crust). Suppose that X is a stellar singular fiber and that $\Theta_0 = \mathbf{P}^1$. A subdivisor Y is called *a crust* if the following two conditions are satisfied:

(CR1) For $j = 1, 2, \cdots, b$, $n_0\Theta_0 + S^{(j)}$ is a subbranch of $m_0\Theta_0 + B^{(j)}$.

(CR2)

$$\frac{\sum_j n_1^{(j)}}{n_0} \ge \frac{\sum_j m_1^{(j)}}{m_0}\,.$$

REMARK. If we don't assume that $\Theta_0 = \mathbf{P}^1$, then the definition of a crust differs. In [T3] the condition (CR1) is removed and another condition called 'tensor condition' is added. We call $n_0\Theta_0 + S^{(j)}$ a branch of a crust Y.

For each branch of a crust Y, the type of the branch is defined as follows.

DEFINITION 2.4 (type of subbranch). Fix j. In this definition m_i (resp. n_i, e) denotes $m_i^{(j)}$ (resp. $n_i^{(j)}$, e_j) for simplicity.

(1) If there exists a positive integer ℓ and the following conditions (TA1) and (TA2) are satisfied, then $n_0\Theta_0 + S^{(j)}$ is a type A_ℓ subbranch.

(TA1) $\ell n_i \leq m_i \ (i = 1, 2, \cdots, e)$.

(TA2) $\dfrac{n_{e-1}}{n_e} \geq r_e$.

(2) If there exists a positive integer ℓ and the following conditions (TB1) and (TB2) are satisfied, then $n_0\Theta_0 + S^{(j)}$ is a type B_ℓ subbranch.

(TB1) $\ell n_i \leq m_i \ (i = 1, 2, \cdots, e)$.

(TB2) $\ell = m_e$ and $n_e = 1$.

(3) If there exists a positive integer ℓ and the following conditions (TC1), (TC2), and (TC3) are satisfied, then $n_0\Theta_0 + S^{(j)}$ is a type C_ℓ subbranch.

(TC1) $\ell n_i \leq m_i \ (i = 1, 2, \cdots, e)$.

(TC2) $\frac{n_{e-1}}{n_e}$ is an integer and less than r_e.

(TC3) Let $u := (m_{e-1} - \ell n_{e-1}) - (r_e - 1)(m_e - \ell n_e)$. Then $\frac{\ell}{u}$ is an integer.

(4) If $S^{(j)} = 0$ then we say that the j-th branch of Y is a type O subbranch for convenience sake.

DEFINITION 2.5 (Simple crust). Suppose that X is a stellar singular fiber and that Y is a crust of X. We assume (ST5-1) $\Theta_0 = \mathbf{P}^1$, $b \geq 3$, and $m_0 \geq 2$. If there exists an integer ℓ such that the following condition (SC) is satisfied, then Y is called *a simple crust* of X. We call ℓ *the crust multiplicity*.

(SC) Each branch of Y is of type A_ℓ or of type B_ℓ or of type C_ℓ or type O.

The following theorem is a result by Takamura [T3].

THEOREM 2.6 (Existence of splitting family). *If a linear degeneration π with a star-shaped singular fiber X has a simple crust Y then there exists a splitting family of degeneration π.*

3. Preparation on continued fractions

In this section we prepare some lemmas on continued fractions. Let $r_1, \cdots, r_\lambda$ be integers greater than 1. $[r_1, \cdots, r_\lambda]$ denotes a continued fraction as follows.

$$\begin{aligned}[r_1, \cdots, r_\lambda] &= \frac{1}{r_1} - \frac{1}{r_2} \cdots - \cdots \frac{1}{r_\lambda} \\ &= \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\cdots - \cfrac{1}{r_\lambda}}}} \ .\end{aligned}$$

Let an integer $\langle r_1, \cdots, r_\lambda \rangle$ be defined as follows:

$$\begin{cases} \langle \mathrm{NULL} \rangle &= 1 \,, \\ \langle r_1 \rangle &= r_1 \,, \\ \langle r_1, r_2 \rangle &= r_1 r_2 - 1 (= r_1 \langle r_2 \rangle - \langle \mathrm{NULL} \rangle) \,, \\ \langle r_1, r_2, \cdots, r_\lambda \rangle &= r_1 \langle r_2, \cdots, r_\lambda \rangle - \langle r_3, \cdots, r_\lambda \rangle \quad (\lambda \geq 3) \,. \end{cases}$$

We have a lemma.

LEMMA 3.1. (1) *Suppose that X is a star-shaped singular fiber and that $m_0, m_i^{(j)}, r_i^{(j)}$ are as in Definition 2.1. If $g = GCD(m_0, m_1^{(j)})$ then*

$$m_i^{(j)} = g \langle r_{i+1}^{(j)}, r_{i+2}^{(j)}, \cdots, r_{\lambda_j}^{(j)} \rangle \,.$$

Remark that if $i = \lambda_j$ then $m_i^{(j)} = g \langle \mathrm{NULL} \rangle = g$.

(2)

$$[r_1, \cdots, r_\lambda] = \frac{\langle r_2, \cdots, r_\lambda \rangle}{\langle r_1, \cdots, r_\lambda \rangle} \,,$$

and the right hand side is irreducible. Remark that if $\lambda = 1$ then $[r_1] = \langle \mathrm{NULL} \rangle / \langle r_1 \rangle = 1/r_1$.

(3)

$$\langle r_1, r_2, \cdots, r_\lambda \rangle = \langle r_\lambda, r_{\lambda-1}, \cdots, r_1 \rangle \,.$$

(4) *If $\lambda > 1$ then*

$$\frac{1}{r_1} - \frac{1}{r_2} \cdots - \cdots \frac{1}{r_\lambda - x} = \frac{-\langle r_2, \cdots, r_{\lambda-1} \rangle x + \langle r_2, \cdots, r_\lambda \rangle}{-\langle r_1, \cdots, r_{\lambda-1} \rangle x + \langle r_1, \cdots, r_\lambda \rangle} \,.$$

Remark that if $\lambda = 2$ then

$$\frac{1}{r_1} - \frac{1}{r_2 - x} = \frac{-\langle \mathrm{NULL} \rangle x + \langle r_2 \rangle}{-\langle r_1 \rangle x + \langle r_1, r_2 \rangle} = \frac{-x + r_2}{-r_1 x + \langle r_1, r_2 \rangle} \,.$$

(5) *If w is a rational number such that $0 < w < 1$, then there exist integers $r_1, \cdots, r_\lambda$ greater than 1 such that*

$$[r_1, \cdots, r_\lambda] = w \,.$$

PROOF. The formula (1) follows the definition of $r_i^{(j)}$ in (ST4). (2) and (4) follow the definition of $\langle r_1, \cdots, r_\lambda \rangle$. (5) follows the Euclidean algorithm. (3) is a well-known fact in number theory.

Next we prove sub-continued-fractions lemma.

LEMMA 3.2 (Sub-continued-fractions lemma). *Let $r_1, \cdots, r_m$ be integers more than 1. $[r_1, \cdots, r_m]$ denotes a finite continued fraction:*

$$[r_1, \cdots, r_m] := \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\cdots - \cfrac{1}{r_m}}}}.$$

Let a, b, c be integers more than 1 and satisfy the following.

$$a > c, \quad a > b, \quad b + c > a\,.$$

Let $r_1, \cdots, r_m, r'_1, \cdots, r'_n$ be integers more than 1 such that

$$\frac{b}{a} = [r_1, \cdots, r_m], \quad \frac{c}{a} = [r'_1, \cdots, r'_n]\,.$$

Then there exist two integers m', n' $(1 \le m' \le m, 1 \le n' \le n)$ such that

$$[r_1, \cdots, r_{m'}] + [r'_1, \cdots, r'_{n'}] = 1\,.$$

In order to show this lemma, we need to prepare some arithmetic lemmas. First we prove the following lemma.

LEMMA 3.3. (1) *If $[r_1, \cdots, r_m] + [r'_1, \cdots, r'_n] = 1$ then $[2, r_1, \cdots, r_m] + [r'_1 + 1, r'_2, \cdots, r'_n] = 1$.*

(2) *If $[r_1, \cdots, r_m] + [r'_1, \cdots, r'_n] = 1$, $m > 1$ and $r_1 = 2$, then $r'_1 \ge 3$ and $[r_2, \cdots, r_m] + [r'_1 - 1, \cdots, r'_n] = 1$.*

PROOF. (1) If $[r_1, \cdots, r_m] = \frac{p}{q}$ then $[2, r_1, \cdots, r_m] = \frac{q}{2q-p}$. From the assumption, $[r'_1, \cdots, r'_n] = \frac{q-p}{q}$ and hence $[r'_1 + 1, r'_2, \cdots, r'_m] = \frac{q-p}{2q-p}$.

(2) Suppose $[r_1, \cdots, r_m] = \frac{p}{q}$. If $r_1 = 2$ then $2p > q$ by definition of continued fraction. It follows that $q - p < \frac{q}{2}$. r'_1 satisfies $(q-p)(r'_1 - 1) < q \le (q-p)r'_1$. Therefore we have $r'_1 \ge 3$. Under the assumption of $r_1 = 2$, we obtain $[r_2, \cdots, r_m] = \frac{2p-q}{p}$, $[r'_1 - 1, \cdots, r'_n] = \frac{q-p}{p}$. q.e.d

If $r_k = r_{k+1} = \cdots = r_{k+a-1} = 2$ in $[r_1, \cdots, r_m]$ then we denote it

$$[r_1, \cdots, r_{k-1}, (2)^a, r_{k+a}, \cdots, r_m]\,.$$

Here we remarkd that $(2)^a$ must be distinguished from a power 2^a of 2. For example, $(2)^2 \neq 4$. If r is odd, $a_1, a_3, \cdots, a_r$ are integers greater than 1, and $a_2, a_4, \cdots, a_{r-1}$ are non-negative integers, then $[a_1, (2)^{a_2}, a_3, (2)^{a_4}, \cdots, (2)^{a_{r-1}}, a_r]$ means

$$[a_1, \overbrace{2, \cdots, 2}^{a_2}, a_3, \overbrace{2, \cdots, 2}^{a_4}, \cdots, \overbrace{2, \cdots, 2}^{a_{r-1}}, a_r]\,.$$

If $a = 0$ then $(2)^0$ means a sequence with no element. For example,

$$[3, (2)^0, 4, (2)^2, 3] = [3, 4, 2, 2, 3]\,.$$

LEMMA 3.4. *Let $a_1, a_2, \cdots, a_r$ be non-negative integers.*
(1)

$$\begin{aligned}&[a_1 + 3, (2)^{a_2}, a_3 + 3, (2)^{a_4}, \cdots, (2)^{a_{r-1}}, a_r + 3]\\ &\quad +[2, (2)^{a_1}, a_2 + 3, (2)^{a_3}, a_4 + 3, \cdots, a_{r-1} + 3, (2)^{a_r}, 2] = 1\,.\end{aligned}$$

(2)

$$\begin{aligned}&[a_1 + 3, (2)^{a_2}, a_3 + 3, (2)^{a_4}, \cdots, a_{r-1} + 3, (2)^{a_r}, 2]\\ &\quad +[2, (2)^{a_1}, a_2 + 3, (2)^{a_3}, a_4 + 3, \cdots, (2)^{a_{r-1}}, a_r + 3] = 1\,.\end{aligned}$$

PROOF. First we have $[3] + [2, 2] = 1$. After that, using Lemma 3.3 inductively, we show these equations. in (1) case, r is an odd number. If $r = 1$ then the equation is $[3 + a_1] + [2, 2^{a_1}, 2] = 1$. If $a_1 = 0$ then $[3] + [2, 2] = 1$ is true. Using Lemma 3.3(1), $[3 + a_1 + 1] + [2, 2, 2^{a_1}, 2] = 1$ follows $[3 + a_1] + [2, 2^{a_1}, 2] = 1$. This complete the case $r = 1$. For $r \geq 3$, assume that

$$\begin{aligned}&[a_3 + 3, 2^{a_4}, \cdots, 2^{a_{r-1}}, a_r + 3]\\ &\quad +[2, 2^{a_3}, a_4 + 3, \cdots, a_{r-1} + 3, 2^{a_r}, 2] = 1\,.\end{aligned}$$

Using Lemma 3.3(1),

$$\begin{aligned}&[\overbrace{2, \cdots, 2}^{a_2+1}, a_3 + 3, 2^{a_4}, \cdots, 2^{a_{r-1}}, a_r + 3]\\ &\quad +[2 + a_2 + 1, 2^{a_3}, a_4 + 3, \cdots, a_{r-1} + 3, 2^{a_r}, 2] = 1\,.\end{aligned}$$

Using Lemma 3.3(1) again,

$$\begin{aligned}&[2 + a_1 + 1, \overbrace{2, \cdots, 2}^{a_2}, a_3 + 3, 2^{a_4}, \cdots, 2^{a_{r-1}}, a_r + 3]\\ &\quad +[\overbrace{2, \cdots, 2}^{a_1+1}, 2 + a_2 + 1, 2^{a_3}, a_4 + 3, \cdots, a_{r-1} + 3, 2^{a_r}, 2] = 1\,,\end{aligned}$$

$$\begin{aligned}&[a_1 + 3, 2^{a_2}, a_3 + 3, 2^{a_4}, \cdots, 2^{a_{r-1}}, a_r + 3]\\ &\quad +[2, 2^{a_1}, a_2 + 3, 2^{a_3}, a_4 + 3, \cdots, a_{r-1} + 3, 2^{a_r}, 2] = 1\,.\end{aligned}$$

For example, we show that

$$[3+0, 2^0, 3+1, 2^2, 3+0] + [2, 2^0, 3+0, 2^1, 3+2, 2^0, 2] = 1$$

The left side is equivalent to

$$[3, 4, 2, 2, 3] + [2, 3, 2, 5, 2].$$

From $[3] + [2, 2] = 1$, we have followings:

$$\begin{aligned} & [3] + [2, 2] = 1 \\ \Rightarrow\ & [2, 3] + [3, 2] = 1 \\ \Rightarrow\ & [2, 2, 3] + [4, 2] = 1 \\ \Rightarrow\ & [2, 2, 2, 3] + [5, 2] = 1 \\ \Rightarrow\ & [3, 2, 2, 3] + [2, 5, 2] = 1 \\ \Rightarrow\ & [4, 2, 2, 3] + [2, 2, 5, 2] = 1 \\ \Rightarrow\ & [2, 4, 2, 2, 3] + [3, 2, 5, 2] = 1 \\ \Rightarrow\ & [3, 4, 2, 2, 3] + [2, 3, 2, 5, 2] = 1\,. \end{aligned}$$

In the same way we can show (2). q.e.d.

As a corollary of Lemma 3.4, we obtain the following Lemma 3.5.

LEMMA 3.5. *If* $[r_1, \cdots, r_m] + [r'_1, \cdots, r'_n] = 1$ *then* $[r_1, \cdots, r_m, 2] + [r'_1, \cdots, r'_{n-1}, r'_n + 1] = 1$.

PROOF. By using Lemma 3.4, We see that if $[r_1, \cdots, r_m] + [r'_1, \cdots, r'_n] = 1$ then $[r_m, \cdots, r_1] + [r'_n, \cdots, r'_1] = 1$. Using 3.3, we have the conclusion easily.

To show sub-continued-fraction lemma, we need a formula of estimation of sub-continued-fractions.

LEMMA 3.6. *Let* $r_1, \cdots, r_m$ *be integers more than 1. Let a set* $S = S(r_1, \cdots, r_m)$ *be defined as follows:*

$$\begin{aligned} & S(r_1, \cdots, r_m) \\ & = \left\{ [r_1, \cdots, r_m, r_{m+1}, \cdots, r_n] \;\middle|\; \begin{array}{l} n > m, \\ r_{m+1}, r_{m+2}, \cdots, r_n \text{ are integers more than } 1 \end{array} \right\}. \end{aligned}$$

(1) *If* $r_m > 2$ *then*

$$S(r_1, \cdots, r_m) = \{x \in \mathbf{Q} \mid [r_1, \cdots, r_m] \le x < [r_1, \cdots, r_m - 1]\}\,.$$

(2) *If there exists an integer* p *(*$2 \le p \le m$*) such that* $r_p = r_{p+1} = \cdots = r_m = 2$ *and* $r_{p-1} > 2$ *then*

$$S(r_1, \cdots, r_m) = \{x \in \mathbf{Q} \mid [r_1, \cdots, r_m] \le x < [r_1, \cdots, r_{p-1} - 1]\}\,.$$

(3) *If* $r_1 = r_2 = \cdots = r_m = 2$ *then*

$$S(r_1, \cdots, r_m) = \{x \in \mathbf{Q} \mid [r_1, \cdots, r_m] \le x < 1\}\,.$$

PROOF. There exists a real number $0 \leq x < 1$ such that

$$[r_1, \cdots, r_m, r_{m+1}, \cdots] = \frac{1}{r_1} - \frac{1}{r_2} - \cdots - \frac{1}{r_m - x}\,.$$

If we regard the right hand side of this equation as a function of x, it is easy to show that it is a increasing function. The conclusion (1) follows. We show (2) and (3) in the same way but we need to check the upper bound (in the case $x = 1$.) It is sufficient to show the following.

$$\frac{1}{r_{p-1}} - \frac{1}{2} - \cdots - \frac{1}{2-1} = \frac{1}{r_{p-1} - 1}\,.$$

This is an easy formula.

PROOF OF LEMMA. 3.2. Let us start the proof of Lemma 3.2. Let $s_1^{(i)}, \cdots, s_{n(i)}^{(i)}$ be integers more than 1 such that

$$1 - [r_1, \cdots, r_i] = [s_1^{(i)}, \cdots, s_{n(i)}^{(i)}]\,.$$

Remark that an integer $n(i)$ is the length of this continued fraction. First we determine the interval $S(s_1^{(1)}, \cdots, s_{n(i)}^{(i)})$.

(Case 1) Suppose that $i = 1$. If $r_1 = 2$ then $1 - [2] = [2]$ and $n(1) = 1$. Hence $s_1^{(1)} = 2$ and $S(s_1^{(1)}) = \{x \in \mathbf{Q} | [s_1^{(1)}] \leq x < 1\}$.

If $r_1 > 2$ then $1 - [r_1] = \frac{r_1 - 1}{r_1} = [2, \cdots, 2]$ and $n(1) = r_1 - 1$. Therefore $S(s_1^{(1)}, \cdots, s_{n(i)}^{(i)}) = [[s_1^{(1)}, \cdots, s_{n(1)}^{(1)}], 1)$.

(Case 2) Suppose that $i > 1$. If $r_i = 2$ then using Lemma 3.4 we obtain that

$$\begin{aligned}
&n(i) = n(i-1),\\
&s_1^{(i)} = s_1^{(i-1)}, \cdots, s_{n(i)-1}^{(i)} = s_{n(i)-1}^{(i-1)}\,,\\
&s_{n(i)}^{(i)} = s_{n(i)}^{(i-1)} + 1\,.
\end{aligned}$$

It follows that

$$S(s_1^{(i)}, \cdots, s_{n(i)}^{(i)}) = [[s_1^{(i)}, \cdots, s_{n(i)}^{(i)}], [s_1^{(i-1)}, \cdots, s_{n(i-1)}^{(i-1)}])\,. \tag{3.1}$$

If $r_i > 2$, we show the same conclusion in the similar way.

The formula (3.1) implies

$$\bigcup_i S(s_1^{(i)}, \cdots, s_{n(i)}^{(i)}) = [1 - \frac{b}{a}, 1)\,. \tag{3.2}$$

Remark that $[s_1^{(m)}, \cdots, s_{n(m)}^{(m)}] = 1 - [r_1, \cdots, r_m] = 1 - \frac{b}{a}$.

From $a - b < c < a$, we have

$$1 - \frac{b}{a} < \frac{c}{a} < 1\,.$$

Hence there exists i such that

$$\frac{c}{a} \in S(s_1^{(i)}, \cdots, s_{n(i)}^{(i)}) .$$

It means that $\frac{c}{a} = [s_1^{(i)}, \cdots, s_{n(i)}^{(i)}, \cdots]$ and it is sufficient to take $m' = i$ and $n' = n(i)$. In fact,

$$\begin{aligned} [r'_1, \cdots, r'_{n'}] &= [s_1^{(i)}, \cdots, s_{n(i)}^{(i)}] \\ &= 1 - [r_1, \cdots, r_i] \\ &= 1 - [r_1, \cdots, r_{m'}] . \end{aligned}$$

This completes the proof of 3.2.

4. Main result

We state our main result.

THEOREM 4.1. *Suppose that $\pi : M \to \Delta$ is a linear degeneration of curves with a stellar singular fiber X such that $\Theta_0 = \mathbf{P}^1$, $b = 3$, $r_0 \geq 2$, and $m_0 \geq 2$. Then there exists a simple crust Y of X. Hence there is a splitting family of X and the degeneration π is not atomic.*

We have the following lemma and it is enough to consider the case $r_0 = 2$.

LEMMA 4.2. *$r_0 < b$. Hence $r_0 \geq 2$ implies $r_0 = 2$ in case $b = 3$.*

PROOF. $\sum_j m_1^{(j)} < bm_0$ follows $m_1^{(j)} < m_0$, and we have $r_0 < b$, So $r_0 \geq 2$ implies $r_0 = 2$ in case $b = 3$. q.e.d.

REMARK. If the singular fiber X satisfies $\Theta_0 = \mathbf{P}^1$, $b = 3$, $r_0 = 1$, and $m_0 \geq 2$, then Takamura [T3] shows that there exists a splitting family of X.

We shall restate our result in a topological way.

COROLLARY 4.3. *Let $\pi : M \to \Delta$ be a degeneration with one singular fiber X and $[\rho] \in \mathcal{M}$ be its monodroy. Here $\mathcal{M}$ is the mapping class group of a general fiber $F = \pi^{-1}(*)$. If the following conditions are satisfied, then there exists a splitting family of the degeneration.*
(1) *A group generated by ρ is a cyclic group of finite order.*
(2) *The orbit space $F/\langle\rho\rangle$ has exactly three singular points and $F/\langle\rho\rangle$ is homeomorphic to a 2-sphere.*

In the sequel, we prove the main theorem 4.1.

We may assume $m_1^{(1)} \geq m_1^{(2)} \geq m_1^{(3)}$. First we consider the case where two of $m_1^{(j)}$ are equal to each other. In this case, we can easily construct a simple crust.

In fact, suppose $m_1^{(1)} = m_1^{(2)}$. Let a crust Y be as follows.

$$Y = \Theta_0 + \Theta_1^{(1)} + \Theta_1^{(2)}$$

and let the crust multiplicity ℓ be $m_1^{(1)}$. The sub-divisor Y has two non-zero subbranches and these two subbranches are of type B_ℓ by definition. Hence Y is a simple crust. In the same way, if $m_1^{(2)} = m_1^{(3)}$ then $Y = \Theta_0 + \Theta_1^{(2)} + \Theta_1^{(3)}$ ($\ell = m_1^{(2)}$) is a simple crust.

In the rest of this section, we assume $m_1^{(1)} > m_1^{(2)} > m_1^{(3)}$. From the formula $m_1^{(1)} + m_1^{(2)} + m_1^{(3)} = 2m_0$, we have:

$$m_1^{(1)} > \frac{2}{3} m_0 \,. \tag{4.1}$$

Let k be $m_0 - m_1^{(1)}$. From (4.1),

$$k < \frac{1}{3} m_0 \,. \tag{4.2}$$

Next let an integer h be $\lfloor m_0/k \rfloor$. Here $\lfloor \cdot \rfloor$ denotes the Gauss's symbol. From (4.2) we obtain

$$h \geq 3 \,, \tag{4.3}$$

and

$$m_0 \geq hk.$$

It follows that

$$m_i^{(1)} = m_0 - ik \quad (i = 1, \cdots, h-1) \,.$$

We state the following lemma.

LEMMA 4.4. *Let j be an integer such that $2 \leq j < h$. Then*

$$j\Theta_0 + \sum_{i=1}^{h-j} j\Theta_i^{(1)}$$

gives a subbranch of type C_k.

PROOF. The condition (TC1) is satisfied clearly. In fact, $m_i^{(1)} - kn_i^{(1)} = m_0 - ik - kj \geq m_0 - (h-j)k - kj = m_0 - hk \geq 0$. The length e of $S^{(1)}$ is $h - j$. Because $n_{e-1}^{(1)}/n_e^{(1)} = j/j = 1$ and $r_e = 2$, the condition (TC2) is satisfied. And we have

$$\begin{aligned} u &= (m_{e-1} - kn_{e-1}) - (r_e - 1)(m_e - kn_e) \\ &= (m_0 - (e-1)k) - kj - (2-1)((m_0 - ek) - kj) \\ &= k \,. \end{aligned}$$

It follows that the condition (TC3) is satisfied. This completes the proof of Lemma 4.4.

We will construct a simple crust Y such that the first subbranch $S^{(1)}$ is given as above. And we will show that we can take the second subbranch $S^{(2)}$ and the third subbranch $S^{(3)}$ of type A_k. In order to show this fact, we need *sub-continued-fractions lemma*.

Using sub-continued-fraction lemma, we show Theorem 4.1 as follows. Let $a = m_0$, $b = m_1^{(2)}$, $c = m_1^{(3)}$. Remark that these are integers greater than 1. In fact, $a \geq 2$ because of the assumption of m_0. In order to show $c > 1$, assume that $c = m_1^{(3)} = 1$. Solving $m_1^{(1)} + m_1^{(2)} + 1 = 2m_0$ and $m_1^{(1)} > m_1^{(2)} > 1$, we have $2m_1^{(1)} \geq 2m_0 - 1$. This contradicts to $m_1^{(1)} < m_0$.

By definition of a stellar sigular fiber, we have $a > b$, and $a > c$. Solving $m_1^{(1)} + m_1^{(2)} + m_1^{(3)} = 2m_0$ and $m_0 > m_1^{(1)}$, we have $b + c > a$. Using Lemma 3.1(2), we get the following lemma:

LEMMA 4.5. *Let $r_i^{(j)}$ be as in Definition 2.1(ST4). If $\frac{b}{a} = [r_1, \cdots, r_m]$, $\frac{c}{a} = [r'_1, \cdots, r'_n]$ as in Lemma 3.2, then we have*

$$r_i = r_i^{(2)}, \quad r'_i = r_i^{(3)}. \tag{4.4}$$

PROOF. $r_i^{(2)}$'s and $r_i^{(3)}$'s are uniquely determined from $m_0 = a$. $m_1^{(2)} = b$, $m_1^{(3)} = c$, by

$$m_{i+1}^{(j)} = \begin{cases} r_i^{(j)} m_i^{(j)} - m_{i-1}^{(j)} & (i = 1, 2, \cdots, \lambda_j - 1), \\ r_i^{(j)} m_i^{(j)} & (i = \lambda_j). \end{cases}$$

On the other hand, if $g = GCD(a, b) = 1$ then $a = g\langle r_1, \cdots, r_{\lambda_2}\rangle$ and $b = g\langle r_2, \cdots, r_{\lambda_2}\rangle$, and r_i satisfy

$$\langle r_i, \cdots, r_{\lambda_2}\rangle = \begin{cases} r_i \langle r_{i+1}, \cdots, r_{\lambda_2}\rangle - \langle r_{i+2}, \cdots, r_{\lambda_2}\rangle & (i = 1, \cdots, \lambda_2 - 2), \\ r_{\lambda_2 - 1}\langle r_{\lambda_2}\rangle - \langle \text{NULL}\rangle & (i = \lambda_2 - 1), \\ r_{\lambda_2}\langle \text{NULL}\rangle = r_{\lambda_2} & (i = \lambda_2). \end{cases}$$

From Lemma 3.1 we obtain that $m_i^{(j)} = g\langle r_{i+1}^{(j)}, r_{i+2}^{(j)}, \cdots, r_{\lambda_j}^{(j)}\rangle$, and this implies $r_i = r_i^{(2)}$. In the similar way we can show $r'_i = r_i^{(3)}$.

Lemma 4.2 implies that there exist two integers m' and n' such that

$$[r_1, \cdots, r_{m'}] + [r'_1, \cdots, r'_{n'}] = 1.$$

We have the followings from Lemma 3.1(2).

$$\langle r_1, \cdots, r_{m'}\rangle = \langle r'_1, \cdots, r'_{n'}\rangle,$$
$$\langle r_2, \cdots, r_{m'}\rangle + \langle r'_2, \cdots, r'_{n'}\rangle = \langle r_1, \cdots, r_{m'}\rangle.$$

Let $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ be defined as follows.

$$\begin{aligned} S^{(1)} &= \textstyle\sum_{i=1}^{h-q} q\Theta_i^{(1)}\,, \\ S^{(2)} &= \textstyle\sum_{i=1}^{m'-1}\langle r_{i+1},\cdots,r_{m'}\rangle\Theta_i^{(2)} + \Theta_{m'}^{(2)}\,, \\ S^{(3)} &= \textstyle\sum_{i=1}^{n'-1}\langle r'_{i+1},\cdots,r'_{n'}\rangle\Theta_i^{(3)} + \Theta_{n'}^{(3)}\,. \end{aligned}$$

Here $q := \langle r_1,\cdots,r_{m'}\rangle$. Let Y be as follows.

$$Y = q\Theta_0 + S^{(1)} + S^{(2)} + S^{(3)}\,. \tag{4.5}$$

To show that Y is a simple crust, we show some lemmas.

LEMMA 4.6. *$q \le h-1$, where $h = \lfloor\frac{m_0}{k}\rfloor$ and $k = m_0 - m_1^{(1)}$*

PROOF. Let $p = \langle r_2,\cdots,r_{m'}\rangle$. We consider the following 4 cases: (case 0) $q = 2$, (Case 1) $1-\frac{b}{a} \le \frac{1}{q+1}$, (Case 2) $p = q-1$ or $p = 1$, (Case 3) $1 < p < q-1$. These conditions are not exclusive to each other, but we will prove in each case under the assumption that the previous cases are proved. The proof in Case 0 is obvious. The proof in Case 1 is obtained from the condition of $m_1^{(j)}$. The proofs in Case 2 and 3 is obtained from the lemma of continued fractions. This proof is only combinatrical, but the authors believe that we have another smart and arithemetic (or algebraic) proof.

CASE 0. $q = 2$.

In this case from (4.3) we obtained $h-1 \ge 2 = q$ clearly.

CASE 1. $1-\frac{b}{a} \le \frac{1}{q+1}$.

First we have $m_1^{(1)} = r_0 m_0 - m_1^{(2)} - m_1^{(3)} = 2a-b-c$. The assumption $m_1^{(1)} > m_1^{(2)}$ implies $2a-b-c > b$. So we have

$$\begin{aligned} h &= \lfloor\tfrac{m_0}{k}\rfloor = \lfloor\tfrac{a}{a-(2a-b-c)}\rfloor \ge \lfloor\tfrac{a}{a-b}\rfloor \\ &\ge q+1. \end{aligned}$$

CASE 2. $p = q-1$ or $p = 1$.

Suppose $p = q-1$. We also assume that $1-\frac{b}{a} > \frac{1}{q+1}$. Let $S(r_1,\cdots)$ be an interval defined in Lemma 3.5, and a power 2^{q-1} denotes a sequence of 2 of $(q-1)$-times. $p = q-1$ follows $\frac{b}{a} \in S(2^{q-1}) = [\frac{q-1}{q}, 1)$. See the proof of Lemma 3.1. In the same say, we have $\frac{c}{a} \in S(q) = [\frac{1}{q}, \frac{1}{q-1})$. The assumption $1-\frac{b}{a} > \frac{1}{q+1}$ follows $\frac{b}{a} < \frac{q}{q+1}$. So we have

$$\begin{aligned} \frac{b}{a} + \frac{c}{a} - 1 &< \frac{q}{q+1} + \frac{1}{q-1} - 1 = \frac{2}{q^2-1} \\ h = \lfloor\frac{a}{b+c-a}\rfloor &\ge \lfloor\frac{q^2-1}{2}\rfloor \ge q+1\,. \end{aligned}$$

To show the last inequality we use the condition $q \ge 3$.

CASE 3. $1 < p < q-1$.

Let q', p' be given by $q' = \langle r_1, \cdots, r_{m'-1}\rangle$, $p' = \langle r_2, \cdots, r_{m'-1}\rangle$. Let a real number x be $1/[r_{m'+1}, \cdots, r_m]$. (If $m' = m$ then $x = 0$.) Then from Lemma 3.1, we get

$$\frac{b}{a} = \frac{1}{r_1} - \frac{1}{r_2} - \cdots - \frac{1}{r_{m'} - x} = \frac{-p'x + p}{-q'x + q} .$$
$$\frac{b}{a} - \frac{p}{q} = \frac{1}{(-q' + q/x)q} .$$

(If $m' = m$ then $\frac{b}{a} - \frac{p}{q} = 0$.) In the same way, let q'' and p'' be defined by

$$q'' = \langle r'_1, \cdots, r'_{n'-1}\rangle, \quad p'' = \langle r'_2, \cdots, r'_{n'-1}\rangle .$$

(Here $\frac{q-p}{q} = [r'_1, \cdots, r'_{n'}]$.) And we obtain

$$\frac{c}{a} - \frac{q-p}{q} = \frac{1}{(-q'' + q/x')q} ,$$

where $x' = 1/[r'_{n'+1}, \cdots, r'_n]$. (If $n' = n$ then $\frac{c}{a} - \frac{q-p}{q} = 0$.) Here we need the following lemmas.

LEMMA 4.7. (1) *$q'' + q' = q$.*
(2) *q and q' are co-prime. q and q'' are co-prime.*
(3) *$1 < q' < q - 1$ and $1 < q'' < q - 1$.*

The proof is easy. Only for (1) we need a comment. Using Lemma 3.4, if $[r_1, \cdots, r_{m'}] + [r'_1, \cdots, r'_{n'}] = 1$ then $[r_{m'}, \cdots, r_1] + [r'_{n'}, \cdots, r'_1] = 1$. Using Lemma 3.1(3), we obtain (1).

We show Lemma 4.6 in case 3.

$$\frac{b}{a} + \frac{c}{a} - 1 < \frac{1}{(-q' + q)q} + \frac{1}{(-q'' + q)q}$$
$$= \frac{1}{q''q'} ,$$
$$h = \lfloor \frac{a}{b + c - a} \rfloor \geq \lfloor q''q' \rfloor \geq q + 1 .$$

This completes the proof of 4.6.

Finally we show that $q\Theta_0 + S^{(2)}$, $q\Theta_0 + S^{(3)}$ are type A_k subbranches.

LEMMA 4.8. *$q\Theta_0 + S^{(2)}$ and $q\Theta_0 + S^{(3)}$ are type A_k subbranches of the singular fiber X.*

PROOF. By definition of $\langle \cdots \rangle$, we have

$$\langle r_i, \cdots, r_{m'}\rangle + \langle r_{i+2}, \cdots, r_{m'}\rangle = r_i \langle r_{i+1}, \cdots, r_{m'}\rangle ,$$
$$\langle r_{m'-1}, r_{m'}\rangle + 1 = r_{m'-1}\langle r_{m'}\rangle ,$$
$$\langle r_{m'}\rangle = r_{m'} .$$

These formulae and (4.4) imply that $q\Theta_0 + S^{(2)}$ gives a subbranch of the singular fiber. (TA1) for $i = 0$ is satisfied since $kn_0 = kq \leq k(h-1) < m_0$. For any $i = 1, \cdots, m'$, inductively we have

$$\begin{aligned} &\tfrac{m_i^{(2)}}{m_{i-1}^{(2)}} = [r_i, \cdots, r_m] \geq [r_i, \cdots, r_{m'}] = \tfrac{n_i^{(2)}}{n_{i-1}^{(2)}} \\ &\Rightarrow m_i^{(2)} \geq \tfrac{n_i^{(2)}}{n_{i-1}^{(2)}} m_{i-1}^{(2)} > kn_i^{(2)} . \end{aligned}$$

Here $m_0^{(2)} = m_0$ and $n_0^{(2)} = n_0$. The length e of this subbranch is m' and we obtain

$$\frac{n_{e-1}^{(2)}}{n_e^{(2)}} = \frac{r_{m'}}{1} = r_{m'} = r_e .$$

Hence the condition (TA2) is satisfied. In the case of $S^{(3)}$, we can prove in the same way.

LEMMA 4.9. *Y as (4.5) is a simple crust of the singular fiber X.*

PROOF. From Lemma 4.4 and Lemma 4.8, $q\Theta_0 + S^{(j)}$ $(j = 1, 2, 3)$ give subbranches of type C_k, A_k, and A_k respectively. And we get

$$\frac{n_1^{(1)} + n_1^{(2)} + n_1^{(3)}}{n_0} = \frac{q + \langle r_2, \cdots, r_{m'} \rangle + \langle r'_2, \cdots, r'_{n'} \rangle}{q} = 2 = r_0 .$$

This completes the proof.

5. The case where a singular fiber has more than three branches

In this section we discuss the splittability problem when a star-shaped singular fiber has more than three branches.

In section 4 we assume that the number of branches equals three. We assume that $r_0 \geq 2$ and we have an inequality $r_0 < b$. (It is easy to show this inequlity from the formulae $m_0 > m_1^{(j)}$ and $m_0 r_0 = \sum_{j=1}^{b} m_1^{(j)}$.) So the condition $b = 3$ implies $r_0 = 2$.

In this section we remove the assumption $b = 3$. That is, we only have $2 \leq r_0 < b$. In this case there exists a stellar singular fiber without a simple crust. If $(r_0, b) = (2, 4)$ then the smallest m_0 of such example is 33. We cannot check this by hand calculation. We need help of computers. To find such singular fiber, the software *Splitica* [A] is very useful.

Splitica is developed by the first author and it allows us to list up all of crusts for a stellar singular fiber.

The following proposition are obtained by computer experiments. As mentioned in section 3, all of the multiplicities of components are determined by m_0 and $m_1^{(j)}$ $(j = 1, 2, \cdots, b)$. So in this section we will represent a stellar singular fiber with genus 0 core by the sequence

$$(m_0; m_1^{(1)}, m_1^{(2)}, \cdots) .$$

PROPOSITION 5.1. *The following star shaped singular fibers with genus 0 core don't have a simple crust.*

(1) *In the case* $(r_0, b) = (2, 4)$

```
(33;25,19,12,10)
(33;24,19,13,10)
(35;27,21,12,10)
(36;27,22,16,7)
(37;29,23,12,10)
(38;29,24,16,7)
(38;24,23,18,11)
```

(If $m_0 \leq 38$ *then there are no other examples.)*

(2) *In the case* $(r_0, b) = (2, 5)$

```
(20;15,11,7,4,3)
(20;13,11,8,5,3)
(21;15,11,7,5,4)
(21;12,11,8,7,4)
(22;15,11,9,6,3)
(22;13,11,8,7,5)
(24;18,14,8,5,3)
(24;17,16,6,5,4)
(24;17,13,9,5,4)
(24;17,13,8,6,4)
(24;17,12,8,6,5)
(24;16,14,7,6,5)
(24;16,11,9,7,5)
(24;14,13,9,7,5)
(24;14,13,8,7,6)
(24;14,12,9,8,5)
```

(If $m_0 \leq 24$ *then there are no other examples.)*

(3) *In the case* $(r_0, b) = (3, 4)$*, there is no example within* $m_0 \leq 24$.

(4) *In the case* $(r_0, b) = (3, 5)$

```
(20;17,15,13,11,4)
(21;17,15,13,11,7)
(21;17,14,13,10,9)
(22;17,15,13,11,10)
```

(If $m_0 \leq 22$ *then there are no other examples.)*

Observing these results, we have the following conjectures. One is in the case $r_0 = b-1$. When $(r_0, b) = (2, 3)$, we have Theorem 4.1. It seems a similar theorem holds when $r_0 = b - 1$. In order to show this conjecture, we need more complicated version of 'sub-continued-fractions lemma.' The other is in the case m_0 is a prime number. For a long time, we have a conjecture that if m_0 is prime then we have the similar criterion. But there is a counter example in the case $(m_0, r_0, b) = (37, 2, 4)$. This is a rare example.

CONJECTURE 5.2. (1) *If $r_0 = b - 1$ then every stellar singular fiber with genus 0 core has a simple crust.*

(2) *If m_0 is prime, it is easier to find out a simple crust in some sense.*

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