Hidden symmetries of hyperbolic links

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Abstract: W. D. Neumann and A. W. Reid conjectured that the figure-eight knot and the two dodecahedral knots are the only hyperbolic knots admitting hidden symmetries. We construct an n-component hyperbolic link whose complement admits hidden symmetries for any $n \geq 4$.

Key words: Hyperbolic link; commensurator; hidden symmetry.

1. Introduction. Hidden symmetry of a hyperbolic manifold $M = \mathbf{H}^3/\Gamma$ is a homeomorphism of finite degree covers of M that does not descend to an automorphism of M. If $\mathrm{Comm}(\Gamma) \neq N(\Gamma)$, we say M admits a hidden symmetry, where $\mathrm{Comm}(\Gamma)$ is the commensurator of Γ and $N(\Gamma)$ is the normalizer of it. We say a link in S^3 admits a hidden symmetry if its complement admits a hidden symmetry. In [6], W. D. Neumann and A. W. Reid conjectured that the figure-eight knot and the two dodecahedral knots are the only hyperbolic knots admitting hidden symmetries. Many researchers concerned with this problem. M. L. Macasieb and T. W. Mattman [4] showed that $(-2, 3, n)$ pretzel knot does not admit a hidden symmetry ($n \in \mathbf{N}$). By using computer, O. Goodman, D. Heard and C. Hodgson [2] have verified for hyperbolic knots with 12 or fewer crossings. A. W. Reid and G. S. Walsh [7] showed that non-arithmetic 2-bridge knots admit no hidden symmetry.

For two-component links, E. Chesebro and J. DeBlois [1] constructed infinitely many two-component non-arithmetic links admitting hidden symmetries. Let L_i $(i = 1, \cdots, 3)$ be the links as in Figure 1. $S^3 - L_2$ is obtained by cutting along the colored twice punctured disk of $S^3 - L_1$, performing π-rotation and regluing it. Repeat this process about the colored twice punctured disk of $S^3 - L_2$. Then we obtain $S^3 - L_3$. O. Goodman, D. Heard and C. Hodgson [2] showed that L_2 and L_3 have hidden symmetries by using computer. J. S. Meyer, C. Millichap and R. Trapp [5] constructed n-component links admitting hidden symmetries ($n \geq 6$). They prove this by analyzing the geometry of those link complements, including their cusp shapes and totally geodesic surfaces inside these manifolds.

In this paper, we generalize the result of O. Goodman, D. Heard and C. Hodgson [2]. Let C be an $(n+1)$-component alternating chain link as in the top picture of Figure 2 ($n \geq 4$). Cut along the colored twice punctured disk of $S^3 - C$, perform π-rotation and reglue it. Denote the resulting n-component link by C_n. In section 3, we prove the following theorem.

Theorem 1. *Let Γ_n be a Kleinian group such that $S^3 - C_n = \mathbf{H}^3/\Gamma_n$. Then Γ_n is non-arithmetic and we have*

$$|\mathrm{Comm}(\Gamma_n) : N(\Gamma_n)| = n + 1.$$

Thus we get the following corollary.

Corollary 1. *The n-component link C_n admits hidden symmetries.*

2. Commensurator and normalizer. Two subgroups G_1, $G_2 < \mathrm{Isom}^+(\mathbf{H}^3)$ are said to be *commensurable* if their intersection $G_1 \cap G_2$ has finite index in both G_1 and G_2. G_1 and G_2 are said to be *commensurable in the wide sense* if there is $h \in \mathrm{Isom}^+(\mathbf{H}^3)$ such that G_1 is commensurable with $h^{-1}G_2h$. The notion of commensurability can be directly transported to hyperbolic orbifolds by considering the respective fundamental groups. Then, commensurable hyperbolic orbifolds admit a finite-sheeted common covering orbifold. Commensurability is an equivalence relation.

For a Kleinian group Γ, the *commensurator* of Γ is defined by

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}^+(\mathbf{H}^3) : g\Gamma g^{-1} \text{ and } \Gamma \text{ are commensurable.}\}.$$

Let Γ be a finitely generated Kleinian group of finite

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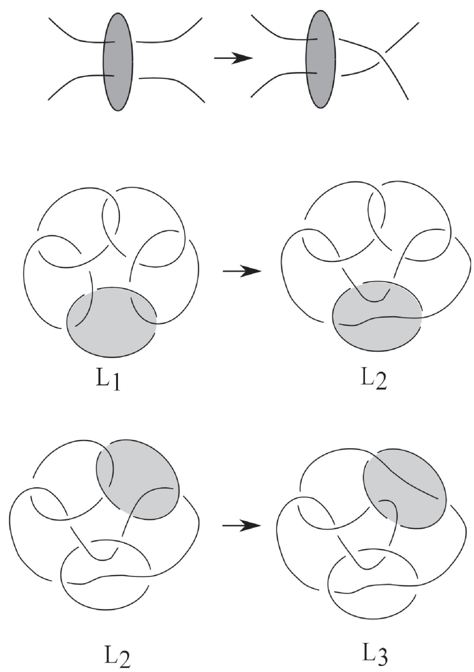

Fig. 1. Chain links.

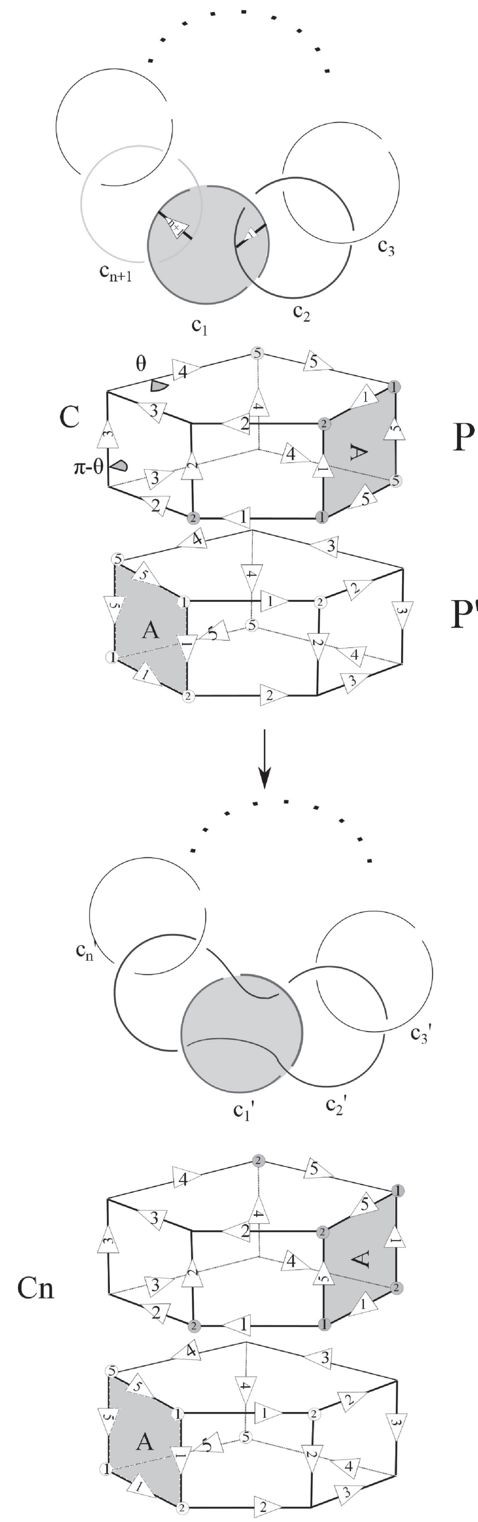

Fig. 2. The link C_n.

co-volume. It is well known that Comm(Γ) is a commensurability invariant (see [10]). Comm(Γ) contains every member of the commensurability class. G. Margulis [3] showed that Comm(Γ) is discrete if and only if Γ is non-arithmetic. For a non-arithmetic Kleinian group Γ, Comm(Γ) contains every member of the commensurability class in finite index.

The normalizer of Γ is

$$N(\Gamma) = \{g \in \mathrm{Isom}^+(\mathbf{H}^3) : g\Gamma g^{-1} = \Gamma\}.$$

Clearly, $N(\Gamma) < \mathrm{Comm}(\Gamma)$. $N(\Gamma)/\Gamma \simeq \mathrm{Isom}^+(\mathbf{H}^3/\Gamma) = \mathrm{Symm}(\mathbf{H}^3/\Gamma)$ and $N(\Gamma)$ is discrete.

For an arithmetic Kleinian group Γ, Comm(Γ) is not discrete. Thus arithmetic Kleinian group always admits a hidden symmetry.

3. Proof of Main Theorem. To prove Theorem 1, we prepare a lemma. Let L be a link such that $S^3 - L$ contains a twice punctured disk S. Cut $S^3 - L$ along S, give a half-twist and reglue them together. Then we get a new link L_S. In general, $S^3 - L$ and $S^3 - L_S$ are not always commensurable ([11]). We have the following Lemma 1.

Lemma 1. *Let L be a link as in Figure* 3. *Assume that a tangle τ is equivalent to the tangle*

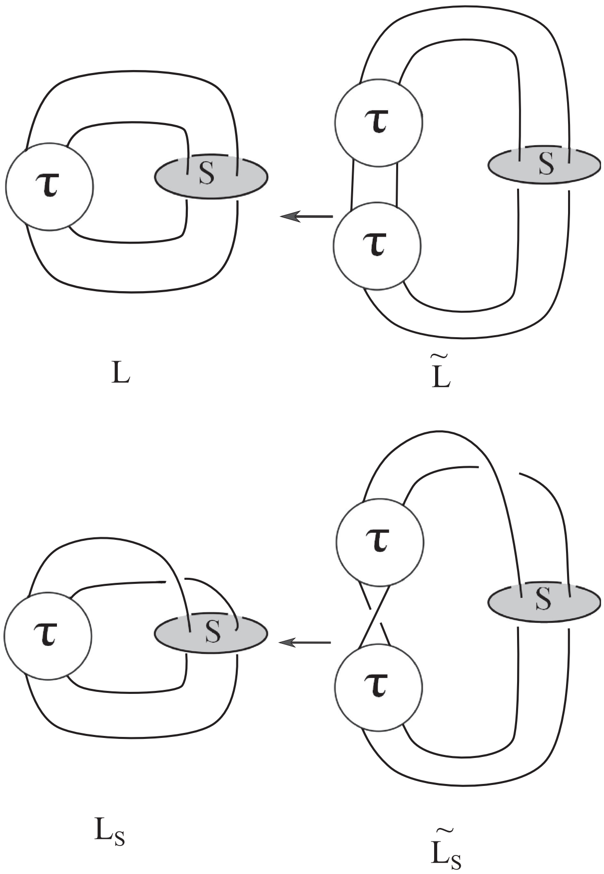

Fig. 3. Double cover of $S^3 - L$ and $S^3 - L_S$.

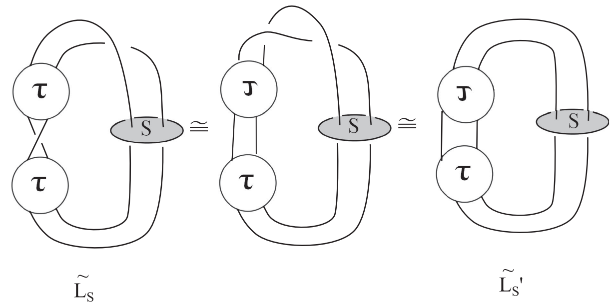

Fig. 4. $\widetilde{L_S}$ and $\widetilde{L'_S}$.

obtained by performing a vertical flip. Then $S^3 - L$ and $S^3 - L_S$ are commensurable.

Proof of Lemma 1. Let $\widetilde{L}$, $\widetilde{L_S}$, $\widetilde{L_S}'$ be links as in Figure 3. Then $S^3 - \widetilde{L}$ (resp. $S^3 - \widetilde{L_S}$) is a double cover of $S^3 - L$ (resp. $S^3 - L_S$).

Cut $S^3 - \widetilde{L_S}$ along S, give a full-twist and reglue them together as in Figure 4. Then we get $S^3 - \widetilde{L_S}'$. Thus $S^3 - \widetilde{L_S}$ is homeomorphic to $S^3 - \widetilde{L_S}'$. Moreover, by the assumption on τ, $S^3 - \widetilde{L_S}'$ is homeomorphic to $S^3 - \widetilde{L}$. Thus $S^3 - \widetilde{L}$ is a common double cover of $S^3 - L$ and $S^3 - L_S$. $\square$

Proof of Theorem 1. Let α_1 and α_2 be the π-rotations of $S^3 - C$ as in Figure 5, and α_3 the $2\pi/(n+1)$-rotation of it. $\mathrm{Symm}(S^3 - C)$ is generated by α_1, α_2, α_3 and $|\mathrm{Symm}(S^3 - C)| = 4(n+1)$ [6]. Let P and P' be ideal polyhedra as in Figure 2 in $\mathbf{H}^3$, the top and bottom faces are regular $(n+1)$-gons, the dihedral angles θ at the edges of $(n+1)$-gons are $\arccos((\cos\pi/(n+1))/\sqrt{2})$ and the other angles are $\pi - 2\theta$. W. Thurston showed that $S^3 - C$ is obtained by glueing P and P' as depicted in Figure 2. (See section 6.8 [9].) Each link component corresponds to four ideal vertices of P and P'. M. Sakuma and J. Weeks [8] showed this ideal polyhedral decomposition is the canonical decomposition of $S^3 - C$. Any symmetry of $S^3 - C$ preserves the canonical decomposition. The twice punctured disk is the common image of the quadrangles labeled with the letter "A" as in Figure 2. We can see that α_1 is the π-rotation around the diagonal of A. α_2 is the π-rotation around the geodesic which is perpendicular to A and which passes through the center of A. α_3 is the $2\pi/(n+1)$-rotation around the geodesic which is perpendicular to the top and bottom faces of P. Assume $S^3 - C = \mathbf{H}^3/\Gamma$. As $\mathrm{Comm}(\Gamma) = N(\Gamma)$ [6], $\mathbf{H}^3/\mathrm{Comm}(\Gamma) \cong (S^3 - C)/\mathrm{Symm}(S^3 - C)$. We have

$$\mathrm{vol}(\mathbf{H}^3/\mathrm{Comm}(\Gamma)) = \mathrm{vol}(P)/2(n+1).$$

The chain link C can be deformed as in Figure 6. The tangle τ in Figure 6 is equivalent to the tangle obtained by performing a vertical flip. By Lemma 1, $S^3 - C$ and $S^3 - C_n$ are commensurable. We have

$$\text{(1)} \quad \begin{aligned} \mathrm{vol}(\mathbf{H}^3/\mathrm{Comm}(\Gamma_n)) &= \mathrm{vol}(\mathbf{H}^3/\mathrm{Comm}(\Gamma)) \\ &= \mathrm{vol}(P)/2(n+1). \end{aligned}$$

Neumann and Reid showed $S^3 - C$ is non-arithmetic [6]. Since commensurability preserves arithmeticity, $S^3 - C_n$ is also non-arithmetic. $\square$

Let c'_i be the cusp corresponding to the component of C_n as depicted in Figure 2. Then c'_2 corresponds to eight ideal vertices of P, P' and c'_i $(i = 1, 3, \cdots, n)$ corresponds to four ideal vertices. Let V_i be the set of ideal points in $\partial\mathbf{H}^3$ which corresponds to c'_i. If $g \in \mathrm{Symm}(S^3 - C_n)$, $g(c'_i) = c'_{\sigma(i)}$ for some $\sigma \in S_n$ where S_n denotes the group of

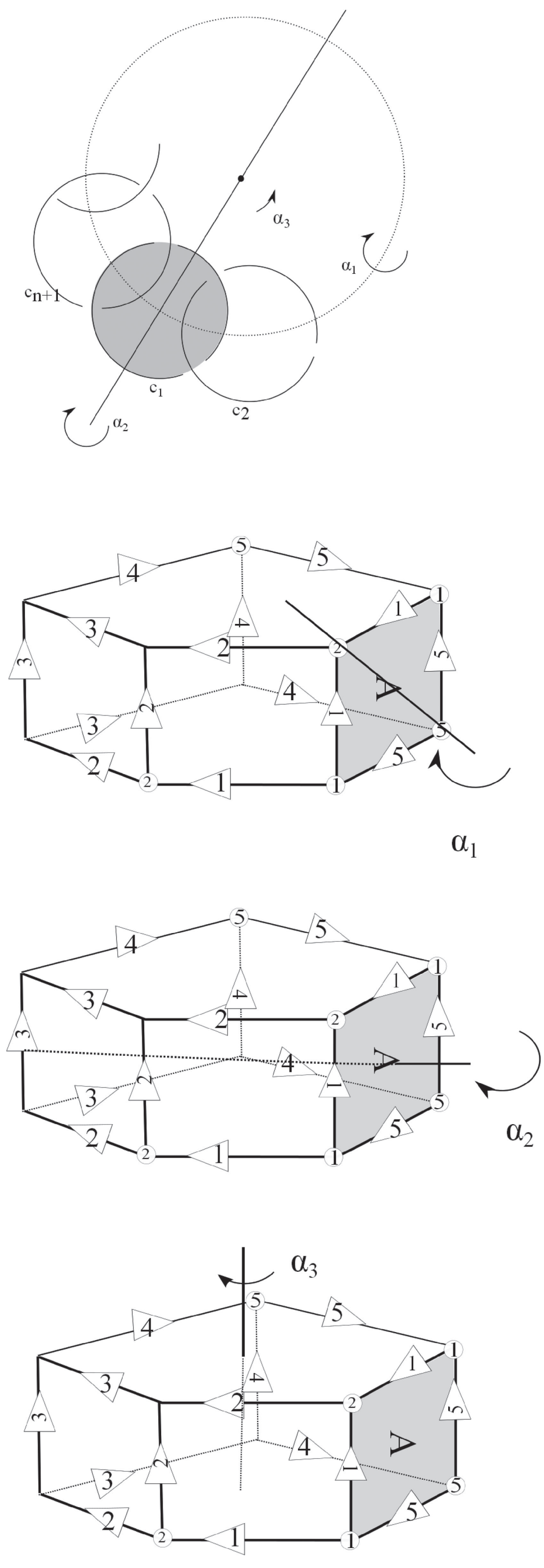

Fig. 5. The symmetries of $S^3 - C$.

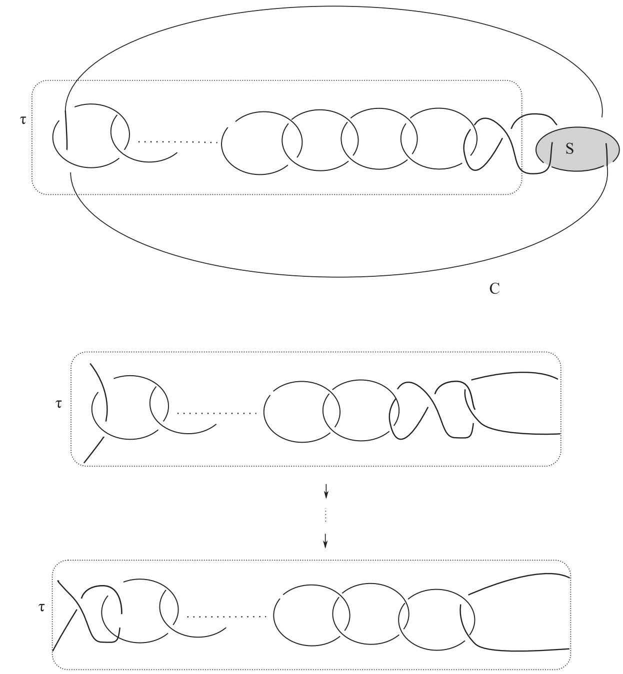

Fig. 6. Deformation of τ.

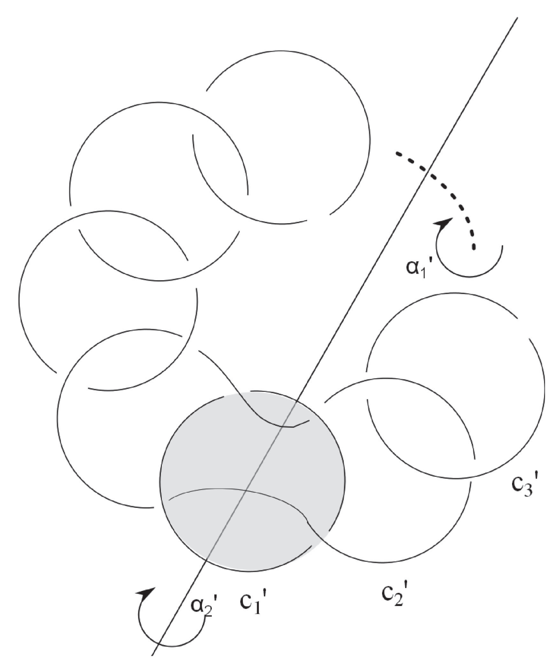

Fig. 7. Symmetry of $S^3 - C_n$.

permutations of $\{1, \cdots, n\}$. We have $N(\Gamma_n) < \{\gamma \in \mathrm{Comm}(\Gamma) | \gamma(V_i) = V_{\sigma(i)} \quad (i = 1, \cdots, n)$ for some $\sigma \in S_n\}$.

Let $\widetilde{P}$ be a lift of P and $\widetilde{A}$ a face of $\widetilde{P}$, which is a lift of A, $\widetilde{P'}$ a lift of P' such that $\widetilde{P} \cap \widetilde{P'} = \widetilde{A}$.

Let $\widetilde{\alpha_1}$, $\widetilde{\alpha_2}$, $\widetilde{\alpha_3} \in \mathrm{Isom}^+(\mathbf{H}^3)$ be a lifts of α_1, α_2, α_3 respectively such that $\widetilde{\alpha_1}$ is the π-rotation around the diagonal of $\widetilde{A}$, $\widetilde{\alpha_2}$ is the π-rotation around the geodesic which is perpendicular to $\widetilde{A}$ and which passes through the center of $\widetilde{A}$, $\widetilde{\alpha_3}$ is the $2\pi/(n+1)$-rotation around the geodesic which is perpendicular to the top and bottom faces of $\widetilde{P}$.

For any $\gamma \in N(\Gamma)$, there exists $\sigma \in S_n$ such that $\gamma(V_i) = V_{\sigma(i)}$ and $\gamma(\widetilde{P})$ is a lift of P or P'. As $\alpha_1(P) = P'$, there is $\gamma_0 \in \langle \widetilde{\alpha_1}, \Gamma \rangle$ such that $\gamma_0 \gamma(\widetilde{P}) = \widetilde{P}$. Since $\gamma_0(V_i) = V_i$, $\gamma_0\gamma(V_i) = V_{\sigma(i)}$ for any i. Hence $\gamma_0\gamma$ fixes the four ideal points of $\widetilde{P}$ which correspond to the cusp c_2'. Hence $\gamma_0\gamma = \widetilde{\alpha_2}$ or identity. We get

$$N(\Gamma_n) < \langle \widetilde{\alpha_1},\ \widetilde{\alpha_2},\ \Gamma \rangle.$$

Thus

$$\begin{aligned} \mathrm{vol}(\mathbf{H}^3/N(\Gamma_n)) &\geq \mathrm{vol}(\mathbf{H}^3/\langle \widetilde{\alpha_1},\ \widetilde{\alpha_2},\ \Gamma \rangle) \\ &= \mathrm{vol}(P)/2. \end{aligned} \tag{2}$$

Let α_1', α_2' be the symmetries of $S^3 - C_n$ as depicted in Figure 7. It is easy to see that $|\langle \alpha_1', \alpha_2' \rangle| = 4$. Thus we obtain

$$\mathrm{vol}(\mathbf{H}^3/N(\Gamma_n)) \leq \mathrm{vol}(P)/2. \tag{3}$$

Therefore

$$\mathrm{vol}(\mathbf{H}^3/N(\Gamma_n)) = \mathrm{vol}(P)/2, \tag{4}$$

by (2), (3). Hence

$$|\mathrm{Comm}(\Gamma_n) : N(\Gamma_n)| = n + 1$$

by (1), (4).

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References

[1] E. Chesebro and J. DeBlois, Hidden symmetries via hidden extensions, Proc. Amer. Math. Soc. **145** (2017), no. 8, 3629–3644.

[2] O. Goodman, D. Heard and C. Hodgson, Commensurators of cusped hyperbolic manifolds, Experiment. Math. **17** (2008), no. 3, 283–306.

[3] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 17, Springer-Verlag, Berlin, 1991.

[4] M. L. Macasieb and T. W. Mattman, Commensurability classes of $(-2, 3, n)$ pretzel knot complements, Algebr. Geom. Topol. **8** (2008), no. 3, 1833–1853.

[5] J. S. Meyer, C. Millichap and R. Trapp, Arithmeticity and hidden symmetries of fully augmented pretzel link complements, New York J. Math. **26** (2020), 149–183.

[6] W. D. Neumann and A. W. Reid, Arithmetic of hyperbolic manifolds, in *Topology* '90 (*Columbus, OH, 1990*), 273–310, Ohio State Univ. Math. Res. Inst. Publ., 1, de Gruyter, Berlin, 1992.

[7] A. W. Reid and G. S. Walsh, Commensurability classes of 2-bridge knot complements, Algebr. Geom. Topol. **8** (2008), no. 2, 1031–1057.

[8] M. Sakuma and J. Weeks, Examples of canonical decompositions of hyperbolic link complements, Japan. J. Math. (N.S.) **21** (1995), no. 2, 393–439.

[9] W. P. Thurston, *The Geometry and Topology of Three-manifolds*, Lecture notes, Princeton University Press, Princeton, 1979.

[10] G. S. Walsh, Orbifolds and commensurability, in *Interactions between hyperbolic geometry, quantum topology and number theory*, Contemp. Math., 541, Amer. Math. Soc., Providence, RI, 2011.

[11] H. Yoshida, Invariant trace fields and commensurability of hyperbolic 3-manifolds, in *KNOTS '96* (*Tokyo, 1996*), 309–318, World Sci. Publ., Singapore, 1997.