Perverse sheaves on $\mathbf{C}^2$ without vanishing cycles at the origin along a general plane curve with singularities

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Abstract: Generalizing MacPherson-Vilonen's method [2] to arbitrary plane curve singularities, we provide a classification of perverse sheaves on the neighborhood of the origin in the complex plane, which are adapted to a germ of a complex analytic plane curve. We rely on the presentation of the fundamental group of the complement of the curve as obtained by Neto and Silva [5]. The main result is an equivalence of categories between the category of perverse sheaves on $\mathbf{C}^2$ stratified with respect to a singular plane curve and the category of n-tuples of finite dimensional vector spaces and linear maps satisfying a finite number of suitable relations. As an application, we classify perverse sheaves with no vanishing cycles at the origin for a special case.

Key words: Perverse sheaves; local systems; plane curve singularity; fundamental group; algebraic link.

Introduction. Let $f : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}, 0)$ be a germ of a complex analytic map, and $C \subset (\mathbf{C}^2, 0)$ be the plane curve defined by the equation $f = 0$. The category $Perv(\mathbf{C}^2)$ of perverse sheaves on $\mathbf{C}^2$ which are constructible relative to the stratification $\{0\} \subset \{C\} \subset \mathbf{C}^2$ is an abelian category. It is a full subcategory of $D^b(\mathbf{C}^2)$ the category of complexes of sheaves of $\mathbf{C}$-vector spaces on $\mathbf{C}^2$ whose cohomology is bounded. In this paper, we classify perverse sheaves on the neighborhood of the origin in $\mathbf{C}^2$, which are adapted to a germ of the complex analytic plane curve C. To do so, we use MacPherson-Vilonen's methods on the elementary construction of perverse sheaves [1] and on the classification of perverse sheaves along the plane curve $x^m = y^n$ [2]. These methods extend to arbitrary plane curve singularities as well as to the global case. The combination of the above methods with the presentation of the fundamental group of the complement of the curve C as obtained by Neto and Silva [5] yields a generalization of the result obtained by MacPherson and Vilonen in the case of monomials plane curves [2]. We prove Theorem 1 stated in section 2. As an application, we classify perverse sheaves with no vanishing cycles at the origin for a special case. Similar descriptions have been obtained by [4], [6].

Notation. Given X a complex manifold with an analytic stratification $\mathcal{S}$, denote by $\Lambda_S = \overline{T^*_S X}$ for $S \in \mathcal{S}$ the closure of $T^*_S X$ in T^*X, and $\widetilde{\Lambda}_S = \Lambda_S \backslash (\bigcup_{R \neq S} \Lambda_R)$. Set $\Lambda = \bigcup_{S \in \mathfrak{S}} \Lambda_S$ and denote by $Perv_\Lambda(X)$ the category whose objects are perverse sheaves on X with characteristic variety contained in Λ. We treat the case of middle perversity.

1. Extending across a smooth curve in $\mathbf{C}^2$. Let C be a smooth curve in $\mathbf{C}^2 \backslash \{0\}$ and let $\Lambda = T^*_{\mathbf{C}^2}\mathbf{C}^2 \cup \overline{T^*_C \mathbf{C}^2 \backslash \{0\}} \cup T^*_{\{0\}}\mathbf{C}^2$ be the union of conormal bundles to the strata $\mathbf{C}^2 \backslash C$, $C \backslash \{0\}$, $\{0\}$. This section deals with an explicit combinatorial description of the category $Perv_\Lambda(\mathbf{C}^2 \backslash \{0\})$ in the general case C given by the equation $f = 0$ where f is a complex analytic map-germ. Recall from [2] that if $C_1, \cdots, C_n$ denote the irreducible components of the curve C, one can identify $T^*_{C_i}\mathbf{C}^2$ with $T_{C_i}\mathbf{C}^2$ using the standard hermitian metric on $\mathbf{C}^2$. Therefore the tubular neighborhood theorem establishes a diffeomorphism between a neighborhood U_j of the zero section of $T^*_{C_i}\mathbf{C}^2$ and a neighborhood V_j of C_j in $\mathbf{C}^2$. Consider the mappings of fundamental groups

$$\pi_1(\widetilde{\Lambda}_{C_j}) \xleftarrow{\simeq} \pi_1(U_j \backslash C_j) \xleftarrow{\simeq} \pi_1(V_j \backslash C_j) \longrightarrow \pi_1(\mathbf{C}^2 \backslash C).$$

Taking the pull backs via these maps of a local system $\mathcal{L}$ on $\mathbf{C}^2 \backslash C$ gives rise to a local system $\widetilde{\mathcal{L}}_j$ on

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$\widetilde{\Lambda}_{C_j}$ that does not depend of the base point chosen. There is an isomorphism $\pi_1(\widetilde{\Lambda}_{C_j}) \cong \pi_1(\mathbf{C}^*) \times \pi_1(C_j)$ obtained by choosing a section of the trivial $\mathbf{C}^*$-bundle $\widetilde{\Lambda}_{C_j}$ on C_j. Let us choose such a section and fix a base point. Let γ_j be the image of the canonical generator of $\pi_1(\mathbf{C}^*)$ in $\pi_1(\widetilde{\Lambda}_{C_j})$.

For $\mathcal{L}$ a local system on $\mathbf{C}^2\backslash C$, the complex $\mathcal{L}[2]$ is a perverse sheaf in $Perv_\Lambda(\mathbf{C}^2\backslash C)$ [2]. Consider the functors ψ and $\psi_c : Perv(\mathbf{C}^2\backslash C) \longrightarrow$ {Local systems on $\widetilde{\Lambda}_{C_j}$} (called the "nearby cycles" and the "nearby cycles with compact support") and the natural transformation $var : \psi \longrightarrow \psi_c$ (called the variation). We recall [2] Lemma 1.1 and Proposition 1.2:

Lemma 1. *For a local system $\mathcal{L}$ on $\mathbf{C}^2\backslash C$, one has $\psi(\mathcal{L}[2]) \cong \psi_c(\mathcal{L}[2]) \cong \widetilde{\mathcal{L}}_j$ on $\widetilde{\Lambda}_{C_j}$, and the variation map $var : \widetilde{\mathcal{L}}_j \to \widetilde{\mathcal{L}}_j$ is given by $var(l) = \gamma_j(l) - l$.*

Proposition 1. *The category $Perv_\Lambda(\mathbf{C}^2\backslash\{0\})$ is equivalent to category G_Λ whose objects are local systems $\mathcal{L}$ on $\mathbf{C}^2\backslash C$ provided with a $\pi_1(C_j)$-module $\mathcal{F}_j$ for each $j = 1, \cdots, k$ such that the diagram*

$$\begin{tikzcd} \widetilde{\mathcal{L}}_j \arrow[r, "var"] \arrow[dr] & \widetilde{\mathcal{L}}_j \\ & \mathcal{F}_j \arrow[u] \end{tikzcd}$$

commutes as a diagram of $\pi_1(C_j)$-modules.

1.1. The fundamental group of $\mathbf{C}^2\backslash C$. We recall a presentation of the fundamental group of the complement of the link of a singular point of a general plane curve obtained by Neto and Silva [5]. This relies on a tree and the vertices correspond to certain tori in the complement of the link. The presentation has two generators for each vertex (a standard meridian and a standard longitude of the corresponding torus) and many relators of six different types.

Let C be a germ of a plane curve at a point o and denote by C_p, $1 \leq p \leq k$, its irreducible components. Consider a local coordinates system (x, y) in an open polydisk Y centered at o such that the tangent cone of C is transverse to $\{x = 0\}$. Let $y = \sum_\epsilon a_{p,\epsilon} x^\epsilon$ be a Puiseux expansion of the component C_p, where $\epsilon \in \mathbf{Q}^+$ and $a_{p,\epsilon} \in \mathbf{C}$. One associates to the curve C a tree Υ_C whose set of vertices is denoted by Ξ_C. Given $1 \leq p, e \leq k$ and $\epsilon \in \mathbf{Q}^+$, one identifies (p, ϵ) with (e, ϵ) if $a_{p,v} = a_{e,v}$ for $v \in \mathbf{Q}$ and $0 \leq v \leq \epsilon$. A pair (p, ϵ) is called a vertex of Υ_C with an exponent ϵ if $\epsilon = 0$ or ϵ is a characteristic exponent of Puiseux of C_p or if there exists $e \neq p$ such that $a_{p,\delta} = a_{e,\delta}$ for $\delta < \epsilon$ and $a_{p,\epsilon} \neq a_{e,\epsilon}$. We denote by $\phi = (p, 0)$ the root of Υ_C. A pair $w = (p, \delta)$ is a descendent of $v = (p, \epsilon)$ $(w > v)$ if $\delta > \epsilon$ with δ minimal. A pair (p, ϵ) is called a shaft if $a_{p,\epsilon} = 0$. One associates to each vertex $v = (p, \epsilon) \in \Xi_C$ which is non-terminal the branch $C_v = \{y = \sum_{\nu<\delta} a_{p,\nu} x^\nu\}$ where $w = (p, \delta)$ is any descendent of v. If v is terminal one associates to it the branch $C_v = C_p$. Given $0 < \eta \ll 1$ let $K_v = C_v \cap (\{x : |x| = \eta\} \times \mathbf{C})$ be the knot of C_v. It is proved that one can find $\eta > 0$ and closed neighborhoods N_w of K_w, $w \in \Xi_C$ such that $N_u \subset \mathrm{int}(N_w \setminus K_w)$ if $w < u$ and $N_w \cap N_u = \emptyset$ if $w \not< u$ and $u \not< w$ (see [5, Lemma 0.1]). The system of neighborhoods (N_w) satisfying the above conditions is called a toric system for C. One call standard meridian and standard longitude of (N_w) a pair of positively oriented simple curves α_w and β_w on ∂N_w such that α_w and β_w are homeomorphic to S^1, $\alpha_w \sim 0$, $\beta_w \sim K_w$ in $H_1(N_w)$, $\ell(\alpha_w, K_w)$. Here $\ell(.,.)$ denotes the linking number in a suitable oriented homology 3-sphere. If w is non-terminal, denote by $w_1, \cdots, w_{b_w}$ the descendents of w that are not shafts. There exist positive integers μ_w and ξ_w that are coprime such that $K_{w_i} \sim \mu_w \alpha_w + \xi_w \beta_w$ in $H_1(N_w \setminus K_w)$. The values of μ_w and ξ_w correspond to the intersection multiplicity and the ramification index of C_w and C_{w_i} respectively. Let r_w, s_w be integers such that $r_w \mu_w = s_w \xi_w + 1$.

Let B be the set of all special paths described in [5] called the base set. We recall Theorem 1.2 of [5]:

Proposition 2. *For each germ of plane curve C, the local fundamental group of C, $\pi_1(Y\backslash C, B)$ is generated by α_w, β_w, w vertex in Ξ_C with the relations $\beta_\phi = 1, [\alpha_w, \beta_w] = 1$ for all w and*

(1) $\alpha_u^{\xi_w\mu_w}\beta_u = \alpha_w^{\mu_w}\beta_w^{\xi_w}$, *if $u > w$ and u is not a shaft,*
(2) $\alpha_u^{\mu_w}\beta_u^{\xi_w} = \alpha_w^{\mu_w}\beta_w^{\xi_w}$, *if $u > w$ and u is a shaft,*
(3) $(\alpha_{w_1}\cdots\alpha_{w_{b_w}}\alpha_w^{s_w}\beta_w^{r_w})^{\xi_w} = (\alpha_w^{\mu_w}\beta_w^{\xi_w})^{r_w}$, *$w$ nonterminal without shaft,*
(4) $\alpha_{w_1}\cdots\alpha_{w_{b_w}}\alpha_w^{s_w}\beta_w^{r_w} = \alpha_{w_0}^{s_w}\beta_{w_0}^{r_w}$, *$w$ nonterminal with shaft w_0.*

We then obtain the description of the fundamental group of $\mathbf{C}^2\backslash C$ as follows:

Proposition 3. *For each germ of plane curve C, the fundamental group of the complement of the curve C in $\mathbf{C}^2$, $\pi_1(\mathbf{C}^2\backslash C, B)$, is given by the generators α_w, β_w, w vertex in Ξ_C with the relations: $\beta_\phi = 1, [\alpha_w, \beta_w] = 1$ for all w and (1)–(4).*

Proof. This result is a consequence of Proposition 2 and the fact that the polydisc Y is a deformation retract of $\mathbf{C}^2$, and therefore $\pi_1(\mathbf{C}^2\backslash C, B)$ is isomorphic to $\pi_1(Y\backslash C, B)$. $\square$

1.2. Combinatorial description of the category $\mathbf{Perv}(\mathbf{C}^2\backslash\{0\})$. If $C_1, \cdots, C_k$ are the irreducible components of the general plane curve C, one knows that $\pi_1(\widetilde{\Lambda}_{C_j}) \cong \pi_1(\mathbf{C}^*) \times \pi_1(C_j) \cong \mathbf{Z} \times \mathbf{Z}$ for all $j \in \{1, \cdots, k\}$. For each vertex $v = (j, \epsilon)$ in Ξ_C, the pair of generators (α_j, β_j) of $\pi_1(\mathbf{C}^2\backslash C, B)$ can be identified with γ_j (the image of) the canonical generator of $\pi_1(\mathbf{C}^*)$ (in $\pi_1(\widetilde{\Lambda}_{C_j})$). We trivialize the fibration $\widetilde{\Lambda}_{C_j} \to C_j$ by choosing another generator γ_j on B for $\pi_1(\widetilde{\Lambda}_{C_j})$. Proposition 1 says that a perverse sheaf $\mathcal{F}$ on $\mathbf{C}^2\backslash\{0\}$ corresponds to a local system $\mathcal{L}$ on $\mathbf{C}^2\backslash C$ equipped with a $\pi_1(C_j)$-module $\mathcal{F}_j$ for every $j = 1, \cdots, k$. For each vertex $v = (j, \epsilon)$, let (a_j, d_j) be the action induced by the generators (α_j, β_j) on $\mathcal{L}$, τ_j be the action induced by γ_j on $\mathcal{F}_j$, and b_j be the action induced by γ_j on $\mathcal{L}$. Denote by R the group generated by the relations of Proposition 3. A more explicit form of the Proposition 1 is:

Proposition 4. *The category $Perv_\Lambda(\mathbf{C}^2\backslash\{0\})$ is equivalent to the category of $(k+1)$-tuples of vector spaces $(\mathcal{L}, \mathcal{F}_1, \cdots, \mathcal{F}_k)$ such that $\mathcal{L}$ has an action by the group $\langle (a_1, d_1), \cdots, (a_k, d_k), b_1, \cdots, b_k\rangle/R$ and each $\mathcal{F}_j$ has an action by τ_j with the $\mathbf{C}$-linear maps $\mathcal{L} \underset{q_j}{\overset{p_j}{\rightleftarrows}} \mathcal{F}_j$ such that for $1 \le j \le k$*

(1) $p_j q_j = a_j - 1$, (2) $p_j q_j = d_j - 1$,
(3) $q_j\tau_j - b_j q_j = 0$, (4) $\tau_j p_j - p_j b_j = 0$.

Proof. One knows from the Proposition 1 that $Perv_\Lambda(\mathbf{C}^2\backslash\{0\})$ is equivalent to the category of local systems $\mathcal{L}$ on $\mathbf{C}^2\backslash C$ with triangles of $\pi_1(C_j)$-modules

$$\begin{tikzcd} \widetilde{\mathcal{L}}_j \arrow[r, "var"] \arrow[dr, "p_j"'] & \widetilde{\mathcal{L}}_j \\ & \mathcal{F}_j \arrow[u, "q_j"'] \end{tikzcd}$$

where $var(l) = \gamma_j(l) - l$ for any $l \in \widetilde{\mathcal{L}}_j$.

Let $p : \widetilde{\Lambda}_C \longrightarrow C$ be the projection. For every element g in $\pi_1(p^{-1}(x, y)), (x, y) \in C$, there is a natural transformation $I_g : \psi_c \longrightarrow \psi$ called the Gabber-Malgrange map [1, Proposition 5.2] or [3]. If μ_g denotes the monodromy of g on $\mathcal{F}_j$, then [1, Theorem 5.3] gives $\mu_g - 1 = p_j \circ I_g \circ q_j$. For every j there is only one Gabber-Malgrange map $I_{(\alpha_j, \beta_j)}$ for the pair of generators (α_j, β_j) and $I_{(\alpha_j, \beta_j)} = Id$. Hence the above relation leads to: $\mu_{\alpha_j} - 1 = p_j \circ q_j$ (resp. $\mu_{\beta_j} - 1 = p_j \circ q_j$). As above, one denotes the action induced by (α_j, β_j) on $\mathcal{L}$ by (a_j, d_j), the action induced by γ_j on $\mathcal{F}_j$ by τ_j, and the action induced by γ_j on $\mathcal{L}$ by b_j. Therefore the relations (1)–(4) are deduced from the above equality. $\square$

A detailed description of this result is given as follows: By [5, p. 145], given $0 < \eta \ll 1$, $0 < \xi \ll 1$, consider $\gamma(t) = \eta^2 \exp(2i\pi nt) - \xi^2 \exp(2i\pi(t(m+1) - \frac{1}{nl}))$, $\xi\eta \exp(2i\pi(t(n+1) - \frac{1}{2nl})) + \xi\eta \exp(2i\pi(tm - \frac{1}{2nl})), t \in [0, 1]$.

Denote by $\mathcal{F}_w$ the vector space associated with the corresponding local system evaluated at the point corresponding to $\gamma(w)$. Let $\theta_w : \mathcal{F}_w \longrightarrow \mathcal{F}_{w+m}$ be the transformation obtained by the local system along γ. We then obtain

Proposition 5. *The category $Perv_\Lambda(\mathbf{C}^2\backslash\{0\})$ is equivalent to the category of $(k+1)$-tuples of vector spaces $(\mathcal{L}, \mathcal{F}_1, \cdots, \mathcal{F}_k)$ with the maps $\mathcal{L} \underset{q_w}{\overset{p_w}{\leftrightarrows}} \mathcal{F}_w$ and $\theta_w : \mathcal{F}_w \longrightarrow \mathcal{F}_{w+m}$ such that:*

(1) $q_w p_w = a_w - 1$ *invertible,*
(2) $q_w p_w = d_w - 1$ *invertible,* (3) θ_w *invertible,*
(4) $q_{w+m}\theta_w = a_u^{\xi_w\mu_w} d_u q_u = a_w^{\mu_w} d_w^{\xi_w} q_w$, *if $u > w$ and u is not a shaft,*
(5) $q_{w+m}\theta_w = a_u^{\mu_w} d_u^{\xi_w} q_u = a_w^{\mu_w} d_w^{\xi_w} q_w$, *if $u > w$ and u is a shaft,*
(6) $q_{w+m}\theta_w = (a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w})^{\xi_w} q_w = (a_w^{\mu_w} d_w^{\xi_w})^{r_w} q_w$, *$w$ nonterminal without shaft,*
(7) $q_{w+m}\theta_w = a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w} q_w = a_{w_0}^{s_w} d_{w_0}^{r_w} q_w$, *$w$ nonterminal with shaft w_0,*
(8) $\theta_w p_w = p_u a_u^{\xi_w\mu_w} d_u = p_w a_w^{\mu_w} d_w^{\xi_w}$, *if $u > w$ and u is not a shaft,*
(9) $\theta_w p_w = p_u a_u^{\mu_w} d_u^{\xi_w} = p_w a_w^{\mu_w} d_w^{\xi_w}$, *if $u > w$ and u is a shaft,*
(10) $\theta_w p_w = p_w(a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w})^{\xi_w} = p_w(a_w^{\mu_w} d_w^{\xi_w})^{r_w}$, *$w$ nonterminal without shaft,*
(11) $\theta_w p_w = p_w a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w} = p_w a_{w_0}^{s_w} d_{w_0}^{r_w}$, *$w$ nonterminal with shaft w_0.*

Proof. We are going to construct two functors establishing an equivalence of categories between the data of Proposition 4 and those of Proposition 5. In order to construct the data of an object of Proposition 4 from that of Proposition 5, consider $\tau_w := \prod_{j=1}^{\frac{nl}{k}-1} \theta_{w+jm}$ and b_w being words on the a_w and d_w such that:

$$b_w := \begin{cases} (a_u^{\xi_w\mu_w} d_u)^{\frac{nl}{k}} = (a_w^{\mu_w} d_w^{\xi_w})^{\frac{nl}{k}}, \\ \quad \text{if } u > w \text{ and } u \text{ is not a shaft;} \\ (a_u^{\mu_w} d_u^{\xi_w})^{\frac{nl}{k}} = (a_w^{\mu_w} d_w^{\xi_w})^{\frac{nl}{k}}, \\ \quad \text{if } u > w \text{ and } u \text{ is a shaft;} \\ (a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w})^{\frac{\xi_w nl}{k}} = (a_w^{\mu_w} d_w^{\xi_w})^{\frac{r_w nl}{k}}, \\ \quad w \text{ nonterminal without shaft;} \\ (a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w})^{\frac{nl}{k}} = (a_{w_0}^{s_w} d_{w_0}^{r_w})^{\frac{nl}{k}}, \\ \quad w \text{ nonterminal with shaft } w_0. \end{cases}$$

Using (4)–(11) of Proposition 5 we obtained (3), (4) of Proposition 4, and moreover the action by $\langle a_1, \cdots, a_k, d_1, \cdots, d_k\rangle$ on $\mathcal{L}$ satisfies the relations of Proposition 3. Conversely, we define a functor from which one constructs the data of an object of Proposition 5 from that of Proposition 4 as follows: choose $\theta_w = id$ when $w = h + jm$ for $1 \le h \le k$, $j < \frac{nl}{k} - 1$ and $\theta_w = b_h$ when $w = h + \frac{lnm}{k}$ for $1 \le h \le k$. We define mappings q_w and p_w thanks to the formulas (4)–(11) of the Proposition 5 for values w not included in $1 \le h \le k$. The above defined functors are inverse to one another. $\square$

2. Extending across the origin. The main theorem is stated as follows:

Theorem 1. *The category $Perv_\Lambda(\mathbf{C}^2)$ of perverse sheaves on $\mathbf{C}^2$ constructible with respect to the stratification $\{0\} \subset \{C\} \subset \mathbf{C}^2$ where C is a general plane curve, is equivalent to the category of $(k+2)$-tuples of finite dimensional vector spaces $(\mathcal{L}, \mathcal{F}_1, \cdots, \mathcal{F}_k, \mathcal{F})$ related by linear maps $\mathcal{L} \underset{q_w}{\overset{p_w}{\leftrightarrows}} \mathcal{F}_w \underset{r_w}{\overset{s_w}{\leftrightarrows}} \mathcal{F}$ and $\theta_w : \mathcal{F}_w \longrightarrow \mathcal{F}_{w+m}$ such as:*

(1) $q_w p_w = a_w - 1$ *invertible,*
(2) $q_w p_w = d_w - 1$ *invertible,* (3) θ_w *invertible,*
(4) $q_{w+m}\theta_w = a_u^{\xi_w\mu_w} d_u q_u = a_w^{\mu_w} d_w^{\xi_w} q_w$, *if $u > w$ and u is not a shaft,*
(5) $q_{w+m}\theta_w = a_u^{\mu_w} d_u^{\xi_w} q_u = a_w^{\mu_w} d_w^{\xi_w} q_w$, *if $u > w$ and u is a shaft,*
(6) $q_{w+m}\theta_w = (a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w})^{\xi_w} q_w = (a_w^{\mu_w} d_w^{\xi_w})^{r_w} q_w$, *$w$ nonterminal without shaft,*
(7) $q_{w+m}\theta_w = a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w} q_w = a_{w_0}^{s_w} d_{w_0}^{r_w} q_w$, *$w$ nonterminal with shaft w_0,*
(8) $\theta_w p_w = p_u a_u^{\xi_w\mu_w} d_u = p_w a_w^{\mu_w} d_w^{\xi_w}$, *if $u > w$ and u is not a shaft,*
(9) $\theta_w p_w = p_u a_u^{\mu_w} d_u^{\xi_w} = p_w a_w^{\mu_w} d_w^{\xi_w}$, *if $u > w$ and u is a shaft,*
(10) $\theta_w p_w = p_w (a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w})^{\xi_w} = p_w (a_w^{\mu_w} d_w^{\xi_w})^{r_w}$, *$w$ nonterminal without shaft,*
(11) $\theta_w p_w = p_w a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w} = p_w a_{w_0}^{s_w} d_{w_0}^{r_w}$, *$w$ nonterminal with shaft w_0,*
(12) $\sum_{i=1}^{k} r_i p_i = 0$, (13) $\sum_{i=1}^{k} q_i s_i = 0$,
(14) $s_j r_l = -\sum_{i=1+l}^{l+w} p_i a_u^{\xi_w\mu_w} d_u q_u + \delta_{l+w,j}^{(nl)} \theta_l - \delta_{lj}^{(nl)}$ *if $u > w$ and u is not a shaft,*
(15) $s_j r_l = -\sum_{i=1+l}^{l+w} p_i a_u^{\mu_w} d_u^{\xi_w} q_u + \delta_{l+w,j}^{(nl)} \theta_l - \delta_{lj}^{(nl)}$ *if $u > w$ and u is a shaft,*
(16) $s_j r_l = -\sum_{i=1+l}^{l+w} p_i (a_{w_1} \cdots a_{w_{b_w}} a_w^{s_w} d_w^{r_w})^{\xi_w} q_w + \delta_{l+w,j}^{(nl)} \theta_l - \delta_{lj}^{(nl)}$, *$w$ nonterminal without shaft where $\delta^{(nl)}$ is the kronecker symbol modulo nl.*

To prove this theorem, we need to consider a perverse sheaf $P^\bullet$ in $Perv_\Lambda(\mathbf{C}^2 \backslash \{0\})$ together with its combinatorial data of Proposition 5, and compute the variation mapping $var : \psi(P^\bullet) \longrightarrow \psi_c(P^\bullet)$ associated to $P^\bullet$. Then we will make another application of Theorem 5.3 of [1]. Note that the extension being done across the origin in $\mathbf{C}^2$ it suffices to evaluate $\psi(P^\bullet)$ and $\psi_c(P^\bullet)$ for one direction in $\widetilde{\Lambda}_{\{0\}}$.

Recall that C is a germ of a general plane curve at the origin o, with irreducible components C_d, $1 \le d \le w$. As in section 1.1, each nonterminal vertex $v = (d, \epsilon)$ is associated with the branch $C_d = \{y = \sum_\epsilon a_{d,\epsilon} x^\epsilon, \epsilon \in \mathbf{Q}^+, a_{d,\epsilon} \in \mathbf{C}\}$. For $0 < \eta \ll 1$ let $K_d = C_d \cap (\{x : |x| = \eta\} \times \mathbf{C})$ be the knot of C_d. Denote by N_d a closed neighborhood of K_d, and consider $N = (N_d)_{d \in \Xi_C}$ the toric system for the curve C. Let $P^\bullet$ be a perverse object in $Perv_\Lambda(\mathbf{C}^2 \backslash \{0\})$. One knows from [1, Remark 1) and 3)] that the stalk of $\psi(P^\bullet)$ at ξ in $\widetilde{\Lambda}_{\{0\}}$ is isomorphic to $\mathbf{H}^{-1}(N, P^\bullet)$, where N is a complex link to the direction ξ, and $\psi_c(P^\bullet)_\xi \simeq \mathbf{H}_c^{-1}(N, P^\bullet)$.

Lemma 2. *For every perverse sheaf $P^\bullet$ in $Perv_\Lambda(\mathbf{C}^2 \backslash \{0\})$ one has:*

(1) $\psi(P^\bullet) = \mathbf{H}^{-1}(N, P^\bullet) = \mathrm{Coker}(\mathcal{L} \xrightarrow{(p_1, \cdots, p_k)} \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k)$,
(2) $\psi_c(P^\bullet) = \mathbf{H}_c^{-1}(N, P^\bullet) = \mathrm{Ker}(\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k \xrightarrow{(q_1, \cdots, q_k)} \mathcal{L})$.

Proof. Let $K = \bigcup_{d \in \Xi_C} K_d \subset N$. Denote by $i : K \hookrightarrow N$ and $j : N \backslash K \hookrightarrow N$. By considering the associated hypercohomology long exact sequence for the distinguished triangle $j_! j^* P^\bullet \to P^\bullet \to i_* i^* P^\bullet \xrightarrow{[1]}$, one gets the long exact sequence

$$\cdots \to \mathbf{H}^{-2}(K,P^\bullet) \xrightarrow{\delta} \mathbf{H}^{-1}(N,j_!j^*P^\bullet) \longrightarrow \mathbf{H}^{-1}(N,P^\bullet) \longrightarrow \mathbf{H}^{-1}(K,P^\bullet) \longrightarrow \cdots.$$

Then using the combinatorial data of Proposition 5: vector spaces $(\mathcal{L},\mathcal{F}_1,\cdots,\mathcal{F}_k)$ and maps $\mathcal{L} \underset{q_w}{\overset{p_w}{\leftrightarrows}} \mathcal{F}_w$ associated to $P^\bullet$, one obtains $\mathbf{H}^{-1}(K,P^\bullet)=0$ and $\mathbf{H}^{-2}(K,P^\bullet)\cong\mathcal{L}$. Giving rise to the canonical isomorphism $\mathbf{H}^{-1}(N,j_!j^*P^\bullet)\cong\bigoplus_{l=1}^{k}\mathcal{F}_l$, and the degree raising map $\delta=(p_1,\cdots,p_k):\mathcal{L}\longrightarrow\mathcal{F}_1\oplus\cdots\oplus\mathcal{F}_k$. So (1) is proved. Similarly, by applying hypercohomology with compact support to the above triangle, one gets the following long exact sequence for compactly supported hypercohomology

$$\cdots \to \mathbf{H}_c^{-1}(N,i_*i^!P^\bullet) \longrightarrow \mathbf{H}_c^{-1}(N,P^\bullet) \longrightarrow \mathbf{H}_c^{-1}(N,Rj_*j^*P^\bullet) \xrightarrow{\delta} \mathbf{H}_c^{0}(N,i_*i^!P^\bullet) \to \cdots.$$

As above, one gets $\mathbf{H}_c^{-1}(N,i_*i^!P^\bullet)=0$ and $\mathbf{H}_c^{0}(N,i_*i^!P^\bullet)\cong\mathcal{L}$. This leads to the canonical isomorphism $\mathbf{H}_c^{-1}(N,Ri_*j^!P^\bullet)\cong\bigoplus_{l=1}^{k}\mathcal{F}_l$ and the map $\delta=(q_1,\cdots,q_k):\mathcal{F}_1\oplus\cdots\oplus\mathcal{F}_k\to\mathcal{L}$. Then (2) is proved. □

Now, to compute the variation map, let us write any element $(b_1,\cdots,b_k)$ in $\mathcal{F}_1\oplus\cdots\oplus\mathcal{F}_k$ as $\sum_{i=1}^{k}b_ie_i$.

Lemma 3. *The variation map* $var:\psi(P^\bullet)\longrightarrow\psi_c(P^\bullet)$ *is defined as follows*:

(1) $var(b_le_l)=-\sum_{i=1}^{l+w}p_ia_u^{\xi_w\mu_w}d_uq_ub_ue_i+\theta(b_l)e_{l+w}-b_le_l=-\sum_{i=1+l}^{l+w}p_ia_w^{\mu_w}d_w^{\xi_w}q_wb_we_i+\theta(b_l)e_{l+w}-b_le_l$, *if* $u>w$ *and* u *is not a shaft,*

(2) $var(b_le_l)=-\sum_{i=1+l}^{l+w}p_ia_u^{\mu_w}d_u^{\xi_w}q_ub_ue_i+\theta(b_l)e_{l+w}-b_le_l=-\sum_{i=i+1}^{l+w}p_ia_w^{\mu_w}d_w^{\xi_w}q_wb_we_i+\theta(b_l)e_{l+w}-b_le_l$, *if* $u>w$ *and* u *is a shaft,*

(3) $var(b_le_l)=-\sum_{i=1+l}^{l+w}p_i(a_{w_1}\cdots a_{w_{b_w}}a_w^{s_w}d_w^{r_w})^{\xi_w}q_wb_we_i+\theta(b_l)e_{l+w}-b_le_l=-\sum_{i=1+l}^{l+w}p_i(a_w^{\mu_w}d_w^{\xi_w})^{r_w}q_wb_we_i+\theta(b_l)e_{l+w}-b_le_l$, *$w$ nonterminal without shaft.*

Proof. If C is a curve with components $(C_d)_{0\le d\le l}$ admitting the Puiseux expansion $y=\exp(2i\pi(\frac{d-1}{nl}))x^{\frac{m}{n}}$ for $d\ge1$ and $y=0$ for $d=0$. Let $Z=\bigcup_{d=1}^{l}Z_d$ where Z_d is defined as follows: Take $0<\eta\ll1$, $0<\tilde{\xi}\ll\xi\ll1$. Following [5], consider $Z_d:=\{(\kappa_d^{-1}\lambda_d^{-1}\tilde{\kappa}_d\tilde{\kappa}_{d-1}^{-1}\lambda_{d-1}\kappa_{d-1})(t),t\in[0,1]\}$ where we have defined, for $j\in\{0,\cdots,nl\}$, $\kappa_j(t)=(\eta,\xi\exp i\frac{\pi}{nl}(-1+2tj))$, $\tilde{\kappa}_j(t)=(\eta,\tilde{\xi}\exp i\frac{\pi}{nl}(-1+2tj))$, $\kappa_{j-1}^{-1}(t)=(\eta,\xi\exp i\frac{\pi}{nl}(-1+2j(1-t)))$, $\tilde{\kappa}_j^{-1}(t)=(\eta,\tilde{\xi}\exp i\frac{\pi}{nl}(-1+2j(1-t)))$, $\lambda_j(t)=(\eta,(\xi(1-t)+t\tilde{\xi})\exp i\frac{\pi}{nl}(-1+2j))$, $\lambda_{j-1}^{-1}(t)=(\eta,(\xi t+(1-t)\tilde{\xi})\exp i\frac{\pi}{nl}(-3+2j))$. Let $i:Z\hookrightarrow N$ and $j:N\backslash Z\hookrightarrow N$. By a calculation one gets $\mathbf{H}^{-1}(N,i_*i^!P^\bullet)\cong\mathcal{F}_1\oplus\cdots\oplus\mathcal{F}_k$ canonically. The variation map is determined as follows: consider a monodromy map $\mu:N\longrightarrow N$ such that $\mu|\partial N=id$, and a function $u:N\longrightarrow\mathbf{R},z\mapsto u(z)=\begin{cases} m & |z|\le\frac12\\ 2m(1-|z|) & \frac12\le|z|\le1\end{cases}$. Then $\mu(z)=e^{2i\pi u(z)}z$. Using the canonical map $\mathcal{F}_r\longrightarrow\psi(P^\bullet)\longrightarrow\psi_c(P^\bullet)\longrightarrow\bigoplus_{l=1}^{k}\mathcal{F}_l$ we obtain the formulas. □

Proof of Theorem 1. Using Proposition 1, Lemma 2, Lemma 3 and applying [1, Theorem 5.3], yields the result. □

3. Perverse sheaves with no vanishing cycles at the origin. This section is an application of our results to a specific example. Let C be a curve in $\mathbf{C}^2$ with a branch of genus 2 given by a Puiseux expansion of type $y=x^{\frac32}+x^{\frac94}$. We study the category $Perv_\Lambda(\mathbf{C}^2)$. Consider $\mathbf{C}\langle\mathcal{L},\mathcal{E},\mathcal{F}\rangle$ the free algebra with three generators, and $K=\mathbf{C}\langle\mathcal{L},\mathcal{E},\mathcal{F}\rangle/(\mathcal{L}\mathcal{E}\mathcal{L}+\mathcal{L}^2+\mathcal{L},\mathcal{E}\mathcal{L}\mathcal{E}+\mathcal{E}^2+\mathcal{E},\mathcal{E}\mathcal{F}\mathcal{E}+\mathcal{E}^2+\mathcal{E},\mathcal{F}\mathcal{E}\mathcal{F}+\mathcal{F}^2+\mathcal{F})$.

Proposition 6. *The category $Perv_\Lambda(\mathbf{C}^2)$ is equivalent to the one of K-modules such that $\mathcal{L}+I,\mathcal{E}+I$ and $\mathcal{F}+I$ are invertible transformations. The equivalence is obtained by associating to the K-module a local system on $\mathbf{C}^2\backslash C$ by choosing $a_1=\mathcal{L}+I,a_2=\mathcal{E}+I$, $a_3=\mathcal{F}+I$ and then taking the intersection homology extension to all of $\mathbf{C}^2$.*

Proof. From Theorem 1, to be in the case of no vanishing cycles at the origin we put $\mathcal{F}=0$. In other words the following system of equations has to be satisfied

$$S=\begin{cases} p_1a_2q_1=-1,\ p_2a_1q_2=-1,\ p_2a_3q_2=-1,\\ p_3a_2q_3=-1,\ \theta_1-p_2q_1-p_2a_1a_2q_1=0,\\ \theta_2-p_1q_2-p_1a_2a_1q_2-p_3q_2-p_3a_2a_3q_2=0,\\ \theta_3-p_2q_3-p_2a_3a_2q_3=0.\end{cases}$$

Since p_1, p_2 and p_3 are surjections and that q_1, q_2 and q_3 are injections, the first two equations are equivalent to

$$\begin{cases} q_1p_1 = -q_1p_1a_2q_1p_1,\ q_2p_2 = -q_2p_2a_1q_2p_2, \\ q_2p_2 = -q_2p_2a_3q_2p_2,\ q_3p_3 = -q_3p_3a_2q_3p_3. \end{cases}$$

Put $\mathcal{L} = a_1 - 1$, $\mathcal{E} = a_2 - 1$ and $\mathcal{F} = a_3 - 1$ therefore these equations take the following form:

$$\begin{cases} \mathcal{L}\mathcal{E}\mathcal{L} + \mathcal{L}^2 + \mathcal{L} = 0,\ \mathcal{E}\mathcal{L}\mathcal{E} + \mathcal{E}^2 + \mathcal{E} = 0, \\ \mathcal{E}\mathcal{F}\mathcal{E} + \mathcal{E}^2 + \mathcal{E} = 0,\ \mathcal{F}\mathcal{E}\mathcal{F} + \mathcal{F}^2 + \mathcal{F} = 0. \end{cases}$$

So for any object in $Perv_\Lambda(\mathbf{C}^2)$ the previous equations are satisfied. Conversely, if they are satisfied, it follows from what was said above that we obtain an object in $Perv_\Lambda(\mathbf{C}^2)$ if we define θ_1, θ_2 and θ_3 by S. □

Let W_γ be a family of representations of K on $\mathbf{C}^2$ given by: $\mathcal{L} = \begin{pmatrix} 0 & 0 \\ 1 & \gamma \end{pmatrix}$, $\mathcal{E} = \begin{pmatrix} \gamma & -(\gamma+1) \\ 0 & 0 \end{pmatrix}$, $\mathcal{F} = \begin{pmatrix} 0 & 0 \\ -1 & \gamma \end{pmatrix}$ with $\gamma \in \mathbf{C}$: W_γ being irreducible for $\gamma \neq \zeta_3, \zeta_3^2$ (ζ_3 is a third root of unity). For $\gamma = \zeta_3$ or $\gamma = \zeta_3^2$, W_γ has the trivial one dimensional-representation and the quotient representation is given by $\mathcal{L} = \mathcal{E} = \mathcal{F} = \zeta_3^2$ or ζ_3. Let $\widetilde{W}_\gamma$ be the family of irreducible representations of K where we quotiented out of trivial representations at $\gamma = \zeta_3$ and $\gamma = \zeta_3^2$ in the family W_γ and add them separately to the family.

Proposition 7. *The irreducible objects in* $Perv_\Lambda(\mathbf{C}^2)$ *correspond to the irreducible representations of* $\widetilde{W}_\gamma$.

Proof. We prove that any irreducible representation of K occurs in $\widetilde{W}_\gamma$. Let W be an arbitrary irreducible representation of K. Note that $\mathcal{L}^2\mathcal{E} = \mathcal{L}\mathcal{E}^2$, $\mathcal{E}^2\mathcal{L} = \mathcal{E}\mathcal{L}^2$; $\mathcal{E}^2\mathcal{F} = \mathcal{E}\mathcal{F}^2$, $\mathcal{F}^2\mathcal{E} = \mathcal{F}\mathcal{E}^2$. Consider
1) Either $\mathcal{L}$ or $\mathcal{E}$ has a non-zero eigenvalue. Suppose there exits a $w \in W$ such that $\mathcal{E}w = \gamma w$, $\gamma \neq 0$. We have two cases:
i) If $\mathcal{L}w = \mu w$, then the relations for W imply that the following equations must be satisfied

$$S = \{\ \mu\gamma\mu + \mu^2 + \mu = 0,\ \gamma\mu\gamma + \gamma^2 + \gamma = 0.$$

If $\mu = 0$ then $\gamma = 0$ or $\gamma = -1$ which is impossible. Thus $\gamma \neq 0$ and we can see that $\gamma = \mu$ and $\gamma^2 + \gamma + 1 = 0$. This yields a one-dimensional representation in $\widetilde{W}_\gamma$.
ii) Assume $\mathcal{L}w = v$ is not a multiple of w. Let V be the subspace of W generated by v and w. V is invariant by K. We have $\mathcal{E}v = \mathcal{E}\mathcal{L}w = \frac{1}{\gamma}\mathcal{E}\mathcal{L}\mathcal{E}w = -\frac{1}{\gamma}(\mathcal{E}^2 + \mathcal{E})w = -(\gamma+1)w$ and $\mathcal{L}v = \mathcal{L}\mathcal{L}w = \frac{1}{\gamma}\mathcal{L}^2\mathcal{E}w = \frac{1}{\gamma}\mathcal{L}\mathcal{E}^2 w = \gamma\mathcal{L}w = \gamma v$ thus $W = V$ occurs in $\widetilde{W}_\gamma$.
2) Either $\mathcal{F}$ or $\mathcal{E}$ has a non-zero eigenvalue. Assume there exists $w \in W$ such that $\mathcal{F}w = \gamma w$, $\gamma \neq 0$. We have two cases:
i) Assume that $\mathcal{E}w = \mu w$. Then the relations for W imply that the system S must be satisfied.
If $\mu = 0$ then $\gamma = 0$ or $\gamma = -1$ which is impossible. So $\gamma \neq 0$ and we see that $\gamma = \mu$ and $\gamma^2 + \gamma + 1 = 0$. This leads to a one-dimensional representation in $\widetilde{W}_\gamma$.
ii) Assume $\mathcal{E}w = v$ is not a multiple of w. Let V be the subspace of W spanned by v and w. V is invariant by K. We have $\mathcal{F}v = \mathcal{F}\mathcal{E}w = \frac{1}{\gamma}\mathcal{F}\mathcal{E}\mathcal{F}w = -\frac{1}{\gamma}(\mathcal{F}^2 + \mathcal{F})w = -(\gamma+1)w$ and $\mathcal{E}v = \mathcal{E}\mathcal{E}w = \frac{1}{\gamma}\mathcal{E}^2\mathcal{F}w = \frac{1}{\gamma}\mathcal{E}\mathcal{F}^2 w = \gamma\mathcal{E}w = \gamma v$ thus $W = V$ occurs in $\widetilde{W}_\gamma$.

Now if neither $\mathcal{L}$ nor $\mathcal{E}$ has non-zero eigenvalues; and neither $\mathcal{F}$ nor $\mathcal{E}$ has non-zero eigenvalues. We can have $\mathcal{L} = \mathcal{E} = \mathcal{F} = 0$ and obtain a trivial representation but if not at least one of them must have a non-trivial Jordan block. Then we can obtain a representation in $\widetilde{W}_\gamma$. □

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