The transcendence of zeros of natural basis elements for the space of the weakly holomorphic modular forms for $\Gamma_0^+(3)$

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(Communicated by Kenji FUKAYA, M.J.A., June 13, 2022)

Abstract: We consider a natural basis for the space of weakly holomorphic modular forms for $\Gamma_0^+(3)$. We prove that for some of the basis elements, if z_0 in the fundamental domain for $\Gamma_0^+(3)$ is one of zeroes of the elements, then either z_0 is transcendental or is in $\{\frac{i}{\sqrt{3}}, \frac{-1+\sqrt{2}i}{3}, \frac{-3+\sqrt{3}i}{6}, \frac{-1+\sqrt{11}i}{6}\}$.

Key words: weakly holomorphic modular form; transcendence.

1. Introduction and statement of a main result. Since Rankin and Swinnerton-Dyer [9], the zeros of weakly holomorphic modular forms has been well-studied. In particular, Duke and Jenkins [4] constructed a natural basis $\{F_{k,m}\}_{m\geq -l}$ for the space of weakly holomorphic modular forms of weight k for $\mathrm{SL}_2(\mathbf{Z})$ and investigated the location of the zeros of the basis elements. The basis elements $F_{k,m}$ have Fourier expansions of the form

$$F_{k,m}(z) = q^{-m} + \sum_{n\geq l+1} a_{k,m}(n)q^n, \tag{1}$$

where $k = 12l + k'$ with $k' \in \{0, 4, 6, 8, 10, 14\}$ and $m \geq -l$. Along with the study of location of the zeros of such forms, the transcendence and algebraicity of the zeros has been investigated. Jennings-Shaffer and Swisher [7] showed that for each $m \geq |l| - l$, the zeros of $F_{k,m}$ in the standard fundamental domain for $SL_2(\mathbf{Z})$ are either transcendental or contained in $\{i, e^{2i\pi/3}\}$. In the higher level cases, Gun and Saha [5] studied the transcendence of zeros of weakly holomorphic modular forms for $\Gamma_0(p)$ under a certain assumption on the location of zeros. Also they studied the nature of the zeros of Eisenstein series for $\Gamma_0^+(p)$ with $p = 2$ or 3. The author and Im [1] extended this result to basis elements of the space of weakly holomorphic modular forms for $\Gamma_0(2)$. Here $\Gamma_0^+(p)$ is the group generated by the Hecke congruence group $\Gamma_0(p)$ and the Fricke involution $W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$.

In this paper we investigate the algebraicity and transcendence of the zeros of natural basis elements of the space of weakly holomorphic modular forms for $\Gamma_0^+(3)$. Also we remark that the zeros of these basis elements don't lie in the region described in the assumption of the result [5, Theorem 4] of Gun and Saha. So our consideration is not covered by [5, Theorem 4].

We let $\mathbf{F}^+(3)$ be the standard fundamental domain for $\Gamma_0^+(3)$ given in [8] by

$$\mathbf{F}^+(3) := \{z \in \mathbf{C} : |z| \geq 1/\sqrt{3}, -1/2 \leq \mathrm{Re}(z) \leq 0\} \cup \{z \in \mathbf{C} : |z| > 1/\sqrt{3}, 0 < \mathrm{Re}(z) < 1/2\}$$

and let

$$V := \left\{\frac{1}{\sqrt{3}} e^{i\theta} : \pi/2 \leq \theta \leq 5\pi/6\right\}.$$

For a given even integer $k \in 2\mathbf{Z}$, we can write

$$k = 12\ell_k + r_k,$$

where $\ell_k \in \mathbf{Z}$, $r_k \in \{0, 4, 6, 8, 10, 14\}$. For integer m with $m \geq -2l_k - \epsilon_k$, there exists a unique weakly holomorphic modular form with the Fourier expansion of the form

$$f_{k,m}(z) = q^{-m} + O(q^{2l_k+\epsilon_k+1}),$$

which they form a basis for the space of weakly holomophic modular forms of weight k for $\Gamma_0^+(3)$. Here ϵ_k is 0 or 1 depending on r_k (see [2,6]).

In particular we have $f_{k,m} = (\Delta_3^+)^{l_k}\Delta_{3,r_k}F_{f_{k,m}}(j_3^+)$, where $F_{f_{k,m}}(x)$ is the monic polynomial in x of degree $2l_k + \epsilon_k + m$ with integer coefficients. Where

2020 Mathematics Subject Classification. Primary 11F03, 11F11.

doi: 10.3792/pjaa.98.009

$$\begin{cases} \eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \\ \Delta_3^+(z) = (\eta(z)\eta(3z))^{12}, \\ E_{r_k}^+(z) = E_{3,r_k}^+(z) = \frac{1}{1+3^{r_k/2}}(E_{r_k}(z) + 3^{r_k/2}E_{r_k}(3z)), \\ j_3^+(z) = \left(\frac{\eta(z)}{\eta(3z)}\right)^{12} + 12 + 3^6\left(\frac{\eta(3z)}{\eta(z)}\right)^{12}, \end{cases}$$

$$\Delta_{3,r_k}(z) := \begin{cases} 1, & \text{if } r_k = 0, \\ E_4^+(z), & \text{if } r_k = 4, \\ E_6^+, & \text{if } r_k = 6, \\ \frac{41}{1728}(E_4^+(z)^2 - E_8^+(z)), & \text{if } r_k = 8, \\ \frac{61}{432}(E_4^+(z)^2E_6^+(z) - E_{10}^+(z)), & \text{if } r_k = 10, \\ \frac{-22427}{272160}(E_6^+(z)E_8^+(z) - E_{14}^+(z)), & \text{if } r_k = 14, \end{cases}$$

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\frac{n^{k-1}q^n}{1-q^n},$$

for $q = e^{2\pi iz}$, $\sigma_{k-1}(n) = \sum_{1\le d|n} d^{k-1}$ and the kth Bernoulli number B_k. In [6] Hanamoto and Kuga showed that if $m \ge 18|l_k| + 23$, then all of the zeros in $\mathbf{F}^+(3)$ of the forms $f_{k,m}$ lie on the circle with radius $1/\sqrt{3}$. In this paper we investigate the algebraicity and transcendence of zeros in $\mathbf{F}^+(3)$ of $f_{k,m}$. In particular, combining the main ideas in [1,5,7] we prove the following result.

Theorem 1.1. *Let k be an even integer and write $k = 12\ell_k + r_k$, for a unique integer $\ell_k \in \mathbf{Z}$ and a unique integer $r_k \in \{0,4,6,8,10,14\}$. If $m \ge 18|l_k| + 23$ and z_0 is a zero in $\mathbf{F}^+(3)$ of $f_{k,m}$, then either z_0 is in $\{\frac{i}{\sqrt{3}}, \frac{-1+\sqrt{2}i}{3}, \frac{-3+\sqrt{3}i}{6}, \frac{-1+\sqrt{11}i}{6}\}$ or z_0 is transcendental.*

This paper is organized as follows: In Sections 2 we prove Theorem 1.1.

2. The proof of Theorem 1.1. We start with a well-known lemma. Let j be the j-invariant defined by

$$j(z) := 1728\frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2}.$$

Lemma 2.1. *If $z \in \mathbf{H}$ and $j(z)$ is algebraic, then either z is transcendental or z is imaginary quadratic.*

Proof. It follows from Schneider's Theorem in [10]. □

Let $|_k$ is the usual slash operator. The q-expansion principle, due to Deligne and Rapoport ([3, Theorem 3.9, p. 304]), implies that if an integral weight modular form f has rational Fourier coefficeients at the cusp infinity, then f has also rational Fourier coefficients at all other cusps. By this fact we have the following lemma.

Lemma 2.2. *For a modular form f of weight k on $\Gamma_0^+(3)$ rational Fourier coefficients, the function*

$$F(z) := \prod_{\gamma\in SL_2(\mathbf{Z})/\Gamma_0(3)} f|_k\gamma$$

which is a modular form of weight $4k$ for $SL_2(\mathbf{Z})$ has also rational Fourier coefficients.

Now, we prove Theorem 1.1. Since Δ_3^+ is a cusp form of weight 12 on $\Gamma_0^+(3)$, $(\Delta_3^+)^n f_{k,m}$ is a modular form on $\Gamma_0^+(3)$ for some positive integer n. We have from Lemma 2.2 that the modular form

$$(2)\qquad F_m := \prod_{\gamma\in SL_2(\mathbf{Z})/\Gamma_0(3)} ((\Delta_3^+)^n f_{k,m})|_{12n+k}\gamma$$

of weight $48n + 4k$ on $SL_2(\mathbf{Z})$ has a rational Fourier expansion at the cusp infinity. Then since z_0 is a zero of $f_{k,m}$, we have $F_m(z_0) = 0$. From the property of the forms $F_{k,m}$ in (1) given by Duke and Jenkins in [4], F_m can be expressed as the product,

$$(3)\qquad F_m(z) = \Delta(z)^l E_{k'}(z)P_m(j(z)),$$

where $48n + 4k = 12l + k'$ for a unique integer $l \in \mathbf{Z}$ and a unique integer $k' \in \{0,4,6,8,10,14\}$ and $P_m(x)$ is a polynomial with rational coefficients. Since $F_m(z_0) = 0$ and $\Delta(z_0) \ne 0$, we see that $E_{k'}(z_0) = 0$ or $P_m(j(z_0)) = 0$. Then we have the following Lemma 2.3.

Lemma 2.3. *If $E_{k'}(z_0) = 0$ for $z_0 \in V$, then $z_0 = \frac{-3+\sqrt{3}i}{6}$.*

Proof. If $E_{k'}(z_0) = 0$ then $z_0 = \gamma i$ or $z_0 = \gamma(-\frac{1}{2} + \frac{\sqrt{3}i}{2})$ for some $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$. If $z_0 = \gamma i \in V$, then $1/(2\sqrt{3}) \le \mathrm{Im}(z_0) = 1/(C^2 + D^2) \le 1/\sqrt{3}$, which implies $C^2 + D^2 = 2$. So $1/3 = |z_0|^2 = (A^2 + B^2)/2$, but there no such integers A and B. If $z_0 = \gamma(-\frac{1}{2} + \frac{\sqrt{3}i}{2}) \in V$, then $1/(2\sqrt{3}) \le \mathrm{Im}(z_0) = \frac{\sqrt{3}/2}{C^2+D^2-CD} \le 1/\sqrt{3}$, which implies $C^2 + D^2 - CD \in \{2,3\}$. Noticing that $1/3 = |z_0|^2 = \frac{A^2+B^2-BA}{C^2+D^2-CD}$ we have $C^2 + D^2 - CD = 3$ and so $z_0 = \frac{-3+\sqrt{3}i}{6}$. □

From Lemma 2.3 it is enough to consider the case when $P_m(j(z_0)) = 0$. Since $P_m(x)$ is a polynomial with rational coefficients, $j(z_0)$ is algebraic and so by Lemma 2.1 z_0 is transcendental or z_0 is imaginary quadratic. Suppose z_0 is imaginary quadratic and z_0 is a root of a polynomial $ax^2 + bx + c$ with discriminant $d = b^2 - 4ac < 0$, where $a > 0$ and $\gcd(a, b, c) = 1$. We now consider the point $w \in \mathbf{C}$ defined by

$$w = \begin{cases} \dfrac{i\sqrt{-d}}{2}, & \text{if } d \equiv 0 \pmod 4, \\ \dfrac{-1 + i\sqrt{-d}}{2}, & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

Following the arguments in [1,5,7], we obtain that $j(z_0)$ and $j(w)$ are conjugate and so we take an automorphism σ of $\mathbf{Q}(\sqrt{d})(j(z_0))$ such that $\sigma(j(z_0)) = j(w)$. Then since σ fixes P_m, we have that $0 = \sigma(P_m(j(z_0))) = P_m(\sigma(j(z_0))) = P_m(j(w))$. As a coset decomposition of $SL_2(\mathbf{Z})$ in $\Gamma_0(3)$, we may choose the 4 elements I, S, ST, and ST^2, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$. Recalling (2) and (3), we have that $0 = F_m(w) = (\Delta_3^+(w)^n f_{k,m}(w))((\Delta_3^+)^n f_{k,m})|_{12n+k} S(w)$ $((\Delta_3^+)^n f_{k,m})|_{12n+k} ST(w)((\Delta_3^+)^n f_{k,m})|_{12n+k} ST^2(w)$. So we see

$$f_{k,m}(w) = 0 \text{ or } f_{k,m}(Sw) = 0 \text{ or}$$
$$f_{k,m}(STw) = 0 \text{ or } f_{k,m}(ST^2 w) = 0$$

because Δ_3^+ has no zeros on the upper half plane. We now let for a positive integer n

$$d = \begin{cases} -4n, & \text{if } d \equiv 0 \pmod 4, \\ -4n + 1, & \text{if } d \equiv 1 \pmod 4, \end{cases}$$

so we have

$$w = \begin{cases} i\sqrt{n}, & \text{if } d \equiv 0 \pmod 4, \\ \dfrac{-1 + i\sqrt{4n-1}}{2}, & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

Since $w \in \mathbf{F}^+(3) - V$, we see from [6] that $f_{k,m}(w) \neq 0$. Now suppose that

$$f_{k,m}(Sw) = 0 \text{ or } f_{k,m}(STw) = 0 \text{ or } f_{k,m}(ST^2 w) = 0.$$

We note the following lemma that is important to prove Proposition 2.5.

Lemma 2.4. *Let $z_0 \in V$ be a root of a polynomial $ax^2 + bx + c$ with discriminant $d = b^2 - 4ac < 0$, where $a > 0$ and $\gcd(a, b, c) = 1$. Then*
(1) if $d = -12$ (so $w = i\sqrt{3}$), then z_0 is $\frac{i}{\sqrt{3}}$ or $\frac{-3+\sqrt{3}i}{6}$,
(2) if $d = -8$ (so $w = i\sqrt{2}$), then $z_0 = \frac{-1+\sqrt{2}i}{3}$,
(3) if $d = -4n + 1$ (so $w = \frac{-1+i\sqrt{4n-1}}{2}$) for $n \in \{1, 3\}$, then $z_0 = \frac{-3+\sqrt{3}i}{6}$ with $n = 1$ or $z_0 = \frac{-1+\sqrt{11}i}{6}$ with $n = 3$.

Proof. Since $z_0 \in V$, the following properties are satisfied:

$$z_0 = \frac{-b + i\sqrt{-d}}{2a} \in V$$

and

$$a \geq b \geq 0, \quad \frac{1}{2\sqrt{3}} \leq \operatorname{Im} z_0 = \frac{\sqrt{-d}}{2a} \leq \frac{1}{\sqrt{3}},$$
$$\text{and} \quad \frac{b^2 - d}{4a^2} = \frac{1}{3}.$$

If $d = -12$ then $3 \leq a \leq 6$ and $b^2 + 12 = 4a^2/3$. So $(a, b) \in \{(3, 0), (6, 6)\}$ which say $z_0 = \frac{i}{\sqrt{3}}$ or $z_0 = \frac{-3+\sqrt{3}i}{6}$.

If $d = -8$ then $a = 3$ and $b^2 + 8 = 4a^2/3$. So $a = 3$ and $b = 2$ which say $z_0 = \frac{-1+\sqrt{2}i}{3}$.

If $d = -3$ then $a = 2$ or 3 and $b^2 + 3 = 4a^2/3$. So $a = 3$ and $b = 3$ which say $z_0 = \frac{-3+\sqrt{3}i}{6}$.

If $d = -11$ then $a \in \{3, 4, 5\}$ and $b^2 + 11 = 4a^2/3$. So $a = 3$ and $b = 1$ which say $z_0 = \frac{-1+\sqrt{11}i}{6}$. □

We complete the proof of Theorem 1.1 by proving Proposition 2.5 explicitly. Note that if $f_{k,m}(u) = 0$ then $\gamma u \in V$ or $\gamma W_3 u \in V$ for some $\gamma \in \Gamma_0(3)$.

Proposition 2.5. *If $f_{k,m}(Sw) = 0$ then we get the following (1)–(4), and if $f_{k,m}(STw) = 0$ then we get the following (5)–(8), and if $f_{k,m}(ST^2 w) = 0$ then we get the following (9)–(12).*

(1) *If $w = i\sqrt{n}$ and $\gamma Sw \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 3$ and $z_0 = \frac{i}{\sqrt{3}}$ or $z_0 = \frac{-3+\sqrt{3}i}{6}$.*

(2) *If $w = i\sqrt{n}$ and $\gamma W_3 Sw \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 3$ and $z_0 = \frac{i}{\sqrt{3}}$ or $z_0 = \frac{-3+\sqrt{3}i}{6}$.*

(3) *If $w = \frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma Sw \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 3$ and $z_0 = \frac{-1+\sqrt{11}i}{6}$.*

(4) *If $w = \frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma W_3 Sw \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 1$ and $z_0 = \frac{-3+\sqrt{3}i}{6}$, or $n = 3$ and $z_0 = \frac{-1+\sqrt{11}i}{6}$.*

(5) *If $w = \sqrt{n}i$ and $\gamma STw \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 2$ and $z_0 = \frac{-1+\sqrt{2}i}{3}$.*

(6) *If $w = \sqrt{n}i$ and $\gamma W_3 STw \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 2$ and $z_0 = \frac{-1+\sqrt{2}i}{3}$.*

(7) *If $w = \frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma STw \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 1$ and $z_0 = \frac{-3+\sqrt{3}i}{6}$, or $n = 3$ and $z_0 = \frac{-1+\sqrt{11}i}{6}$.*

(8) *If $w = \frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma W_3 STw \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 1$ and $z_0 = \frac{-3+\sqrt{3}i}{6}$, or $n = 3$ and $z_0 = \frac{-1+\sqrt{11}i}{6}$.*

(9) *If $w = \sqrt{n}i$ and $\gamma ST^2 w \in V$ for some $\gamma \in \Gamma_0(3)$, then $n = 2$ and $z_0 = \frac{-1+\sqrt{2}i}{3}$.*

(10) *If* $w=\sqrt{n}i$ *and* $\gamma W_3ST^2w\in V$ *for some* $\gamma\in\Gamma_0(3)$, *then* $n=2$ *and* $z_0=\frac{-1+\sqrt{2}i}{3}$.

(11) *If* $w=\frac{-1+i\sqrt{4n-1}}{2}$ *and* $\gamma ST^2w\in V$ *for some* $\gamma\in\Gamma_0(3)$, *then* $n=1$ *and* $z_0=\frac{-3+\sqrt{3}i}{6}$.

(12) *If* $w=\frac{-1+i\sqrt{4n-1}}{2}$ *and* $\gamma W_3ST^2w\in V$ *for some* $\gamma\in\Gamma_0(3)$, *then* $n=1$ *and* $z_0=\frac{-3+\sqrt{3}i}{6}$.

Proof. Let $\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_0(3)$. Then for u in the upper half plane, $\gamma u\in V$ satisfies the following

(a) $AD-BC=1$ and $3|C$, so $AD\neq 0$.
(b) $-1/2\le \mathrm{Re}(\gamma u)\le 0$.
(c) $1/(2\sqrt{3})\le \mathrm{Im}(\gamma u)\le 1/\sqrt{3}$.
(d) $|\gamma u|^2=1/3$.

For convenience we give the following list:

$$\gamma S=\begin{pmatrix}-B & A\\ -D & C\end{pmatrix},\quad \gamma ST=\begin{pmatrix}-B & -B+A\\ -D & -D+C\end{pmatrix},$$
$$\gamma ST^2=\begin{pmatrix}-B & -2B+A\\ -D & -2D+C\end{pmatrix}$$
$$W_3S=\begin{pmatrix}1/\sqrt{3} & 0\\ 0 & \sqrt{3}\end{pmatrix},$$
$$W_3ST=\begin{pmatrix}1/\sqrt{3} & 1/\sqrt{3}\\ 0 & \sqrt{3}\end{pmatrix},$$
$$W_3ST^2=\begin{pmatrix}1/\sqrt{3} & 2/\sqrt{3}\\ 0 & \sqrt{3}\end{pmatrix}.$$

For (1), if $w=\sqrt{n}i$ and $\gamma Sw=\frac{-B\sqrt{n}i+A}{-D\sqrt{n}i+C}\in V$, then the condition (c) gives

$$\sqrt{3}\le\frac{C^2}{\sqrt{n}}+D^2\sqrt{n}\le 2\sqrt{3},$$

which says $D^2=1$ and $1\le n\le 12$. There is no non-zero integer C satisfying

$$\sqrt{3n}\le C^2+n\le 2\sqrt{3n}\quad\text{and}\quad 3|C.$$

Thus $C=0$. By (a) we have that $AD=1$ and $\gamma Sw=\pm B+i/\sqrt{n}\in V$ which implies $n=3$. So by Lemma 2.4 $z_0=\frac{i}{\sqrt{3}}$ or $z_0=\frac{-3+\sqrt{3}i}{6}$.

For (2), if $w=\sqrt{n}i$ and $\gamma W_3Sw=\frac{A\sqrt{n}i/3+B}{C\sqrt{n}i/3+D}\in V$, then the condition (c) gives

$$\sqrt{3}\le\frac{D^2+nC^2/9}{\sqrt{n}/3}\le 2\sqrt{3},$$

which says $C^2<6\sqrt{\frac{3}{n}}\le 6\sqrt{3}$. So by (a) we have that $C=0$ or $C^2=9$. If $C^2=9$ then

$$3\sqrt{n}<\frac{D^2+nC^2/9}{\sqrt{n}/3}=\frac{3D^2}{\sqrt{n}}+3\sqrt{n}\le 2\sqrt{3}$$

gives $n=1$ and so $3D^2+3\le 2\sqrt{3}$. This is a contradiction. Thus $C=0$ and $AD=1$. Moreover, $1/3=|\gamma W_3Sw|^2=|\sqrt{n}i/3\pm B|^2$ gives that $n=3$ and by Lemma 2.4 $z_0=\frac{i}{\sqrt{3}}$ or $z_0=\frac{-3+\sqrt{3}i}{6}$.

For (3), if $w=\frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma Sw=\frac{(B-B\sqrt{4n-1}i)/2+A}{(D-D\sqrt{4n-1}i)/2+C}\in V$, then the condition (c) is equivalent to

$$\sqrt{3}\le\frac{2(C+D/2)^2}{\sqrt{4n-1}}+\frac{D^2\sqrt{4n-1}}{2}=\frac{2CD+2C^2+2nD^2}{\sqrt{4n-1}}\le 2\sqrt{3},\tag{4}$$

which implies that $\frac{\sqrt{4n-1}}{2}<2\sqrt{3}$, $1\le n\le 12$ and $D^2<4\sqrt{\frac{3}{4n-1}}\le 4$. So $D^2=1$. For each $n\in\{1,2,\cdots,12\}$, the inequality (4) which says $\sqrt{12n-3}\le 2CD+2C^2+2n\le 2\sqrt{12n-3}$ and the condition (a) give $C=0$. By condition (d) we have $1/3=|\gamma Sw|^2=(1/(2n)\pm B)^2+(4n-1)/(4n^2)$, which gives $3(1\pm 2nB)^2=4n^2-12n+3$ implying that $B=0$ and $n=3$. By Lemma 2.4 we obtain $z_0=\frac{-1+\sqrt{11}i}{6}$.

For (4), if $w=\frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma W_3Sw=\frac{(-A+A\sqrt{4n-1}i)+6B}{(-C+C\sqrt{4n-1}i)+6D}\in V$, then the condition (c) is equivalent to

$$\sqrt{3}\le\frac{(6D-C)^2}{6\sqrt{4n-1}}+\frac{C^2\sqrt{4n-1}}{6}=\frac{(6D-C)^2+C^2(4n-1)}{6\sqrt{4n-1}}\le 2\sqrt{3},\tag{5}$$

which implies that $C^2\sqrt{4n-1}\le 12\sqrt{3}$. So by (a) we have $C^2=0$ or $C^2=9$. If $C^2=9$ then $n=1$ and by Lemma 2.4 we obtain $z_0=\frac{-3+\sqrt{3}i}{6}$. If $C^2=0$ then the inequality (5) is equivalent to $\sqrt{12n-3}\le 6\le 2\sqrt{12n-3}$, which says $1\le n\le 3$. By condition (d) we have $1/3=|\gamma W_3Sw|^2=((6B\pm 1)^2+4n-1)/36$. So $B=0$ and $n=3$. By Lemma 2.4 we obtain $z_0=\frac{-1+\sqrt{11}i}{6}$.

For (5), if $w=\sqrt{n}i$ and $\gamma STw=\gamma W_3(W_3STw)\in V$, then $1\le n\le 2$. Indeed, if $n>2$, then $W_3STw=1/3+\sqrt{n}i/3$ lies in the interior of $\mathbf{F}^+(3)$ and so $\gamma STw\notin V$ since $\gamma W_3\in\Gamma_0^+(3)$. If $n=1$ then $z_0=\frac{-b+2i}{2a}\in V$. But there are no integers a,b satisfying the conditions (c) and (d). Thus $n=2$ and by Lemma 2.4 we obtain $z_0=\frac{-1+\sqrt{2}i}{3}$.

For (6), if $w=\sqrt{n}i$ and $\gamma W_3STw\in V$, then as in the proof of (5), if $n>2$, then W_3STw lies in the interior of $\mathbf{F}^+(3)$ and so $\gamma W_3STw\notin V$ since $\gamma\in\Gamma_0^+(3)$. Thus $n=2$ and by Lemma 2.4 we obtain $z_0=\frac{-1+\sqrt{2}i}{3}$.

For (7), if $w=\frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma STw=\gamma W_3(W_3STw)\in V$, then $1\le n\le 3$. Indeed, if $n>3$, then $W_3STw=1/6+\sqrt{4n-1}i/6$ lies in the interior of $\mathbf{F}^+(3)$ and so $\gamma STw\notin V$ since $\gamma W_3\in\Gamma_0^+(3)$. If $n=2$ then $z_0=\frac{-b+\sqrt{7}i}{2a}\in V$. But there are no integers a,b satisfying the conditions (c) and (d). Thus by Lemma 2.4, if $n=1$ then $z_0=\frac{-3+\sqrt{3}i}{6}$ and if $n=3$ then $z_0=\frac{-1+\sqrt{11}i}{6}$.

For (8), if $w=\frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma W_3STw\in V$, then as in the proof of (7), if $n=1$ then $z_0=\frac{-3+\sqrt{3}i}{6}$ and if $n=3$ then $z_0=\frac{-1+\sqrt{11}i}{6}$.

For (9), if $w=\sqrt{n}i$ and $\gamma ST^2w=\gamma W_3T(T^{-1}W_3ST^2w)\in V$, then $1\le n\le 2$. Indeed, if $n>2$, then $T^{-1}W_3ST^2w=-1/3+\sqrt{n}i/3$ lies in the interior of $\mathbf{F}^+(3)$ and so $\gamma STw\notin V$ since $\gamma W_3T\in\Gamma_0^+(3)$. If $n=1$ then $z_0=\frac{-b+2i}{2a}\in V$. But there are no integers a,b satisfying the conditions (c) and (d). Thus $n=2$ and by Lemma 2.4 we obtain $z_0=\frac{-1+\sqrt{2}i}{3}$.

For (10), if $w=\sqrt{n}i$ and $\gamma W_3ST^2w=\gamma T(T^{-1}W_3ST^2w)\in V$, then as in the proof of (9) we obtain $n=2$ and $z_0=\frac{-1+\sqrt{2}i}{3}$.

For (11), if $w=\frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma ST^2w=\gamma W_3T(T^{-1}W_3ST^2w)\in V$, then $n=1$. Indeed, if $n>1$, then $T^{-1}W_3ST^2w=-1/2+\sqrt{4n-1}i/6$ lies on the left vertical boundary of $\mathbf{F}^+(3)$ and so $\gamma STw\notin V$ since $\gamma W_3T\in\Gamma_0^+(3)$. Thus $n=1$ and $z_0=\frac{-3+\sqrt{3}i}{6}$.

For (12), if $w=\frac{-1+i\sqrt{4n-1}}{2}$ and $\gamma W_3ST^2w\in V$, then as in the proof of (11) we obtain $n=1$ and $z_0=\frac{-3+\sqrt{3}i}{6}$. □

Remark 2.6. Let $\mathbf{F}^+(p)$ be the standard fundamental domain for $\Gamma_0^+(p)$. For a prime p such that the genus of $\Gamma_0^+(p)$ is zero, the space of weakly holomorphic modular forms for $\Gamma_0^+(p)$ has a natural basis [2]. If all the zeros in $\mathbf{F}^+(p)$ of elements $f_{k,m}$ of the basis lie on the lower boundary of the fundamental domain, then we can generalize our results to the case $\Gamma_0^+(p)$. For a further research, we would like to find the location of the zeros of $f_{k,m}$ and investigate the algebracity and transcendence of them by a unified method.

Acknowledgments. I would like to express my thanks to the referee for his/her valuable comments.

I was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1A2C1A01007112).

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