Some remarks on finiteness of extremal rays of divisorial type

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Abstract: Let X be a normal $\mathbf{Q}$-factorial projective variety with at most log canonical singularities. We shall give a sufficient condition for the existence of at most finitely many K_X-negative extremal rays $R(\subset \overline{\mathrm{NE}}(X))$ of divisorial type. As an application, we show that for a nonisomorphic surjective endomorphism $f: X \to X$ of a normal projective $\mathbf{Q}$-factorial terminal 3-fold X with $\kappa(X) > 0$, a suitable power f^k $(k > 0)$ of f descends to a nonisomorphic surjective endomorphism $g: X_{min} \to X_{min}$ of a minimal model X_{min} of X.

Key words: Endomorphism; extremal ray; termination; divisorial contraction; flip.

1. Introduction. The main purpose of this note is to give the following theorem concerning finiteness of extremal rays of divisorial type on a normal projective variety with at most log canonical singularities.

Theorem 1.1. *Let X be a normal $\mathbf{Q}$-factorial projective variety with at most log canonical singularities. Suppose that there exists an effective divisor D on X such that for any K_X-negative extremal ray $R(\subset \overline{\mathrm{NE}}(X))$ of divisorial type, the exceptional divisor E_R of the contraction morphism $\mathrm{Cont}_R: X \to X'$ is contained in $\mathrm{Supp}(D)$. Then there exist at most finitely many K_X-negative extremal rays R of divisorial type.*

Corollary 1.2. *Let X be a normal $\mathbf{Q}$-factorial projective variety with at most canonical singularities. Suppose that $\kappa(X) \geq 0$. Then there exist at most finitely many K_X-negative extremal rays $R(\subset \overline{\mathrm{NE}}(X))$ of divisorial type.*

Let us explain briefly our motivations. Let $f: X \to X$ be a nonisomorphic étale endomorphism of a normal projective variety X with only canonical singularities. Then it is not necessarily true that for a K_X-negative extremal ray $R(\subset \overline{\mathrm{NE}}(X))$, there exists a positive integer k such that $(f^k)_*(R) = R$ for the automorphism $(f^k)_*: \mathrm{N_1(X)} \simeq \mathrm{N_1(X)}$ induced from the k-th power $f^k = f \circ \cdots \circ f$. Thus, if we apply the minimal model program (MMP, for short, cf. [6], [7]) to the study of nonisomorphic surjective endomorphisms of projective varieties, this phenomenon causes serious troubles. We cannot always apply the MMP working compatibly with étale endomorphisms. Thus it is an interesting problem to give a sufficient condition for a K_X-negative extremal ray R to be preserved under a suitable power of f. For example, if there exist at most finitely many K_X-negative extremal rays of divisorial type, then by replacing f by its suitable power $f^k (k > 0)$, we can apply the MMP working compatibly with nonisomorphic surjective endomorphisms (cf. [1], [2]).

2. Notations and preliminaries. In this paper, we work over the complex number field $\mathbf{C}$. A projective variety is a complex variety embedded in a projective space. By an endomorphism $f: X \to X$, we mean a morphism from a projective variety X to itself.

The following symbols are used for a variety X.

K_X: the canonical divisor of X.

$\mathrm{Aut}(X)$: the algebraic group of automorphisms of X.

$N_1(X) := (\{\text{1-cycles on } X\}/\equiv) \otimes_{\mathbf{Z}} \mathbf{R}$, where $\equiv$ means a numerical equivalence.

$N^1(X) := (\{\text{Cartier divisors on } X\}/\equiv) \otimes_{\mathbf{Z}} \mathbf{R}$, where $\equiv$ means a numerical equivalence.

$\mathrm{NE}(X)$: the smallest convex cone in $N_1(X)$ containing all effective 1-cycles.

$\overline{\mathrm{NE}}(X)$: the Kleiman-Mori cone of X, i.e., the closure of $\mathrm{NE}(X)$ in $N_1(X)$ for the metric topology.

$\rho(X) := \dim_{\mathbf{R}} N_1(X)$, the Picard number of X.

$[C]$: the numerical equivalence class of a 1-cycle C.

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cl(D): the numerical equivalence class of a Cartier divisor D.

$\sim_{\mathbf{Q}}$: the $\mathbf{Q}$-linear equivalence of $\mathbf{Q}$-divisors of X.

For an endomorphism $f\colon X \to X$ and an integer $k > 0$, f^k stands for the k-times composite $f \circ \cdots \circ f$ of f.

Extremal rays: For a normal projective $\mathbf{Q}$-factorial variety X with at most log canonical singularities, an extremal ray R means a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$, i.e., a 1-dimensional face of $\overline{\mathrm{NE}}(X)$ with $K_X R < 0$. An extremal ray R defines a proper surjective morphism $\pi_R := \mathrm{Cont}_R\colon X \to Y$ with connected fibers such that, for an irreducible curve $C \subset X$, $\pi_R(C)$ is a point if and only if $[C] \in R$ (cf. [3]). This is called the contraction morphism associated to R. If π_R is birational and contracts a divisor, then π_R is called a divisorial contraction and R is called of divisorial type. In this case, the exceptional set $\mathrm{Exc}(\pi_R)$ of π_R is a prime divisor and we denote it by E_R. If π_R is birational and $\mathrm{Exc}(\pi_R)$ has codimension ≥ 2 (i.e., π_R is small), then π_R is called a flipping contraction and R is called of flipping type.

For more details and terminologies of the minimal model program, the reader can consult [6] or [7].

3. Proof of Theorem 1.1. We shall give a proof of Theorem 1.1.

Proof of Theorem 1.1. We set $D = \sum_{i=1}^k a_i D_i$, where each a_i is a positive integer and each D_i is a prime divisor such that $D_i \neq D_j$ for any $i \neq j$. Suppose that there exist infinitely many K_X-negative extremal rays $R(\subset \overline{\mathrm{NE}}(X))$ of divisorial type and we shall derive a contradiction. We follow the idea of the proof of [9, Lemma 6.2]. Then there exists a prime divisor D_i such that $D_i = E_R$ for infinitely many extremal rays R of divisorial type. Let $\mathcal{S}$ be an infinite set consisting of extremal rays $R(\subset \overline{\mathrm{NE}}(X))$ such that $E_R = D_i$. For $R \in \mathcal{S}$, let $\pi_R := \mathrm{Cont}_R\colon X \to Y_R$ be the divisorial contraction morphism associated to R. We set $N^1_{\mathbf{C}}(X) := N^1(X) \otimes_{\mathbf{R}} \mathbf{C}$. We have the following commutative diagram

$$\begin{array}{ccc} \pi_R^* N^1_{\mathbf{C}}(Y_R) & \longrightarrow & N^1_{\mathbf{C}}(X) \\ \downarrow & & \downarrow \\ \pi_R^* N^1_{\mathbf{C}}(Y_R)|_{D_i} & \longrightarrow & N^1_{\mathbf{C}}(X)|_{D_i} \end{array}$$

where both horizontal arrows are inclusions and both vertical arrows are surjections. Then by the cone theorem (cf. [3], [4], [6], [7]), $\pi_R^* N^1_{\mathbf{C}}(Y_R) \hookrightarrow N^1_{\mathbf{C}}(X)$ is a linear subspace of codimension one. Hence $\Delta_R := \pi_R^* N^1_{\mathbf{C}}(Y_R)|_{D_i} \hookrightarrow N^1_{\mathbf{C}}(X)|_{D_i}$ is also a linear subspace of codimension at most one. On the other hand, $H|_{D_i}$ is not contained in Δ_R for an ample divisor H of X. Hence Δ_R is of codimension one in $N^1_{\mathbf{C}}(X)|_{D_i}$. If we set $V := \{v \in N^1_{\mathbf{C}}(X)|_{D_i};\ v^{\dim X - 1} = 0\}$, then V is an affine hypersurface of degree $\dim X - 1$ in the complex vector space $N^1_{\mathbf{C}}(X)|_{D_i}$. Since $\dim \pi_R(D_i) \leq \dim X - 2$, the complex vector space Δ_R is contained in V. Since $\dim V = \dim \Delta_R = \dim(N^1_{\mathbf{C}}(X)|_{D_i}) - 1$, Δ_R is an irreducible component of V. Let C_R be an extremal curve on X whose numerical class $[C_R]$ spans R. Then $[C_R]$ is orthogonal to Δ_R via the intersection pairing. If $R \neq R' \in \mathcal{S}$, then C_R is not contracted to a point by $\pi_{R'}$ and $[C_R]$ is not orthogonal to $\Delta_{R'}$. Hence $\Delta_R \neq \Delta_{R'}$ and V has an infinite number of irreducible components Δ_R $(R \in \mathcal{S})$. Since the number of all the irreducible components of V is finite, this is a contradiction. Thus the proof is finished. □

Proof of Corollary 1.2. Since $\kappa(X) \geq 0$, mK_X is a Cartier divisor and $|mK_X| \neq \emptyset$ for some positive integer m. Take a member $D \in |mK_X|$ and we set $D = \sum_{i=1}^k a_i D_i$, where each a_i is a positive integer and each D_i is a prime divisor such that $D_i \neq D_j$ for any $i \neq j$. For any K_X-negative extremal ray R of divisorial type, take an extremal curve C_R whose numerical class $[C_R]$ spans R. Since $0 > m(K_X, C_R) = \sum_i a_i (D_i, C_R)$, we have $(D_i, C_R) < 0$ for some i. Hence C_R is contained in D_i. Since C_R sweeps out E_R, we have $E_R \subset D_i$. Hence $E_R = D_i$, since D_i is irreducible. Then applying Theorem 1.1 to D, the proof follows immediately. □

Next, we shall consider extremal rays of an almost homogeneous variety.

Definition 3.1. Let X be an irreducible normal algebraic variety. Suppose that a connected algebraic group G acts algebraically on X. If the group G has an open dense orbit in X, then X is called almost homogeneous (with respect to the action of G), or the G-action on X is almost transitive. In particular, if $\mathrm{Aut}^0(X)$ has an open dense orbit in X, then we say that X is almost homogeneous.

Corollary 3.2. *Let G be a connected positive dimensional algebraic group which acts regularly on a smooth projective variety X. Suppose that*

X is almost homogeneous with respect to the G-action (cf. Definition 3.1). Then the number of K_X-negative extremal rays of divisorial type on $\overline{\mathrm{NE}}(X)$ is finite.

Proof. Let X^0 be an open dense orbit of G and $S := X \setminus X^0$ its complement. For any extremal ray R of divisorial type, let E_R be the exceptional divisor of the contraction morphism Cont_R associated to R. First we show that $E_R \subset S$. The proof is by contradiction. Assume the contrary. Then, there exists some point $P \in E_R \cap X^0$. Let ℓ be an extremal rational curve on X which passes through P and its numerical class $[\ell]$ spans R. By assumption, for any $Q \in X^0$ there exists some $g \in G$ such that $g(P) = Q$. Since G is connected, it acts trivially on the homology group $H_2(X, \mathbf{Z})$ which is discrete, and hence on $H_2(X, \mathbf{R})$. Thus the action of G on $\overline{\mathrm{NE}}(X)$ is also trivial. Hence $g(\ell)$ is an extremal rational curve passing through Q and its numerical class $[g(\ell)] = [\ell]$ also spans the same extremal ray R. Thus $Q(\in g(\ell))$ is contained in E_R. Hence the open dense G-orbit X^0 is contained in the exceptional divisor E_R, which derives a contradiction. Let D be a reduced divisor on X which is a sum of all the prime divisors contained in S. Then $E_R \subset \mathrm{Supp}(D)$ for any K_X-negative extremal ray R of divisorial type. Hence applied Theorem 1.1, we see that the number of all the K_X-negative extremal rays R of divisorial type is finite. □

4. Applications to endomorphisms. In this section, as an application of Therem 1.1, we shall apply the MMP to a nonisomorphic surjective endomorphism $f\colon X \to X$ of a normal $\mathbf{Q}$-factorial projective 3-fold X with only terminal singularities and $\kappa(X) > 0$. We recall the following fundamental result.

Lemma 4.1. *Let $f\colon X \to X$ be a surjective endomorphism of a normal $\mathbf{Q}$-factorial projective variety X. Suppose that K_X is pseudo-effective. Then f is a finite morphism which is étale in codimension one.*

Proof. The proof follows immediately by the same argument as in the proof of [1, Lemma 2.3]. □

Lemma 4.2 (cf. [1, Propositions 4.2 and 4.12])**.** *Let $f\colon Y \to X$ be a surjective morphism between normal, $\mathbf{Q}$-factorial projective log canonical n-folds with $\rho(X) = \rho(Y)$. Then the following hold.*

(1) *f is a finite morphism and the push-forward map $f_*\colon N_1(Y) \to N_1(X)$ is an isomorphism and $f_*\overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(X)$.*
(2) *Let $f_*\colon N^1(Y) \to N^1(X)$ be the map induced from the push-forward map $D \mapsto f_*D$ of divisors. Then the dual map $f^*\colon N_1(X) \to N_1(Y)$ is an isomorphism and $f^*\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(Y)$.*
(3) *If f is étale in codimension one and K_X is not nef, then f^* and f_* above give a one-to-one correspondence between the set of extremal rays of X and Y.*
(4) *Under the same assumption as in (3), for an extremal ray $R(\subset \overline{\mathrm{NE}}(Y))$, and for the contraction morphisms $\mathrm{Cont}_R\colon Y \to Y'$ and $\mathrm{Cont}_{f_*R}\colon X \to X'$, there exists a finite surjective morphism $f'\colon Y' \to X'$ such that $f' \circ \mathrm{Cont}_R = \mathrm{Cont}_{f_*R} \circ f$.*

Proof. Since the cone and contraction theorem holds if X is a $\mathbf{Q}$-factorial log canonical n-fold (cf. [3]), the proof follows immediately by the same argument as in the proof of [1, Propositions 4.2 and 4.12]. □

Lemma 4.3. *Let $f\colon X \to X$ be a nonisomorphic surjective endomorphism of a normal, $\mathbf{Q}$-factorial projective n-fold X with only canonical singularities and $\kappa(X) \geq 0$. Suppose that K_X is not nef and there exists a K_X-negative extremal ray $R(\subset \overline{\mathrm{NE}}(X))$ of divisorial type. Then replacing f by its suitable power $f^k (k > 0)$, there exits the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{g} & Y \end{array}$$

which is almost Cartesian (i.e., the fiber product when restricted over a Zariski open subset Y^0 of Y) such that the following hold:

(1) *$\pi\colon X \to Y$ is an extremal divisorial contraction associated to R and contracts a prime divisor on X to a positive-dimensional subvariety on Y.*
(2) *$g\colon Y \to Y$ is a a nonisomorphic surjective endomorphism of a $\mathbf{Q}$-factorial variety Y with at most canonical singularities.*

Proof. Let $\pi := \mathrm{Cont}_R\colon X \to Y$ be an extremal divisorial contraction associated to R. By Corollary 1.2 we see that $(f^k)_*R = R$ for some integer $k > 0$. Hence, if we replace f by its power f^k and applied Lemma 4.2, f descends to a nonisomorphic surjective endomorphism g of Y. By Lemma 4.2, we see that g is finite and étale in codimension one. If we set $E := \mathrm{Exc}(\pi)$, then E is a prime divisor on X and

$K_X \sim_{\mathbf{Q}} \pi^* K_Y + aE$ for a positive rational number $a > 0$. We have $K_X \sim_{\mathbf{Q}} f^* K_X$ and $K_Y \sim_{\mathbf{Q}} g^* K_Y$, since both f and g are finite morphisms étale in codimension one. Since $\pi \circ f = g \circ \pi$, we have $f^* E \sim_{\mathbf{Q}} E$. Suppose that $\pi(E)$ is a point on Y. Since $-E$ is π-ample, we have $(-E|_E)^{(n-1)} > 0$. Since $(f|_E)^*(-E|_E) \sim_{\mathbf{Q}} -E|_E$, we have $(-E|_E)^{n-1} = \deg(f|_E)(-E|_E)^{n-1}$. Then we have $(-E|_E)^{n-1} = 0$, since $\deg(f|_E) = \deg(f) \geq 2$. Thus a contradiction is derived and $\pi(E)$ is not a point on Y. □

Proposition 4.4. *Let $f: X \to X$ be a nonisomorphic surjective endomorphism of a normal $\mathbf{Q}$-factorial projective variety X with at most canonical singularities. Suppose that $\kappa(X) \geq 0$ and K_X is not nef. Then replacing f by its suitable power $f^k (k > 0)$, there exits the following finite sequence of birational morphisms*

$$X = X_1 \xrightarrow{\pi_1} \cdots \to X_i \xrightarrow{\pi_i} X_{i+1} \to \cdots \to X_k = Y$$

such that

(1) *each π_i is an extremal divisorial contraction which contracts a prime divisor E_i on X_i to a positive-dimensional subvariety on X_{i+1},*
(2) *$f = f_1$ descends to a nonisomorphic surjective endomorphism $f_i: X_i \to X_i$ of a $\mathbf{Q}$-factorial normal projective variety X_i with at most canonical singularities, and*
(3) *any K_Y-negative extremal ray $R(\subset \overline{\mathrm{NE}}(Y))$ is of flipping type, i.e., the contraction morphism associated to R is small.*

Proof. We may assume that there exists some K_X-negative extremal ray $R_1(\subset \overline{\mathrm{NE}}(X_1))$ of divisorial type. Let $\pi_1: X = X_1 \to X_2$ be the extremal divisorial contraction associated to R_1. Then Lemma 4.3 shows that if we replace f by its suitable power $f^\ell (\ell > 0)$, then f descends to a nonisomorphic surjective endomorphism $f_2: X_2 \to X_2$ of X_2. If there exists some K_{X_2}-negative extremal ray $R_2(\subset \overline{\mathrm{NE}}(X_2))$ of divisorial type, then we repeat the same procedure and obtain the following sequence

$$X = X_1 \xrightarrow{\pi_1} X_2 \xrightarrow{\pi_2} \cdots \to X_i \xrightarrow{\pi_i} X_{i+1} \to \cdots \to \cdots,$$

where

- each π_i is an extremal divisorial contraction which contracts a prime divisor on X_i to a positive-dimensional subvariety on X_{i+1}, and
- f descends to a nonisomorphic surjective endomorphism $f_i: X_i \to X_i$ of X_i.

Since $\rho(X_{i+1}) = \rho(X_i) - 1$, these procedures eventually stop. Hence there exists no K_{X_k}-negative extremal ray of divisorial type for some $k > 0$ and we set $Y := X_k$. Then any K_Y-negative extremal ray $R(\subset \overline{\mathrm{NE}}(Y))$ is of flipping type and we are done. □

Remark 4.5 (cf. [1, Theorem 4.8, Proposition 4.9, and Definition 4.15])**.** Let $f: X \to X$ be a nonisomorphic surjective endomorphism of a smooth projective 3-fold X with $\kappa(X) \geq 0$. Then, for any i, X_i is nonsingular and $\pi_{i-1}: X_{i-1} \to X_i$ is the blowing-up of an elliptic curve $C_i(\subset X_i)$ such that $f_i^{-1}(C_i) = C_i$. Note that there exists no K_Y-negative extremal ray of flipping type on $\overline{\mathrm{NE}}(Y)$, since Y is a smooth projective 3-fold (cf. [10]). In this case, $Y = X_k$ is the unique minimal model of X and $f_k: Y \to Y$ is called the minimal reduction of $f: X \to X$.

Next, we shall apply the MMP to a nonisomorphic surjective endomorphism $f: X \to X$ of a normal projective $\mathbf{Q}$-factorial terminal 3-fold X with $\kappa(X) > 0$.

Theorem 4.6. *Let $f: X \to X$ be a nonisomorphic surjective endomorphism of a normal projective $\mathbf{Q}$-factorial 3-fold X with only terminal singularities. Suppose that $\kappa(X) > 0$. Then if we replace f by its suitable power f^k $(k > 0)$, there exists the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X_{min} & \xrightarrow{g} & X_{min} \end{array}$$

which satisfies the following

(1) *X_{min} is a minimal model of X, i.e., X_{min} is a normal, projective $\mathbf{Q}$-factorial terminal 3-fold which is birational to X and $K_{X_{min}}$ is nef.*
(2) *π' is a composition of a finite number of divisorial contractions contracting a prime divisor to a curve, and a finite number of terminal flips.*
(3) *$\pi = w \circ \mu$, where $\mu: X \cdots \to X'$ is a composition of a finite number of divisorial contractions contracting a prime divisor to a curve and a finite number of terminal flips, and $w: X' \simeq X_{min}$ is an isomorphism.*
(4) *g is a nonisomorphic surjective endomorphism of X_{min}.*

Proof. We may assume that K_X is not nef. Then applied Proposition 4.4 and replacing f by its suitable power $f^k (k > 0)$, there exits the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ V & \xrightarrow{h} & V \end{array}$$

which is almost Cartesian (i.e., the fiber product when restricted over a Zariski open subset V^0 of V) such that the following hold:

(1) $\tau: X \to V$ is a succession of extremal divisorial contractions which contracts a prime divisor to a curve.

(2) $h: V \to V$ is a nonisomorphic surjective endomorphism of a normal $\mathbf{Q}$-factorial projective 3-fold V with only terminal singularities.

(3) Any K_V-negative extremal ray $R(\subset \overline{\mathrm{NE}}(V))$ is of flipping type.

Hereafter, we may assume that K_V is not nef. Take a K_V-negative extreal ray $R^{(1)}(\subset \overline{\mathrm{NE}}(V))$. We set $R_0^{(1)} := R^{(1)}$ and $R_n^{(1)} := (f^n)_*(R^{(1)}), R_{-n}^{(1)} := (f^n)^* R^{(1)}$ for a positive integer n. Then by Lemma 4.2, we see that $R_n^{(1)}(\subset \overline{\mathrm{NE}}(V))$ is a K_V-negative extremal ray of flipping type for any $n \in \mathbf{Z}$. Let $u_n: V \to W_n$ be the small birational contraction associated to $R_n^{(1)}$. Then for any $n \in \mathbf{Z}$, there exits the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ V & \xrightarrow{h} & V \\ {\scriptstyle u_n}\downarrow & & \downarrow{\scriptstyle u_{n+1}} \\ W_n & \xrightarrow{\rho_n} & W_{n+1} \end{array}$$

where ρ_n is a nonisomorphic finite morphism étale in codimension one. The first (resp. the second) commutative diagram from the top is almost Cartesian, i.e., the fiber product when restricted over a Zariski open subset V^0 of V (resp. W_{n+1}^0 of W_{n+1}). Then by [11], the canonical ring $R_n := \oplus_{m \geq 0} u_{n*}(\mathcal{O}_V(mK_V))$ is a finitely generated $\mathcal{O}_{W_n}$-algebra and set $V_n^+ := \mathrm{Proj}_{W_n}(R_n)$. Then $u_n^+: V_n^+ \to W_n$ is a flip of $u_n: V \to W_n$. Let U_n be the normalization of $V_{n+1}^+ \times_{W_{n+1}} W_n$. Then K_{U_n} is a well-defined $\mathbf{Q}$-Cartier divisor since $U_n \to V_{n+1}^+$ is finite and étale in codimension one. Note that K_{U_n} is the pull-back of $K_{V_{n+1}^+}$ by construction. Therefore, K_{U_n} is ample over W_n and $U_n \to W_n$ is small by construction. Hence U_n is a flip of $V \to W_n$ and $U_n \simeq V_n^+$ (cf. [7, Lemma 6.2]). By this observation, for any $n \in \mathbf{Z}$, we can construct the commutative diagram of flip

$$\begin{array}{ccc} V_n^+ & \xrightarrow{v_n} & V_{n+1}^+ \\ {\scriptstyle u_n^+}\downarrow & & \downarrow{\scriptstyle u_{n+1}^+} \\ W_n & \xrightarrow{\rho_n} & W_{n+1}, \end{array}$$

where V_n^+ is a normal $\mathbf{Q}$-factorial projective 3-fold with only terminal singularities and the natural projection v_n is a nonisomorphic finite morphism which is étale in codimension one. If $K_{V_n^+}$ is nef, then we stop. If $K_{V_n^+}$ is not nef, then we repeat the same procedure. Because of the termination of 3-fold flips (cf. [11]), these procedures eventually stop after finitely many times and for any $n \in \mathbf{Z}$, we obtain the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & V \\ {\scriptstyle \mu_n}\downarrow & & \downarrow{\scriptstyle \mu_{n+1}} \\ Z_n & \xrightarrow{\nu_n} & Z_{n+1} \end{array}$$

which satisfies the following

(1) Z_n is a minimal model of V (hence of X), i.e., K_{Z_n} is nef.

(2) μ_n is a composition of finitely many terminal flips.

(3) ν_n is a nonisomorphic finite morphism which is étale in codimension one.

Since $\kappa(X) > 0$, [5, Theorem 4.5] shows that there exist only finitely many minimal models of X up to isomorphisms. Hence there exists an isomorphism $w: Z_p \simeq Z_q$ for some integers $p < q$. Thus we have the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{h^{q-p}} & V \\ {\scriptstyle \mu_p}\downarrow & & \downarrow{\scriptstyle \mu_q} \\ Z_p & \xrightarrow{\psi} & Z_q \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ Z_q & \xrightarrow{\psi\circ w^{-1}} & Z_q \end{array}$$

where we set $\psi := \nu_{q-1} \circ \cdots \nu_p$. Hence if we further replace f (resp. h) by its positive power f^{q-p} (resp. h^{q-p}) and set $X_{min} := Z_q, X' := Z_p, v = w \circ \mu_p, v' = \mu_q$, and $g := \psi \circ w^{-1}$, then we obtain the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ V & \xrightarrow{h} & V \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ X_{min} & \xrightarrow{g} & X_{min} \end{array}$$

which satisfies the following

(1) τ is a succession of extremal divisorial contractions which contracts a prime divisor to a curve.

(2) $v = w \circ \mu_p$, where $\mu_p : X \cdots \to X'$ is a composition of finitely many terminal flips and $w : X' \simeq X_{min}$ is an isomorphism.

(3) v' is a composition of finitely many terminal flips.

(4) g is a nonisomorphic surjective endomorphism of X_{min}.

Thus if we set $\mu := \mu_p \circ \tau : X \cdots \to X', \pi := w \circ \mu\ (= v \circ \tau)$ and $\pi' := v' \circ \tau$, then the proof is finished. □

Remark 4.7. (1) In [5], the finiteness of minimal models of X is not established in the case of $\kappa(X) = 0$. Thus by the proof of Theorem 4.6, we can show the following

'Suppose that $\kappa(X) = 0$ in the assumption of Theorem 4.6. Then, after a finite number of divisorial contractions and terminal flips, an endomorphism $f : X \to X$ induces a tower of nonisomorphic finite morphisms $\{Z_n \to Z_{n+1}\}_{n\in\mathbf{Z}}$ between minimal models Z_n of X which is étale in codimension one.'

(2) The conclusion of Lemma 4.3 does not necessarily hold for a K_X-negative extremal ray $R(\subset \overline{\mathrm{NE}}(X))$ of flipping type. We shall give such an example. [8, Theorem 7.1] shows the existence of a terminal, projective 3-fold Y of nonnegative Kodaira dimension with infinitely many K_Y-negative extremal rays of flipping type. Y has a fiber space structure $\varphi : Y \to \Gamma$ over a curve Γ of genus $g(\Gamma) \geq 1$ whose general fiber is isomorphic to the product $E \times E$ of an elliptic curve E. Moreover, a K_Y-negative flipping curve ℓ is contained in a fiber of $\varphi : Y \to \Gamma$. The relative automorphism group $\mathrm{Aut}(Y/\Gamma)$ of Y over Γ contains a subgroup G which is isomorphic to $\mathrm{SL}(2, \mathbf{Z})$. The G-orbit of ℓ all give K_Y-negative extremal curves of flipping type. Let C be an elliptic curve and $\mu_n : C \to C$ be a multiplication mapping by a positive integer $n > 1$. We take an element $g \in G$ of infinite order. Let $X := Y \times C$ be the product of Y and C. Then $\tau := g \times \mu_n : X \to X$ gives a nonisomorphic surjective endomorphism of a terminal 4-fold X with $\kappa(X) = \kappa(Y) \geq 0$. The numerical class $[\gamma]$ of a curve $\gamma := \ell \times \{o\}\ (o \in C)$ also spans a K_X-negative extremal ray $L(\subset \overline{\mathrm{NE}}(X))$ of flipping type. By construction, $(\tau^k)_* L \neq L$ for any positive integer $k > 0$.

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