A note on operator perturbation of woven frames

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Abstract: In this paper, we give a necessary and sufficient condition under which a woven frame can be preserved when applying an operator to it, which provides a correction to one existing corresponding result due to Rahimi *et al.* We also present some new results on the stability of woven frames under perturbation of two different operators.

Key words: Weaving frame; operator perturbation; woven frame operator; pseudo-inverse.

1. Introduction. This paper adopts the following notations: $\mathscr{H}$ and $\mathscr{K}$ are complex separable Hilbert spaces, $\mathbf{J}$ is a countable index set. We use $\mathcal{B}(\mathscr{H}, \mathscr{K})$ to denote the family of bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$, which, as usual, is abbreviated to $\mathcal{B}(\mathscr{H})$, if $\mathscr{K} = \mathscr{H}$. For a given number $m \in \mathbf{N}$, let $[m] = \{1, 2, \cdots, m\}$, where $\mathbf{N} = \{1, 2, \cdots\}$. Also, the notation $\text{Id}_{\mathscr{H}}$ is used to denote the identical operator on $\mathscr{H}$.

One calls a sequence $\Phi = \{\phi_j\}_{j \in \mathbf{J}} \subset \mathscr{H}$ a frame for $\mathscr{H}$, if there are constants $0 < C_\Phi \leq D_\Phi < \infty$ such that

$$C_\Phi \|f\|^2 \leq \sum_{j \in \mathbf{J}} |\langle f, \phi_j \rangle|^2 \leq D_\Phi \|f\|^2, \quad \forall f \in \mathscr{H}.$$

The numbers C_Φ and D_Φ are called, respectively, the lower and upper frame bounds of Φ.

Related to a given frame $\Phi = \{\phi_j\}_{j \in \mathbf{J}}$ for $\mathscr{H}$, there are two famous bounded linear operators, namely, analysis operator and frame operator, defined respectively by

$$U_\Phi : \mathscr{H} \to \ell^2(\mathbf{J}), \quad U_\Phi f = \{\langle f, \phi_j \rangle\}_{j \in \mathbf{J}}.$$
$$S_\Phi : \mathscr{H} \to \mathscr{H}, \quad S_\Phi f = \sum_{j \in \mathbf{J}} \langle f, \phi_j \rangle \phi_j.$$

Recall that two frames $\Phi = \{\phi_j\}_{j \in \mathbf{J}}$ and $\Gamma = \{\gamma_j\}_{j \in \mathbf{J}}$ with analysis operators U_Φ and U_Γ respectively are said to be orthogonal, if $U_\Phi^* U_\Gamma = 0$.

A frame in a Hilbert space is some kind of "redundant basis", which was introduced by Duffin and Schaeffer [12] in the early 1950s to investigate

some deep problems about nonharmonic Fourier series. Daubechies *et al.* [9] brought frames back to people's mind in 1986, when they discovered the close relationship between frame theory and wavelet theory. Today, frames have become to be a research hotspot because they served as an useful tool not only in theoretical work [8,17], but also in dozens of application areas [1,5,13,14,18].

Bemrose *et al.* [2] put forward a new problem in frame theory called woven frames, motivated by a question arising in distributed signal processing. Because of some potential applications such as in wireless sensor networks and pre-processing of signals, woven frames have attracted many researchers' attention [4,6,10,19]. And some variations of woven frames were also considered [3,11,15,16].

The aim of this paper is to further study the operator perturbation of woven frames, and the motivation derives from an examination of one known result on this topic, Theorem 3.7 in [19], which stated that bounded linear operators preserve woven frames if and only if they are invertible. However, a counterexample (see Example 2.2) shows that the "only if" part of the result is not true. We provide a correction to the result (see Theorem 2.3), and some other new results on the stability of woven frames under operator perturbation are also given.

We continue with this section to collect some definitions, and basic facts on pseudo-inverse operators which will be used in next section.

Definition 1.1 (see [2]). A family of frames $\mathscr{F} = \{f_{ij}\}_{j \in \mathbf{J}, i \in [m]} \subset \mathscr{H}$ is said to be woven for $\mathscr{H}$, if there are universal constants $C_\mathscr{F}$ and $D_\mathscr{F}$ such that for any partition $\{\sigma_i\}_{i \in [m]}$ of $\mathbf{J}$, the family $\cup_{i \in [m]} \{f_{ij}\}_{j \in \sigma_i}$ is a frame for $\mathscr{H}$ with lower and upper

frame bounds $C_{\mathscr{F}}$ and $D_{\mathscr{F}}$, respectively. Each family $\cup_{i\in[m]}\{f_{ij}\}_{j\in\sigma_i}$ is called a weaving.

Suppose that the family of frames $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J},i\in[m]}$ is woven for $\mathscr{H}$. The woven frame operator associated with $\mathscr{F}$, introduced in [19], is defined by

$$S_{\mathscr{F}} : \mathscr{H} \to \mathscr{H}, \quad S_{\mathscr{F}}f = \sum_{i\in[m]}\sum_{j\in\mathbf{J}}\langle f, f_{ij}\rangle f_{ij}.$$

Lemma 1.2 (see [7]). *Suppose that* $\Theta \in \mathcal{B}(\mathscr{H},\mathscr{K})$ *has closed range, then there exists a unique operator* $\Theta^{\dagger} \in B(\mathscr{K},\mathscr{H})$, *called the pseudo-inverse of* Θ, *satisfying*

$$\Theta\Theta^{\dagger} = P_{Ran(\Theta)}, \quad \Theta^{\dagger}\Theta = P_{Ran(\Theta^{\dagger})},$$
$$(\Theta\Theta^{\dagger})^* = \Theta\Theta^{\dagger}, \quad (\Theta^{\dagger}\Theta)^* = \Theta^{\dagger}\Theta,$$
$$Ker\Theta^{\dagger} = Ran^{\perp}(\Theta), \quad Ran(\Theta^{\dagger}) = Ker^{\perp}\Theta,$$

where $P_{Ran(\Theta)}$ *denotes the orthogonal projection onto* $Ran(\Theta)$.

2. Main results. The purpose of this section is to examine the stability of woven frames under operator perturbation. We begin with the following assertion which is stated in [19] as Theorem 3.7.

Assertion 2.1. Suppose that $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J},i\in[m]}$ is a woven frame for $\mathscr{H}$ with woven frame operator $S_{\mathscr{F}}$ and universal frame bounds $C_{\mathscr{F}}$ and $D_{\mathscr{F}}$, and that $U \in \mathcal{B}(\mathscr{H})$. Then the operator U is invertible if and only if $U\mathscr{F} = \{Uf_{ij}\}_{j\in\mathbf{J},i\in[m]}$ is woven for $\mathscr{H}$. In this case, the universal frame bounds for $U\mathscr{F}$ are $C_{\mathscr{F}}\|U^{-1}\|^{-2}$ and $D_{\mathscr{F}}\|U\|^2$, and the woven frame operator is $US_{\mathscr{F}}U^*$.

It is really true that a woven frame can be preserved by an invertible operator. But, if $U\mathscr{F} = \{Uf_{ij}\}_{j\in\mathbf{J},i\in[m]}$, the operator U is applied to the woven frame $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J},i\in[m]}$, is a woven frame, one can not conclude, in general, that U is invertible. Although in the proof the authors proved that $US_{\mathscr{F}}U^*$ is invertible, it does not imply that U is invertible, as shown in the following example.

Example 2.2. Let $\{e_j\}_{j\in\mathbf{N}}$ be an orthonormal basis of $\mathscr{H}$, which, as we all known, is a frame with frame operator $S = \mathrm{Id}_{\mathscr{H}}$. We define a shift operator U on $\mathscr{H}$ by $Ue_j = e_{j-1}, j \geq 2$, and $Ue_1 = 0$. Then, it is easily seen that, U^*, the adjoint operator of U, is given by $U^*f = \sum_{j\geq 2}\langle f, e_{j-1}\rangle e_j, \ f \in \mathscr{H}$. Thus

$$\sum_{j\in\mathbf{N}}|\langle f, Ue_j\rangle|^2 = \sum_{j\in\mathbf{N}}|\langle U^*f, e_j\rangle|^2$$
$$= \|U^*f\|^2 = \sum_{j\in\mathbf{N}}|\langle f, e_j\rangle|^2 = \|f\|^2,$$

meaning that $\{Ue_j\}_{j\in\mathbf{N}}$ is a frame for $\mathscr{H}$. But U is not invertible although

$$USU^*f = UU^*f = \sum_{j\geq 2}\langle f, e_{j-1}\rangle e_{j-1}$$
$$= \sum_{j\in\mathbf{N}}\langle f, e_j\rangle e_j = f$$

yields $USU^* = \mathrm{Id}_{\mathscr{H}}$.

Assertion 2.1 can be corrected as follows:

Theorem 2.3. *Suppose that the family of frames* $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J},i\in[m]}$ *is woven for* $\mathscr{H}$ *with woven frame operator* $S_{\mathscr{F}}$ *and universal frame bounds* $C_{\mathscr{F}}$ *and* $D_{\mathscr{F}}$, *and that* $U \in \mathcal{B}(\mathscr{H})$. *Then* $U\mathscr{F} = \{Uf_{ij}\}_{j\in\mathbf{J},i\in[m]}$ *is woven for* $\mathscr{H}$ *if and only if* U *is surjective. In this case, the universal frame bounds and woven frame operator for* $U\mathscr{F}$ *are, respectively,* $C_{\mathscr{F}}\|U^{\dagger}\|^{-2}$ *and* $D_{\mathscr{F}}\|U\|^2$, *and* $US_{\mathscr{F}}U^*$.

Proof. "$\Leftarrow$". Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathbf{J}$. Then

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, Uf_{ij}\rangle|^2 = \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle U^*f, f_{ij}\rangle|^2$$
$$\leq D_{\mathscr{F}}\|U^*f\|^2$$
$$\leq D_{\mathscr{F}}\|U\|^2\|f\|^2, \quad \forall f \in \mathscr{H}.$$

Since each $f \in \mathscr{H}$ can be expressed as $f = Ug = UU^{\dagger}Ug = (UU^{\dagger})^*Ug$ for some $g \in \mathscr{H}$ by the surjectivity of U and Lemma 1.2, it follows that

$$\|f\|^2 = \|(U^{\dagger})^*U^*Ug\|^2 \leq \|U^{\dagger}\|^2\|U^*Ug\|^2$$
$$\leq \frac{\|U^{\dagger}\|^2}{C_{\mathscr{F}}}\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle U^*Ug, f_{ij}\rangle|^2$$
$$= \frac{\|U^{\dagger}\|^2}{C_{\mathscr{F}}}\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle Ug, Uf_{ij}\rangle|^2$$
$$= \frac{\|U^{\dagger}\|^2}{C_{\mathscr{F}}}\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, Uf_{ij}\rangle|^2,$$

and, consequently,

$$C_{\mathscr{F}}\|U^{\dagger}\|^{-2}\|f\|^2 \leq \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, Uf_{ij}\rangle|^2, \quad \forall f \in \mathscr{H}.$$

Altogether we know that $U\mathscr{F}$ is woven for $\mathscr{H}$ with universal frame bounds $C_{\mathscr{F}}\|U^{\dagger}\|^{-2}$ and $D_{\mathscr{F}}\|U\|^2$.

"$\Rightarrow$". As shown in the proof of Theorem 3.7 in [19], the woven frame operator of $U\mathscr{F}$, $S_{U\mathscr{F}}$, an

invertible operator on $\mathscr{H}$, is given by $S_{U\mathscr{F}} = US_{\mathscr{F}}U^*$, implying that U is surjective. $\qquad\square$

Corollary 2.4. *Suppose that the family of frames $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J}, i\in[m]}$ is woven for $\mathscr{H}$ and that $U \in \mathcal{B}(\mathscr{H})$. Then $U\mathscr{F} = \{Uf_{ij}\}_{j\in\mathbf{J}, i\in[m]}$ and $U^*\mathscr{F} = \{U^*f_{ij}\}_{j\in\mathbf{J}, i\in[m]}$ are both woven frames for $\mathscr{H}$ if and only if U is invertible.*

Proof. By Theorem 2.3,

$$U\mathscr{F} = \{Uf_{ij}\}_{j\in\mathbf{J}, i\in[m]} \text{ is woven for } \mathscr{H}$$
$$\Leftrightarrow U \text{ is surjective,}$$

and

$$U^*\mathscr{F} = \{U^*f_{ij}\}_{j\in\mathbf{J}, i\in[m]} \text{ is woven for } \mathscr{H}$$
$$\Leftrightarrow U^* \text{ is surjective} \Leftrightarrow U \text{ is injective.}$$

Therefore,

$$U\mathscr{F} = \{Uf_{ij}\}_{j\in\mathbf{J}, i\in[m]} \text{ and } U^*\mathscr{F} = \{U^*f_{ij}\}_{j\in\mathbf{J}, i\in[m]}$$

are both woven for $\mathscr{H} \Leftrightarrow U$ is invertible.

$\qquad\square$

Corollary 2.5. *Suppose that $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J}, i\in[m]}$ is a family in $\mathscr{H}$, that $\sum_{i\in[m]}\sum_{j\in\mathbf{J}}\langle f, f_{ij}\rangle f_{ij}$ converges for all $f \in \mathscr{H}$, and that $U \in \mathcal{B}(\mathscr{H})$. If $U\mathscr{F} = \{Uf_{ij}\}_{j\in\mathbf{J}, i\in[m]}$ and $U^*\mathscr{F} = \{U^*f_{ij}\}_{j\in\mathbf{J}, i\in[m]}$ are woven frames for $\mathscr{H}$, then $\mathscr{F}$ is a woven frame for $\mathscr{H}$.*

Proof. Let us define

$$S : \mathscr{H} \to \mathscr{H}, \quad Sf = \sum_{i\in[m]}\sum_{j\in\mathbf{J}}\langle f, f_{ij}\rangle f_{ij}.$$

If we denote by $S_{U\mathscr{F}}$, the woven frame operator of $U\mathscr{F}$, then it is easy to check that $S_{U\mathscr{F}} = USU^*$. From which we conclude that U is surjective. A similar discussion can show that U^* is also surjective. Hence U is invertible. Now the result follows from Theorem 2.3, since $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J}, i\in[m]} = \{U^{-1}Uf_{ij}\}_{j\in\mathbf{J}, i\in[m]}$. $\qquad\square$

The following two results present more conditions for the sum of a woven frame and its perturbation under an operator to be still a woven frame.

Proposition 2.6. *Suppose that the family of frames $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J}, i\in[m]}$ is woven for $\mathscr{H}$ with universal frame bounds $C_{\mathscr{F}}$ and $D_{\mathscr{F}}$, and that $U \in \mathcal{B}(\mathscr{H})$. If U is a positive operator, then $\mathscr{F} + U\mathscr{F}$ is woven for $\mathscr{H}$ with universal frame bounds $C_{\mathscr{F}}$ and $D_{\mathscr{F}}\|\mathrm{Id}_{\mathscr{H}} + U\|^2$.*

Proof. For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbf{J}$ and any $f \in \mathscr{H}$ we obtain

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, f_{ij} + Uf_{ij}\rangle|^2$$
$$= \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle(\mathrm{Id}_{\mathscr{H}} + U)^*f, f_{ij}\rangle|^2$$
$$\leq D_{\mathscr{F}}\|\mathrm{Id}_{\mathscr{H}} + U\|^2\|f\|^2, \quad \forall f \in \mathscr{H}.$$

Now, noting that U is positive, we get

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, f_{ij} + Uf_{ij}\rangle|^2$$
$$\geq C_{\mathscr{F}}\|(\mathrm{Id}_{\mathscr{H}} + U)^*f\|^2$$
$$= C_{\mathscr{F}}(\|f\|^2 + 2\,\mathrm{Re}\langle Uf, f\rangle + \|U^*f\|^2)$$
$$\geq C_{\mathscr{F}}\|f\|^2,$$

and we arrive at the result. $\qquad\square$

Theorem 2.7. *Suppose that the family of frames $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J}, i\in[m]}$ is woven for $\mathscr{H}$ with universal frame bounds $C_{\mathscr{F}}$ and $D_{\mathscr{F}}$, and that $U \in \mathcal{B}(\mathscr{H})$. If the composition of U and the frame operator of each weaving is positive, then $\mathscr{F} + U\mathscr{F}$ is woven for $\mathscr{H}$ with universal frame bounds $C_{\mathscr{F}}$ and $D_{\mathscr{F}}\|\mathrm{Id}_{\mathscr{H}} + U\|^2$.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathbf{J}$. Since the universal upper frame bound is always given, we only need to show the universal lower frame bound condition. Denote by $S_{\mathscr{F}}^\sigma$ and $S_{\mathscr{F}+U\mathscr{F}}^\sigma$ be respectively the frame operators of $\cup_{i\in[m]}\{f_{ij}\}_{j\in\sigma_i}$ and $\cup_{i\in[m]}\{f_{ij} + Uf_{ij}\}_{j\in\sigma_i}$. The assumption tells us that $US_{\mathscr{F}}^\sigma$ is positive, then so is $S_{\mathscr{F}}^\sigma U^*$. Therefore,

$$S_{\mathscr{F}+U\mathscr{F}}^\sigma f$$
$$= \sum_{i\in[m]}\sum_{j\in\sigma_i}\langle f, f_{ij} + Uf_{ij}\rangle(f_{ij} + Uf_{ij})$$
$$= \sum_{i\in[m]}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij} + U\sum_{i\in[m]}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij}$$
$$+ \sum_{i\in[m]}\sum_{j\in\sigma_i}\langle U^*f, f_{ij}\rangle f_{ij}$$
$$+ U\sum_{i\in[m]}\sum_{j\in\sigma_i}\langle U^*f, f_{ij}\rangle f_{ij}$$
$$= S_{\mathscr{F}}^\sigma f + US_{\mathscr{F}}^\sigma + S_{\mathscr{F}}^\sigma U^*f + US_{\mathscr{F}}^\sigma U^*f$$
$$\geq S_{\mathscr{F}}^\sigma f$$

for each $f \in \mathscr{H}$. This leads to

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, f_{ij} + Uf_{ij}\rangle|^2 = \langle S_{\mathscr{F}+U\mathscr{F}}^\sigma f, f\rangle$$
$$\geq \langle S_{\mathscr{F}}^\sigma f, f\rangle \geq C_{\mathscr{F}}\|f\|^2,$$

as desired. $\qquad\square$

The following operator perturbation type for

woven frames is more general, comparing to the ones in Proposition 2.6 and Theorem 2.7.

Theorem 2.8. *Suppose that $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J},i\in[m]}$ and $\mathscr{G} = \{g_{ij}\}_{j\in\mathbf{J},i\in[m]}$ are woven frames for $\mathscr{H}$ with universal frame bounds $C_{\mathscr{F}}, D_{\mathscr{F}}$ and $C_{\mathscr{G}}, D_{\mathscr{G}}$ respectively, and that $T_1, T_2 \in \mathcal{B}(\mathscr{H})$. If for any partition of $\mathbf{J}$, the two weaving frames generated respectively by $\mathscr{F}$ and $\mathscr{G}$ are orthogonal, and T_1 or T_2 is surjective, then $T_1\mathscr{F} + T_2\mathscr{F}$ is woven for $\mathscr{H}$.*

Proof. For any given partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbf{J}$, let $U_{\mathscr{F}}^{\sigma}$ and $U_{\mathscr{G}}^{\sigma}$ be, respectively, the analysis operators of $\cup_{i\in[m]}\{f_{ij}\}_{j\in\sigma_i}$ and $\cup_{i\in[m]}\{g_{ij}\}_{j\in\sigma_i}$. We assume, without loss of generality, that T_1 is surjective. Then, by the same way as that presented in the proof of Theorem 2.3 we can show that $\|T_1^*f\|^2 \geq \|T_1^\dagger\|^{-2}\|f\|^2$ for each $f \in \mathscr{H}$. Noting also that $(U_{\mathscr{F}}^{\sigma})^*U_{\mathscr{G}}^{\sigma} = (U_{\mathscr{G}}^{\sigma})^*U_{\mathscr{F}}^{\sigma} = 0$, we have

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_1f_{ij} + T_2g_{ij}\rangle|^2$$

$$= \|U_{\mathscr{F}}^{\sigma}T_1^*f + U_{\mathscr{G}}^{\sigma}T_2^*f\|^2$$
$$= \langle(U_{\mathscr{F}}^{\sigma}T_1^* + U_{\mathscr{G}}^{\sigma}T_2^*)^*(U_{\mathscr{F}}^{\sigma}T_1^* + U_{\mathscr{G}}^{\sigma}T_2^*)f, f\rangle$$
$$= \langle T_1(U_{\mathscr{F}}^{\sigma})^*U_{\mathscr{F}}^{\sigma}T_1^*f, f\rangle$$
$$\quad + \langle T_1(U_{\mathscr{F}}^{\sigma})^*U_{\mathscr{G}}^{\sigma}T_2^*f, f\rangle$$
$$\quad + \langle T_2(U_{\mathscr{G}}^{\sigma})^*U_{\mathscr{F}}^{\sigma}T_1^*f, f\rangle$$
$$\quad + \langle T_2(U_{\mathscr{G}}^{\sigma})^*U_{\mathscr{G}}^{\sigma}T_2^*f, f\rangle$$
$$= \langle(U_{\mathscr{F}}^{\sigma})^*U_{\mathscr{F}}^{\sigma}T_1^*f, T_1^*f\rangle + \langle(U_{\mathscr{G}}^{\sigma})^*U_{\mathscr{G}}^{\sigma}T_2^*f, T_2^*f\rangle$$
$$\geq \langle(U_{\mathscr{F}}^{\sigma})^*U_{\mathscr{F}}^{\sigma}T_1^*f, T_1^*f\rangle$$
$$= \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle T_1^*f, f_{ij}\rangle|^2$$
$$\geq C_{\mathscr{F}}\|T_1^*f\|^2 \geq C_{\mathscr{F}}\|T_1^\dagger\|^{-2}\|f\|^2$$

for each $f \in \mathscr{H}$. It is trivial to show that

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_1f_{ij} + T_2g_{ij}\rangle|^2$$

$$= \|U_{\mathscr{F}}^{\sigma}T_1^*f + U_{\mathscr{G}}^{\sigma}T_2^*f\|^2$$
$$\leq 2(\|U_{\mathscr{F}}^{\sigma}T_1^*f\|^2 + \|U_{\mathscr{G}}^{\sigma}T_2^*f\|^2)$$
$$= 2\left(\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle T_1^*f, f_{ij}\rangle|^2\right.$$
$$\left. + \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle T_2^*f, g_{ij}\rangle|^2\right)$$
$$\leq 2(D_{\mathscr{F}}\|T_1^*f\|^2 + D_{\mathscr{G}}\|T_2^*f\|^2)$$
$$\leq 2(D_{\mathscr{F}}\|T_1\|^2 + D_{\mathscr{G}}\|T_2\|^2)\|f\|^2,$$

and the proof is completed. □

An alternative condition under which the conclusion of Theorem 2.8 remains true is given below.

Theorem 2.9. *Suppose that $\mathscr{F} = \{f_{ij}\}_{j\in\mathbf{J},i\in[m]}$ and $\mathscr{G} = \{g_{ij}\}_{j\in\mathbf{J},i\in[m]}$ are woven frames for $\mathscr{H}$ with universal frame bounds $C_{\mathscr{F}}, D_{\mathscr{F}}$ and $C_{\mathscr{G}}, D_{\mathscr{G}}$ respectively, and that $T_1, T_2 \in \mathcal{B}(\mathscr{H})$. If T_1 is surjective with $\frac{1}{2}\|T_1^\dagger\|^{-2}C_{\mathscr{F}} - D_{\mathscr{G}}\|T_2\|^2 > 0$ or T_2 is surjective with $\frac{1}{2}\|T_2^\dagger\|^{-2}C_{\mathscr{G}} - D_{\mathscr{F}}\|T_1\|^2 > 0$, then $T_1\mathscr{F} + T_2\mathscr{F}$ is woven for $\mathscr{H}$.*

Proof. Without loss of generality, let T_1 be surjective with $\frac{1}{2}\|T_1^\dagger\|^{-2}C_{\mathscr{F}} - D_{\mathscr{G}}\|T_2\|^2 > 0$. For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbf{J}$ and any $f \in \mathscr{H}$ we have

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_1f_{ij}\rangle|^2$$

$$= \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_1f_{ij} + T_2g_{ij} - T_2g_{ij}\rangle|^2$$
$$\leq 2\left(\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_1f_{ij} + T_2g_{ij}\rangle|^2\right.$$
$$\left. + \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_2g_{ij}\rangle|^2\right).$$

Therefore,

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_1f_{ij} + T_2g_{ij}\rangle|^2$$

$$\geq \frac{1}{2}\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_1f_{ij}\rangle|^2$$
$$\quad - \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f, T_2g_{ij}\rangle|^2$$
$$\geq \frac{1}{2}C_{\mathscr{F}}\|T_1^*f\|^2 - D_{\mathscr{G}}\|T_2^*f\|^2$$
$$\geq \frac{1}{2}C_{\mathscr{F}}\|T_1^\dagger\|^{-2}\|f\|^2 - D_{\mathscr{G}}\|T_2\|^2\|f\|^2$$
$$= \left(\frac{1}{2}C_{\mathscr{F}}\|T_1^\dagger\|^{-2} - D_{\mathscr{G}}\|T_2\|^2\right)\|f\|^2.$$

The universal upper frame bound condition is straightforward, or check the second part, proof of Theorem 2.8. □

In [2, Example 2], the authors told us that applying two different operators to woven frames can give frames that are not woven. So a natural problem arises: To make woven frames to be still woven when applying two different operators to them, what conditions need to be attached? To this problem, we have the following

Theorem 2.10. *Suppose the frames* $\mathscr{F}' = \{f_j\}_{j\in\mathbf{J}}$ *and* $\mathscr{G}' = \{g_j\}_{j\in\mathbf{J}}$ *with frame bounds* $C_{\mathscr{F}'}, D_{\mathscr{F}'}$ *and* $C_{\mathscr{G}'}, D_{\mathscr{G}'}$ *respectively are woven for* $\mathscr{H}$ *with universal frame bounds* $C_{\mathscr{F}'\mathscr{G}'}$ *and* $D_{\mathscr{F}'\mathscr{G}'}$, *and that* $T_1, T_2 \in \mathcal{B}(\mathscr{H})$. *If* T_1 *is surjective and* $\sqrt{C_{\mathscr{F}'\mathscr{G}'}}\|T_1^\dagger\|^{-1} > \sqrt{D_{\mathscr{G}'}}\|T_1 - T_2\|$, *then* $T_1\mathscr{F}'$ *and* $T_2\mathscr{G}'$ *are woven for* $\mathscr{H}$ *with universal frame bounds* $(\sqrt{C_{\mathscr{F}'\mathscr{G}'}}\|T_1^\dagger\|^{-1} - \sqrt{D_{\mathscr{G}'}}\|T_1 - T_2\|)^2$ *and* $(D_{\mathscr{F}'}\|T_1\|^2 + D_{\mathscr{G}'}\|T_2\|^2)$.

Proof. Let $\{\sigma_1, \sigma_2\}$ be any partition of $\mathbf{J}$. The universal upper frame bound condition follows from the following calculation:

$$\sum_{j\in\sigma_1} |\langle f, T_1 f_j\rangle|^2 + \sum_{j\in\sigma_2} |\langle f, T_2 g_j\rangle|^2$$

$$= \sum_{j\in\sigma_1} |\langle T_1^* f, f_j\rangle|^2 + \sum_{j\in\sigma_2} |\langle T_2^* f, g_j\rangle|^2$$

$$\leq \sum_{j\in\mathbf{J}} |\langle f, T_1 f_j\rangle|^2 + \sum_{j\in\mathbf{J}} |\langle f, T_2 g_j\rangle|^2$$

$$\leq D_{\mathscr{F}'}\|T_1^* f\|^2 + D_{\mathscr{G}'}\|T_2^* f\|^2$$

$$\leq (D_{\mathscr{F}'}\|T_1\|^2 + D_{\mathscr{G}'}\|T_2\|^2)\|f\|^2, \quad \forall f \in \mathscr{H}.$$

For the universal lower frame bound condition, we have by Minkowski's inequality and subadditivity of the square root function that

$$\left(\sum_{j\in\sigma_1} |\langle f, T_1 f_j\rangle|^2 + \sum_{j\in\sigma_2} |\langle f, T_2 g_j\rangle|^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_{j\in\sigma_1} |\langle T_1^* f, f_j\rangle|^2 \right.$$

$$\left. + \sum_{j\in\sigma_2} |\langle T_1^* f + (T_2^* - T_1^*)f, g_j\rangle|^2\right)^{\frac{1}{2}}$$

$$\geq \left(\sum_{j\in\sigma_1} |\langle T_1^* f, f_j\rangle|^2 + \sum_{j\in\sigma_2} |\langle T_1^* f, g_j\rangle|^2\right)^{\frac{1}{2}}$$

$$- \left(\sum_{j\in\sigma_2} |\langle (T_2^* - T_1^*)f, g_j\rangle|^2\right)^{\frac{1}{2}}$$

$$\geq \sqrt{C_{\mathscr{F}'\mathscr{G}'}}\|T_1^* f\| - \left(\sum_{j\in\mathbf{J}} |\langle (T_2^* - T_1^*)f, g_j\rangle|^2\right)^{\frac{1}{2}}$$

$$\geq \sqrt{C_{\mathscr{F}'\mathscr{G}'}}\|T_1^\dagger\|^{-1}\|f\| - \sqrt{D_{\mathscr{G}'}}\|T_1 - T_2\|\|f\|$$

$$= (\sqrt{C_{\mathscr{F}'\mathscr{G}'}}\|T_1^\dagger\|^{-1} - \sqrt{D_{\mathscr{G}'}}\|T_1 - T_2\|)\|f\|,$$

where in the last inequality we apply the fact that $\|T_1^* f\| \geq \|T_1^\dagger\|^{-1}\|f\|$ for any $f \in \mathscr{H}$, and the proof is finished. $\qquad\square$

Corollary 2.11. *Suppose that* $\mathscr{F}' = \{f_j\}_{j\in\mathbf{J}}$ *is a frame for* $\mathscr{H}$ *with frame bounds* $C_{\mathscr{F}'}$ *and* $D_{\mathscr{F}'}$, *and that* $T_1, T_2 \in \mathcal{B}(\mathscr{H})$. *If* T_1 *is surjective and* $\sqrt{C_{\mathscr{F}'}}\|T_1^\dagger\|^{-1} > \sqrt{D_{\mathscr{F}'}}\|T_1 - T_2\|$, *then* $T_1\mathscr{F}'$ *and* $T_2\mathscr{F}'$ *are woven for* $\mathscr{H}$ *with universal frame bounds* $(\sqrt{C_{\mathscr{F}'}}\|T_1^\dagger\|^{-1} - \sqrt{D_{\mathscr{F}'}}\|T_1 - T_2\|)^2$ *and* $D_{\mathscr{F}'}(\|T_1\|^2 + \|T_2\|^2)$.

Proof. Since $\mathscr{F}'$ is woven to itself, the result follows immediately from Theorem 2.10. $\quad\square$

Corollary 2.12. *Suppose that* $\mathscr{F}' = \{f_j\}_{j\in\mathbf{J}}$ *is a frame for* $\mathscr{H}$ *with frame bounds* $C_{\mathscr{F}'}$ *and* $D_{\mathscr{F}'}$, *and that* $T \in \mathcal{B}(\mathscr{H})$. *If* $\|\mathrm{Id}_{\mathscr{H}} - T\| < \sqrt{\frac{C_{\mathscr{F}'}}{D_{\mathscr{F}'}}}$, *then* $\mathscr{F}'$ *and* $T\mathscr{F}'$ *are woven for* $\mathscr{H}$.

Proof. The result follows if in Corollary 2.11 we take $T_1 = \mathrm{Id}_{\mathscr{H}}$ and $T_2 = T$. $\quad\square$

Remark 2.13. Corollary 2.12 is just Proposition 6.2 in [2].

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