On the sign ambiguity in equivariant cohomological rigidity of GKM graphs

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Abstract: This is the sequel to the author's previous paper [1] with Matthias Franz. In the present paper, we introduce the notion of equivariant total Chern class of a GKM graph and show that the pair of graph equivariant cohomology and the equivariant total Chern class determines the GKM graph completely. We also show that for a torus graph in the sense of Maeda–Masuda–Panov, the pair of graph equivariant cohomology and the equivariant 1-st Chern class determines the torus graph completely.

Key words: GKM graph; torus graph; equivariant cohomological rigidity; equivariant total Chern class.

1. Introduction. This is the sequel to the author's previous paper [1] with Matthias Franz. We quickly recall the main result in *loc.cit.*

Let $\mathcal{G}$ be a finite n-valent graph (multi-edges are allowed, but loops are not) with vertex set $\mathcal{V}$. We denote by $\mathcal{E}$ the set of *directed* edges of $\mathcal{G}$. For a direct edge $e \in \mathcal{E}$, we denote by $i(e)$ and $t(e)$ the initial and terminal point of e, respectively. We set

$$\mathcal{E}_p := \{e \in \mathcal{E} \mid i(e) = p\}$$

for each $p \in \mathcal{V}$.

Let T be the compact torus of rank r. An isomorphism $T \to (S^1)^r$ of Lie groups induces a graded ring isomorphism from $H_T^*(pt) = H^*(BT)$ to polynomial ring $\mathbf{Z}[x_1, \ldots, x_r]$ with the grading $\deg x_i = 2$, where BT is the base space of the universal T-bundle. For two polynomials $P, Q \in H^*(BT)$, we write $P \mid Q$ if $Q = RP$ for some $R \in H^*(BT)$.

Definition 1.1. An **axial function** on $\mathcal{G}$ is a map

$$\alpha : \mathcal{E} \to H^2(BT)$$

satisfying the following conditions:

(i) $\alpha(\overline{e}) = \pm\alpha(e)$ where $\overline{e}$ is the directed edge obtained by reversing the direction of e.

(ii) $\alpha(e)$ and $\alpha(e')$ are linearly independent over $\mathbf{Z}$ if $e \neq e'$ and $i(e) = i(e')$.

(iii) The greatest common divisor of the coefficients of $\alpha(e)$ is 1.

For a vertex $p \in \mathcal{V}$ of $\mathcal{G}$, we often denote

$$\alpha_{p,1}, \ldots, \alpha_{p,n}$$

the elements of $\{\alpha(e) \mid e \in \mathcal{E}, i(e) = p\}$.

In [4] Guillemin–Zara found the following important notion:

Definition 1.2. A **parallel transport** of $(\mathcal{G}, \alpha)$ is a family $\mathcal{P} = \{\mathcal{P}_e\}_{e \in \mathcal{E}}$ of bijections

$$\mathcal{P}_e : \mathcal{E}_{i(e)} \to \mathcal{E}_{t(e)}$$

satisfying the following conditions for all $e \in \mathcal{E}$ and $e' \in \mathcal{E}_{i(e)}$:

(i) $\mathcal{P}_{\overline{e}} = \mathcal{P}_e^{-1}$.

(ii) $\mathcal{P}_e(e) = \overline{e}$.

(iii) $\alpha(\mathcal{P}_e(e')) - \alpha(e') \in \mathbf{Z}\,\alpha(e)$.

Note that a parallel transport is called a connection in [4]. See [1, Remark 2.4 (iii)].

A pair $(\mathcal{G}, \alpha)$ (or $\mathcal{G}$ for simplicity) is called an **abstract GKM graph** (**GKM graph** for short) of type (r, n) if there exists at least one parallel transport of $(\mathcal{G}, \alpha)$.

For a GKM graph $\mathcal{G}$ one can attach a graded $H^*(BT)$-algebra $H_T^*(\mathcal{G})$ as follows:

$$H_T^*(\mathcal{G}) := \big\{ f : \mathcal{V} \to H^*(BT) \;\big|\; \alpha(e) \mid (f(i(e)) - f(t(e)))\ (e \in \mathcal{E}) \big\}.$$

This is called **graph equivariant cohomology** of the GKM graph $\mathcal{G}$. We denote by $H_T^{2i}(\mathcal{G})$ its degree $2i$ component (an element $f \in H_T^*(\mathcal{G})$ is of degree $2i$ if $f(p)$ is so for any $p \in \mathcal{V}$).

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Recall that the $H^*(BT)$-algebra structure on $H_T^*(\mathcal{G})$ is defined by the ring monomorphism

$$H^*(BT) \to H_T^*(\mathcal{G}), \quad P \mapsto f_P$$

where $f_P(p) := P$ for any $p \in \mathcal{V}$.

Definition 1.3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha)$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \alpha')$ be GKM graphs of type (r, n). We say that $\mathcal{G}$ and $\mathcal{G}'$ are **isomorphic** in the sense of [1] if there exist bijections

$$\varphi_0 : \mathcal{V} \to \mathcal{V}', \quad \varphi_1 : \mathcal{E} \to \mathcal{E}'$$

satisfying the following conditions (i) and (ii) for any $e \in \mathcal{E}$:

(i) $\varphi_0(i(e)) = i(\varphi_1(e))$.
(ii) $\alpha'(\varphi_1(e)) = \pm\alpha(e)$.

Such a pair (φ_0, φ_1) is called an **isomorphism** from $\mathcal{G}$ to $\mathcal{G}'$.

Note that an isomorphism

$$g = (\varphi_0, \varphi_1) : \mathcal{G} \to \mathcal{G}'$$

of GKM graphs induces a graded $H^*(BT)$-algebra isomorphism

$$g^* : H_T^*(\mathcal{G}') \to H_T^*(\mathcal{G})$$

defined by $(g^*(f))(p) := f(\varphi_0(p))$. The assignment

$$g \mapsto g^*$$

is functorial in the sense that $\mathrm{id}_{\mathcal{G}}^* = \mathrm{id}_{H_T^*(\mathcal{G})}$ and $(g' \circ g)^* = g^* \circ (g')^*$ for isomorphisms $g : \mathcal{G} \to \mathcal{G}'$, $g' : \mathcal{G}' \to \mathcal{G}''$. Here the composition of isomorphisms of GKM graphs is defined in an obvious way.

Then the main result of [1] is the following:

Theorem 1.4 ([1]). *$H_T^*(\mathcal{G})$ and $H_T^*(\mathcal{G}')$ are isomorphic as $H^*(BT)$-algebras if and only if $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic as GKM graphs.*

Note that very recently, in case of GKM graphs arising from GKM manifolds Goertsches-Zoller [3] found a vast generalization of Theorem 1.4.

The following remark tells us that in Theorem 1.4 one can not remove the sign ambiguity in the values of the axial functions:

Remark 1.5. Consider the following two GKM graphs of type $(1, 1)$:

$$\mathcal{G} = (\{p, q\}, \{e, \overline{e}\}, \alpha), \quad \mathcal{G}' = (\{p, q\}, \{e, \overline{e}\}, \alpha')$$

where $i(e) = p, t(e) = q$ and

$$\alpha(e) = x_1, \alpha(\overline{e}) = -x_1, \alpha'(e) = \alpha'(\overline{e}) = x_1.$$

The GKM graphs $\mathcal{G}$ and $\mathcal{G}'$ have the same graph equivariant cohomology, but the signs appearing in the value of the axial functions are different. In particular one can not recover the GKM graph $\mathcal{G}$ from the graded $H^*(BT)$-algebra $H_T^*(\mathcal{G})$.

Theorem 1.4 was motivated by Masuda's result on toric manifolds [7]. Recently, Hiraku Abe pointed out that in [7] one needs additional assumptions on the graded $H^*(BT)$-algebra isomorphism. As explained in [5, Remark 2.5], one needs the assumption that the graded $H^*(BT)$-algebra isomorphism preserves equivariant 1-st Chern classes. The aim of the present paper is to reveal that as well as toric case, equivariant *total* Chern classes resolve the sign ambiguity in Theorem 1.4.

To state our result more precisely, we make the following definition:

Definition 1.6. We say that GKM graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha)$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \alpha')$ are **geometrically isomorphic** if there exist bijections

$$\varphi_0 : \mathcal{V} \to \mathcal{V}', \quad \varphi_1 : \mathcal{E} \to \mathcal{E}'$$

satisfying the following conditions (i) and (ii′) for any $e \in \mathcal{E}$:

(i) $\varphi_0(i(e)) = i(\varphi_1(e))$.
(ii′) $\alpha'(\varphi_1(e)) = \alpha(e)$.

Such a pair (φ_0, φ_1) is called a **geometric isomorphism** from $\mathcal{G}$ to $\mathcal{G}'$.

The difference with an isomorphism of GKM graphs is in the condition (ii′).

In Section 2 we introduce the notion of equivariant total Chern class $c^T(\mathcal{G})$ of a GKM graph $\mathcal{G}$. This is an element of the graph equivariant cohomology $H_T^*(\mathcal{G})$. The i-th equivariant Chern class $c_i^T(\mathcal{G})$ is defined by the homogeneous decomposition

$$c^T(\mathcal{G}) = c_0^T(\mathcal{G}) + c_1^T(\mathcal{G}) + \cdots + c_n^T(\mathcal{G})$$

of the equivariant total Chern class.

The following is the main result in the present paper:

Theorem 1.7. *Let $\mathcal{G}$ and $\mathcal{G}'$ be GKM graphs. If there exists an isomorphism $\varphi : H_T^*(\mathcal{G}') \to H_T^*(\mathcal{G})$ of graded $H^*(BT)$-algebras which preserves equivariant total Chern classes, then there exists a geometric isomorphism $g : \mathcal{G} \to \mathcal{G}'$ which induces the isomorphism φ.*

Corollary 1.8. *Let $\mathcal{G}$ and $\mathcal{G}'$ be GKM graphs. Then, there exists a graded $H^*(BT)$-algebra isomorphism $H_T^*(\mathcal{G}) \to H_T^*(\mathcal{G}')$ preserving equivariant total Chern classes if and only if $\mathcal{G}$ and $\mathcal{G}'$ are geometrically isomorphic as GKM graphs.*

Remark 1.9. In view of abstract algebra and Theorem 1.4, it is natural to focus on the notion of an isomorphism of GKM graphs. However, Corollary 1.8 indicates that *in view of geometry it is natural to regard the pair* $(H_T^*(\mathcal{G}), c^T(\mathcal{G}))$ *as a single object* (like as symplectic manifold (X, ω)). See also Remark 3.4 (ii).

In [6] Maeda–Masuda–Panov introduced the notion of a torus graph. A GKM graph of type (n, n) is called a **torus graph** of degree n if the set

$$\{\alpha(e) \mid i(e) = p\}$$

is linearly independent over $\mathbf{Z}$ for any $p \in \mathcal{V}$.

In case of torus graphs, one can show the following result which is analogues to Theorem 1.7.

Theorem 1.10. *Let* $\mathcal{T}$ *and* $\mathcal{T}'$ *be torus graphs. If there exists an isomorphism* $\varphi : H_T^*(\mathcal{T}') \to H_T^*(\mathcal{T})$ *of graded* $H^*(BT)$*-algebras which preserves equivariant* 1*-st Chern classes, then there exists a geometric isomorphism* $g : \mathcal{T} \to \mathcal{T}'$ *which induces the isomorphism* φ.

Corollary 1.11. *Let* $\mathcal{T}$ *and* $\mathcal{T}'$ *be torus graphs of degree* n*. Then, there exists a graded* $H^*(BT)$*-algebra isomorphism* $H_T^*(\mathcal{T}) \to H_T^*(\mathcal{T}')$ *preserving equivariant* 1*-st Chern classes if and only if* $\mathcal{T}$ *and* $\mathcal{T}'$ *are geometrically isomorphic as GKM graphs.*

2. Equivariant cohomology classes arizing from symmetric polynomials. In this section we introduce the notion of equivariant total Chern classes of GKM graphs. We first show the following lemma:

Lemma 2.1. *For each symmetric polynomial*

$$P(y_1, \dots, y_n) \in \mathbf{Z}[y_1, \dots, y_n]^{\mathfrak{S}_n}$$

in n*-variables, the map*

$$f_P : \mathcal{V} \to H^*(BT), \quad f_P(p) := P(\alpha_{p,1}, \dots, \alpha_{p,n})$$

provides a well-defined element in $H_T^*(\mathcal{G})$.

Proof. Note that since $P(y_1, \dots, y_n)$ is symmetric, the polynomial

$$P(\alpha_{p,1}, \dots, \alpha_{p,n}) \in \mathbf{Z}[x_1, \dots, x_n]$$

is independent of numbering of elements of the set $\{\alpha(e) \mid e \in \mathcal{E}_p\}$.

Take a parallel transport $\mathcal{P} = \{\mathcal{P}_e\}_{e \in \mathcal{E}}$ of $\mathcal{G}$. Let e be a directed edge of $\mathcal{G}$. We set

$$p := i(e), \quad q := t(e).$$

For each $e' \in \mathcal{E}_{i(e)}$, by the definition of parallel transport we have a relation

$$\alpha(\mathcal{P}_e(e')) = \alpha(e') + k_{e'}\alpha(e)$$

for some $k_{e'} \in \mathbf{Z}$. In other words, there exists such a permutation

$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$$

that $\alpha_{q,\sigma(i)} = \alpha_{p,i} + k_i\alpha(e)$ for some $k_i \in \mathbf{Z}$. Then one can calculate as follows:

$$\begin{aligned} f_P(p) &= P(\alpha_{p,1}, \dots, \alpha_{p,n}) \\ &= P(\alpha_{q,\sigma(1)} - k_1\alpha(e), \dots, \alpha_{q,\sigma(n)} - k_n\alpha(e)) \\ &= P(\alpha_{q,\sigma(1)}, \dots, \alpha_{q,\sigma(n)}) \\ &\quad + (\text{terms divisible by } \alpha(e)) \\ &= P(\alpha_{q,1}, \dots, \alpha_{q,n}) + (\text{terms divisible by } \alpha(e)) \\ &= f_P(q) + (\text{terms divisible by } \alpha(e)). \end{aligned}$$

Thus $f_P(p) - f_P(q)$ is divisible by $\alpha(e)$ in $H^*(BT)$. The proof is now complete. □

Using Lemma 2.1 we define equivariant Chern classes of GKM graphs:

Definition 2.2. Let $\mathcal{G}$ be a GKM graph of type (r, n).

(1) The ***T*-equivariant total Chern class** $c^T(\mathcal{G})$ of $\mathcal{G}$ is the element f_P in $H_T^*(\mathcal{G})$ attached to the symmetric polynomial

$$P(y_1, \dots, y_n) = \prod_{i=1}^{n}(1 + y_i).$$

(2) For each $0 \le i \le n$, the ***T*-equivariant *i*-th Chern class** $c_i^T(\mathcal{G})$ of $\mathcal{G}$ is defined by the homogeneous decomposition

$$c^T(\mathcal{G}) = c_0^T(\mathcal{G}) + c_1^T(\mathcal{G}) + \cdots + c_n^T(\mathcal{G}).$$

Remark 2.3.

(i) For any equivariantly formal GKM manifold X having an invariant almost complex structure and connected stabilizers, through the GKM localization

$$H_T^*(X) \cong H_T^*(\mathcal{G}_X),$$

where $\mathcal{G}_X$ is the GKM graph of X, the T-equivariant total Chern class of X corresponds to $c^T(\mathcal{G}_X)$ (see [2, Proposition 3.5]).

(ii) In Definition 2.2, we allow the case that $\mathcal{G}$ is the GKM graph of a GKM manifold having no almost complex structure. For example we define a GKM graph $\mathcal{G}$ of type $(2, 2)$ by

$$\mathcal{V} = \{p, q\}, \quad \mathcal{E} = \{e_1, \overline{e_1}, e_2, \overline{e_2}\},$$

where $i(e_1) = i(e_2) = p, t(e_1) = t(e_2) = q$,

$$\alpha(e_1) = x_1, \alpha(\overline{e_1}) = x_1, \alpha(e_2) = x_2, \alpha(\overline{e_2}) = x_2.$$

Note that the GKM graph $\mathcal{G}$ naturally arises from the canonical $(S^1)^2$-action on the 4-sphere S^4, which does not admit almost complex structures. In this case we have

$$c^T(\mathcal{G}) = ((1 + x_1)(1 + x_2), (1 + x_1)(1 + x_2)),$$
$$c_0^T(\mathcal{G}) = (1, 1), \quad c_1^T(\mathcal{G}) = (x_1 + x_2, x_1 + x_2),$$
$$c_2^T(\mathcal{G}) = (x_1x_2, x_1x_2).$$

Note that $H^2(S^4) = 0$, but $H_T^2(S^4) \neq 0$.

3. GKM case. In this section we give the proof of Theorem 1.7. Note that for each $p \in \mathcal{V}$, the map

$$\tau_p : \mathcal{V} \to H^*(BT), \quad \tau_p(q) := \begin{cases} \prod_{e \in \mathcal{E}_p} \alpha(e) & \text{if } q = p, \\ 0 & \text{if } q \neq p \end{cases}$$

defines a well-defined element in $H_T^*(\mathcal{G})$. In [4] this is called **equivariant Thom class** associated with p. By [1, Proposition 3.4, Theorem 1.1] we have the following lemma:

Lemma 3.1. *For any $H^*(BT)$-algebra isomorphism $\varphi : H_T^*(\mathcal{G}') \to H_T^*(\mathcal{G})$, there exists an isomorphism $(\varphi_0, \varphi_1) : \mathcal{G} \to \mathcal{G}'$ of GKM graphs satisfying $\varphi(\tau_{\varphi_0(p)}) = \pm\tau_p$ for any $p \in \mathcal{V}$.*

The following is the main result in the present paper:

Theorem 3.2. *Let $\mathcal{G}$ and $\mathcal{G}'$ be GKM graphs. If there exists an isomorphism $\varphi : H_T^*(\mathcal{G}') \to H_T^*(\mathcal{G})$ of graded $H^*(BT)$-algebras which preserves equivariant total Chern classes, then there exists a geometric isomorphism $g : \mathcal{G} \to \mathcal{G}'$ which induces the isomorphism φ.*

Proof. We take an isomorphism

$$g = (\varphi_0, \varphi_1) : \mathcal{G} \to \mathcal{G}'$$

as in Lemma 3.1. Note that $\varphi(\tau_{\varphi_0(q)}) = \eta_q\tau_q$ for some $\eta_q \in \{\pm 1\}$ (as we will see in Remark 3.4 (i), one can show that $\eta_p = 1$ for any $p \in \mathcal{V}$. However we do not need this fact here).

Let f be an arbitrary element in $H_T^*(\mathcal{G}')$. By the definition of equivariant Thom classes associated with vertices we have

$$\begin{aligned}\varphi(f\tau_{\varphi_0(p)}) &= \varphi(f(\varphi_0(p))\tau_{\varphi_0(p)}) \\ &= f(\varphi_0(p))\varphi(\tau_{\varphi_0(p)}) \\ &= \eta_p f(\varphi_0(p))\tau_p.\end{aligned}$$

On the other hand, since φ is a homomorphism of $H^*(BT)$-algebras, we have

$$\begin{aligned}\varphi(f\tau_{\varphi_0(p)}) &= \varphi(f)\varphi(\tau_{\varphi_0(p)}) \\ &= \eta_p\varphi(f)\tau_p \\ &= \eta_p(\varphi(f))(p)\tau_p.\end{aligned}$$

Comparing above computations one has

$$(\varphi(f))(p) = f(\varphi_0(p))$$

for all $p \in \mathcal{V}$. Then we have

$$(g^*(f))(p) = f(\varphi_0(p)) = (\varphi(f))(p).$$

Thus φ is induced from the isomorphism g. In the rest of the proof, we show that g is a geometric isomorphism.

By taking f as $c^T(\mathcal{G}')$ we have

$$c^T(\mathcal{G})(p) = \varphi(c^T(\mathcal{G}'))(p) = c^T(\mathcal{G}')(\varphi_0(p)).$$

In terms of the axial functions, the equality is rephrased as follows:

$$\prod_{e \in \mathcal{E}_p}(1 + \alpha(e)) = \prod_{e' \in \mathcal{E}'_{\varphi_0(p)}}(1 + \alpha'(e')).$$

Since both

$$\{\alpha(e) \mid e \in \mathcal{E}_p\}$$

and

$$\{\alpha'(e') = \pm\alpha(\varphi_1^{-1}(e')) \mid e' \in \mathcal{E}'_{\varphi_0(p)}\}$$

are 2-linearly independent over $\mathbf{Z}$, we finally get

$$\alpha'(e') = \alpha(\varphi_1^{-1}(e')).$$

The proof is now complete. □

Corollary 3.3. *Let $\mathcal{G}$ and $\mathcal{G}'$ be GKM graphs of type (r, n). Then, there exists a graded $H^*(BT)$-algebra isomorphism $H_T^*(\mathcal{G}) \to H_T^*(\mathcal{G}')$ preserving equivariant total Chern classes if and only if $\mathcal{G}$ and $\mathcal{G}'$ are geometrically isomorphic as GKM graphs.*

Remark 3.4.

(i) In Theorem 3.2, one can show that the isomorphism $\varphi : H_T^*(\mathcal{G}') \to H_T^*(\mathcal{G})$ preserves equivariant Thom classes associated with vertices. The proof goes as follows:
We focus on equivariant top Chern class $c_n^T(\mathcal{G}')$. Since φ preserves the grading and equivariant total Chern classes, we have

$$\varphi(c_n^T(\mathcal{G}')) = c_n^T(\mathcal{G}).$$

On the other hand, the equivariant top Chern class $c_n^T(\mathcal{G})$ can be described in terms of equivariant Thom classes as follows:

$$c_n^T(\mathcal{G}') = \sum_{q\in\mathcal{V}} \tau_{\varphi_0(q)}.$$

Therefore we have

$$\sum_{q\in\mathcal{V}} \varphi(\tau_{\varphi_0(q)}) = \sum_{q\in\mathcal{V}} \tau_q.$$

Since $\varphi(\tau_{\varphi_0(q)}) = \eta_q\tau_q$ for some $\eta_q \in \{\pm 1\}$, we also have

$$\sum_{q\in\mathcal{V}} \eta_q\tau_q = \sum_{q\in\mathcal{V}} \tau_q.$$

By taking the value of both hand side at p, one finds that $\eta_p = 1$. Thus we have $\varphi(\tau_{\varphi_0(p)}) = \tau_p$ as desired.

(ii) Recall that $H_T^*(\mathcal{G})$ has the structure of an $H^*(BT)$-algebra:

$$H^*(BT) \to H_T^*(\mathcal{G}).$$

On the other hand, Lemma 2.1 defines a ring homomorphism

$$\mathbf{Z}[y_1,\dots,y_n]^{\mathfrak{S}_n} \to H_T^*(\mathcal{G}).$$

Since the tensor product

$$H^*(BT)\otimes_{\mathbf{Z}} \mathbf{Z}[y_1,\dots,y_n]^{\mathfrak{S}_n}$$

is isomorphic to $H^*(BT)[y_1,\dots,y_n]^{\mathfrak{S}_n}$, by combining above two homomorphisms, we have a ring homomorphism

$$H^*(BT)[y_1,\dots,y_n]^{\mathfrak{S}_n} \to H_T^*(\mathcal{G})$$

which makes graph equivariant cohomology $H_T^*(\mathcal{G})$ into a graded $H^*(BT)[y_1,\dots,y_n]^{\mathfrak{S}_n}$-algebra.

Since $\mathbf{Z}[y_1,\dots,y_n]^{\mathfrak{S}_n}$ is generated by elementary symmetric polynomials, an isomorphism

$$\varphi : H_T^*(\mathcal{G}') \to H_T^*(\mathcal{G})$$

of graded $H^*(BT)$-algebras preserves equivariant total Chern classes if and only if it respects the $H^*(BT)[y_1,\dots,y_n]^{\mathfrak{S}_n}$-algebra structures.

(iii) In Theorem 3.2, the assignment $g\mapsto g^*$ is not injective in general: let us consider the GKM graph $\mathcal{G}'$ in Remark 1.5. Then, the non-trivial geometric automorphism

$$\varphi_0 : \{p,q\}\to\{p,q\},\quad \varphi_1 : \{e,\overline{e}\}\to\{e,\overline{e}\}$$

of $\mathcal{G}'$ defined by

$$\varphi_0(p) = q, \varphi_0(q) = p, \varphi_1(e) = \overline{e}, \varphi_1(\overline{e}) = e$$

induces the identity map of $H_T^*(\mathcal{G}')$.

4. Toric case. We next give a proof of Theorem 1.10. To this end we work with equivariant Thom classes associated with facets of torus graphs. Let $\mathcal{T} = (\mathcal{V},\mathcal{E},\alpha)$ be a torus graph of degree n. Recall from [6] that a ***k*-face** of $\mathcal{T}$ is a k-valent connected subgraph $F = (\mathcal{V}_F,\mathcal{E}_F,\alpha|_{\mathcal{E}_F})$ of $\mathcal{T}$ which is closed under the parallel transport of $\mathcal{T}$ (i.e., if $e,e'\in\mathcal{E}_F$ and $i(e) = i(e')$, then $\mathcal{P}_e(e')\in\mathcal{E}_F$). An $(n-1)$-face is called a **facet** of $\mathcal{T}$.

For each k-face F, we define **equivariant Thom class** $\tau_F\in H_T^{2(n-k)}(\mathcal{T})$ as follows:

$$\tau_F(p) := \begin{cases} \prod_{e\in\mathcal{E}_p\setminus(\mathcal{E}_F)_p} \alpha(e) & \text{if } p\in\mathcal{V}_F, \\ 0 & \text{if } p\notin\mathcal{V}_F. \end{cases}$$

Thanks to [6, Lemma 4.1] τ_F gives a well-defined element in $H_T^{2(n-k)}(\mathcal{T})$.

Let $F_1,\dots,F_m$ be the collection of facets of $\mathcal{T}$. We denote by $\tau_1,\dots,\tau_m$ the corresponding equivariant Thom classes. By [6, Lemma 5.2, Theorem 5.5] $\tau_1,\dots,\tau_m$ forms a $\mathbf{Z}$-basis of $H_T^2(\mathcal{T})$. In particular the number of facets of $\mathcal{T}$ is equal to $\mathrm{rk}_{\mathbf{Z}} H_T^2(\mathcal{T})$.

Theorem 4.1. *Let $\mathcal{T}$ and $\mathcal{T}'$ be torus graphs. If there exists an isomorphism $\varphi : H_T^*(\mathcal{T}')\to H_T^*(\mathcal{T})$ of graded $H^*(BT)$-algebras which preserves equivariant 1-st Chern classes, then there exists a geometric isomorphism $g : \mathcal{T}\to\mathcal{T}'$ which induces the isomorphism φ.*

Proof. We take an isomorphism

$$g = (\varphi_0,\varphi_1) : \mathcal{T}\to\mathcal{T}'$$

as in Lemma 3.1. Then the argument in the proof of Theorem 3.2 shows that g induces the isomorphism φ. In addition one has

$$c_1^T(\mathcal{T})(p) = c_1^T(\mathcal{T}')(\varphi_0(p))$$

as in the proof of Theorem 3.2.

Note that equivariant total Chern classes can be expressed as

$$c_1^T(\mathcal{T}) = \sum_{i=1}^m \tau_i.$$

This follows from [6, Lemma 4.2] and the definition of $c_1^T(\mathcal{T})$. Thus we have

$$\sum_{i=1}^{m} \tau_i(p) = \sum_{i=1}^{m} \tau'_i(\varphi_0(p)).$$

In terms of axial function, this equality is rephrased as follows:

$$\sum_{e \in \mathcal{E}_p} \alpha(e) = \sum_{e \in \mathcal{E}_p} \alpha'(\varphi_1(e)).$$

Since both

$$\{\alpha(e) \mid e \in \mathcal{E}_p\}$$

and

$$\{\alpha'(\varphi_1(e)) = \pm\alpha(e) \mid e \in \mathcal{E}_p\}$$

are linearly independent over $\mathbf{Z}$, we have

$$\alpha'(\varphi_1(e)) = \alpha(e)$$

as desired.

The proof is now complete. □

Corollary 4.2. *Let $\mathcal{T}$ and $\mathcal{T}'$ be torus graphs. Then, there exists a graded $H^*(BT)$-algebra isomorphism $H_T^*(\mathcal{T}) \to H_T^*(\mathcal{T}')$ preserving equivariant 1-st Chern classes if and only if $\mathcal{T}$ and $\mathcal{T}'$ are geometrically isomorphic as GKM graphs.*

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