PBW theoretic approach to the module category of quantum affine algebras

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Abstract: Let $U_q'(\mathfrak{g})$ be a quantum affine algebra of untwisted affine ADE type and let $\mathscr{C}_{\mathfrak{g}}^0$ be Hernandez-Leclerc's category. For a duality datum $\mathcal{D}$ in $\mathscr{C}_{\mathfrak{g}}^0$, we denote by $\mathcal{F}_{\mathcal{D}}$ the quantum affine Weyl-Schur duality functor. We give a sufficient condition for a duality datum $\mathcal{D}$ to provide the functor $\mathcal{F}_{\mathcal{D}}$ sending simple modules to simple modules. Moreover, under the same condition, the functor $\mathcal{F}_{\mathcal{D}}$ has compatibility with the new invariants introduced by the authors. Then we introduce the notion of cuspidal modules in $\mathscr{C}_{\mathfrak{g}}^0$, and show that all simple modules in $\mathscr{C}_{\mathfrak{g}}^0$ can be constructed as the heads of ordered tensor products of cuspidal modules. We next state that the ordered tensor products of cuspidal modules have the unitriangularity property.

Key words: Cuspidal modules; quantum affine Weyl-Schur duality; Hernandez-Leclerc category; quantum affine algebra; quiver Hecke algebra.

1. Introduction. Let q be an indeterminate and let $\mathscr{C}_{\mathfrak{g}}$ be the category of finite-dimensional integrable modules over a quantum affine algebra $U_q'(\mathfrak{g})$. The category $\mathscr{C}_{\mathfrak{g}}$ occupies an important position in the representation theory of quantum affine algebras because of its rich structure. The simple modules in $\mathscr{C}_{\mathfrak{g}}$ are parameterized using n-tuples of polynomials with constant term 1 (called *Drinfeld polynomials*), which was proved in [2–4] for the untwisted cases and in [5] for the twisted cases. Any simple module can be obtained as the head of ordered tensor product of *fundamental modules*, which was shown in [1,13,27]. A geometric approach to simple modules was also studied in [21,27].

Let $\mathfrak{g}_0$ be a finite-dimensional simple Lie algebra of ADE type and $\mathfrak{g}$ the untwisted affine Lie algebra associated with $\mathfrak{g}_0$. Hernandez and Leclerc defined the full subcategory $\mathscr{C}_{\mathfrak{g}}^0$ of $\mathscr{C}_{\mathfrak{g}}$ such that all simple subquotients of its objects appear in tensor products of certain fundamental representations ([7]). Because any simple module in $\mathscr{C}_{\mathfrak{g}}$ can be obtained as a tensor product of suitable parameter shifts of simple modules in $\mathscr{C}_{\mathfrak{g}}^0$, the category $\mathscr{C}_{\mathfrak{g}}^0$ contains an essential information of $\mathscr{C}_{\mathfrak{g}}$. For each Q-*data* $\mathcal{Q} = (Q, \phi)$ of $\mathfrak{g}_0$, which is a pair of a Dynkin quiver Q of $\mathfrak{g}_0$ and its height function ϕ, Hernandez and Leclerc defined a monoidal subcategory $\mathscr{C}_{\mathcal{Q}}$ of $\mathscr{C}_{\mathfrak{g}}^0$ and proved that its complexified Grothendieck ring $\mathbf{C} \otimes_{\mathbf{Z}} K(\mathscr{C}_{\mathcal{Q}})$ is isomorphic to the coordinate ring $\mathbf{C}[N]$ of the unipotent group associated with $\mathfrak{g}_0$ ([8]). Under this isomorphism, the set of isomorphism classes of simple modules in $\mathscr{C}_{\mathcal{Q}}$ corresponds to the upper global base of $\mathbf{C}[N]$.

The *quantum affine Weyl-Schur duality* was introduced in [10]. This duality tells us that, for a *duality datum* $\mathcal{D} = \{L_i\}_{i\in J} \subset \mathscr{C}_{\mathfrak{g}}$, there exists a monoidal functor $\mathcal{F}_{\mathcal{D}}$ from the finite-dimensional graded module category R-gmod of the *quiver Hecke algebra* R ([19,25]) determined by $\mathcal{D}$ to the category $\mathscr{C}_{\mathfrak{g}}$. In general, it is hard to find conditions for $\mathcal{D}$ to provide the functor $\mathcal{F}_{\mathcal{D}}$ with good properties. But, to each choice of Q-data $\mathcal{Q}$, we can associate a quantum affine Weyl-Schur duality functor $\mathcal{F}_{\mathcal{Q}}$ with good properties ([6,9]):

$$\mathcal{F}_{\mathcal{Q}}\colon R_{\mathfrak{g}_0}\text{-gmod} \longrightarrow \mathscr{C}_{\mathcal{Q}} \subset \mathscr{C}_{\mathfrak{g}}^0,$$

where $R_{\mathfrak{g}_0}$ is the symmetric quiver Hecke algebra associated with $\mathfrak{g}_0$. This functor $\mathcal{F}_{\mathcal{Q}}$ sends simple

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modules of $R_{\mathfrak{g}_0}$-gmod to simple modules of $\mathscr{C}_{\mathcal{Q}}$ and gives an isomorphism at the Grothendieck ring level.

One of the main results of this paper is to describe a sufficient condition for a duality datum $\mathcal{D}$ to provide the functor $\mathcal{F}_{\mathcal{D}}$ with good properties. Let $U_q'(\mathfrak{g})$ be an *arbitrary* quantum affine algebra. Let $\mathcal{D} = \{L_i\}_{i\in J} \subset \mathscr{C}_{\mathfrak{g}}$ be a duality datum associated with a simply-laced finite Cartan matrix C and let R_{C} be the symmetric quiver Hecke algebra associated with C. If L_i are root modules (see (2.1)) and $\mathcal{D}$ satisfies condition (2.2) below, we say that $\mathcal{D}$ is a *strong duality datum*. We show that, for a strong duality datum $\mathcal{D}$, the duality functor $\mathcal{F}_{\mathcal{D}}$ sends simple modules to simple modules and it induces an injective ring homomorphism from $K_{q=1}(R_{\mathsf{C}}\text{-gmod})$ to $K(\mathscr{C}_{\mathfrak{g}})$ (see Theorem 2.1). Moreover, the duality functor is compatible with the new invariants:

$$(1.1)\qquad \begin{aligned} &\Lambda(M,N) = \Lambda(\mathcal{F}_{\mathcal{D}}(M), \mathcal{F}_{\mathcal{D}}(N)),\\ &\mathfrak{d}(M,N) = \mathfrak{d}(\mathcal{F}_{\mathcal{D}}(M), \mathcal{F}_{\mathcal{D}}(N)),\\ &(\mathrm{wt}(M), \mathrm{wt}(N)) = -\Lambda^{\infty}(\mathcal{F}_{\mathcal{D}}(M), \mathcal{F}_{\mathcal{D}}(N)) \end{aligned}$$

for any simple modules M, N in R_{C}-gmod. Here, Λ, $\mathfrak{d}$ and Λ^{∞} are new invariants for pairs of objects of $\mathscr{C}_{\mathfrak{g}}$ introduced in [14]. These invariants are quantum affine algebra analogues of the invariants (with the same notations) for the quiver Hecke algebras (see [10,12]). Note that the block decompositions for $\mathscr{C}_{\mathfrak{g}}$ and $\mathscr{C}_{\mathfrak{g}}^0$ were given by using the new invariant Λ^{∞} in [15].

The other main theorem of this paper is to construct all simple modules in $\mathscr{C}_{\mathfrak{g}}^0$ as the heads of ordered tensor products of *cuspidal modules*, which can be understood as a generalization of the simple module construction using ordered tensor products of fundamental modules ([4,22]).

Suppose that $\mathfrak{g}_0$ is a finite-dimensional simple Lie algebra of *ADE* type and $\mathfrak{g}$ is the untwisted affine Lie algebra associated with $\mathfrak{g}_0$. Let $\mathcal{Q}$ be a Q-data of $\mathfrak{g}_0$ and let $\mathcal{D}_{\mathcal{Q}}$ be the duality datum induced by $\mathcal{Q}$. Let $\mathcal{F}_{\mathcal{Q}} := \mathcal{F}_{\mathcal{D}_{\mathcal{Q}}}$ be the duality functor associated with $\mathcal{D}_{\mathcal{Q}}$. Since $\mathcal{D}_{\mathcal{Q}}$ is a strong duality datum, $\mathcal{F}_{\mathcal{Q}}$ sends simples to simples and preserves the new invariants, i.e., (1.1) holds.

Let w_0 be the longest element of the Weyl group W_0 of $\mathfrak{g}_0$, and ℓ the length of w_0. We choose an arbitrary reduced expression $\underline{w_0}$ of w_0. We define the cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbf{Z}} \subset \mathscr{C}_{\mathfrak{g}}^0$ to be the simple $U_q'(\mathfrak{g})$-modules given by

(a) $\mathsf{S}_k = \mathcal{F}_{\mathcal{Q}}(V_k)$ for any $k = 1, \ldots, \ell$,

(b) $\mathsf{S}_{k+\ell} = \mathscr{D}(\mathsf{S}_k)$ for any $k \in \mathbf{Z}$,

where $\{V_k\}_{k=1,\ldots,\ell} \subset R_{\mathfrak{g}_0}$-gmod are the cuspidal modules associated with $\underline{w_0}$, and $\mathscr{D}(X)$ and $\mathscr{D}^{-1}(X)$ denote the right dual and the left dual of a module $X \in \mathscr{C}_{\mathfrak{g}}$, respectively. Note that the cuspidal module V_k corresponds to the *dual PBW vectors* associated with $\underline{w_0}$ under the categorification using quiver Hecke algebras.

For any $\mathbf{a} = (a_k)_{k\in\mathbf{Z}} \in \mathbf{Z} := \mathbf{Z}_{\geq 0}^{\oplus\mathbf{Z}}$, we define the ordered tensor product by

$$\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a}) := \cdots \otimes \mathsf{S}_1^{\otimes a_1} \otimes \mathsf{S}_0^{\otimes a_0} \otimes \mathsf{S}_{-1}^{\otimes a_{-1}} \otimes \cdots.$$

We prove that the head $\mathrm{hd}(\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a}))$ of the ordered tensor product $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$ is simple and the simple module $\mathrm{hd}(\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a}))$ appears only once in $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$. Moreover, for any simple module $V \in \mathscr{C}_{\mathfrak{g}}^0$, there exists a unique $\mathbf{a} \in \mathbf{Z}$ such that V is isomorphic to the head $\mathrm{hd}(\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a}))$. Such an $\mathbf{a}$ is denoted by $\mathbf{a}_{Q,\underline{w_0}}(V)$.

Thus, setting $\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a}) := \mathrm{hd}(\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a}))$, the set $\{\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{Z}\}$ is a complete and irredundant set of simple modules of $\mathscr{C}_{\mathfrak{g}}^0$ up to isomorphisms (see Theorem 3.4). We prove further that, if V is a simple subquotient of $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$ which is not isomorphic to $\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a})$, then

$$\mathbf{a}_{Q,\underline{w_0}}(V) \prec \mathbf{a},$$

where $\prec$ is the bi-lexicographic order on $\mathbf{Z}$ (see (3.2)). Thus the family of modules $\{\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})\}_{\mathbf{a}\in\mathbf{Z}}$ has the unitriangularity property with respect to $\prec$ (see Theorem 3.5).

We can generalize the above results to an arbitrary quantum affine algebra $U_q'(\mathfrak{g})$ including *twisted type* by using certain strong duality datum in $\mathscr{C}_{\mathfrak{g}}$ (see Remark 3.6).

This paper is an announcement whose details will appear elsewhere.

2. Quantum affine Weyl-Schur duality. Let $\mathbf{k}$ be the algebraic closure of the subfield $\mathbf{Q}(q)$ in the algebraically closed field $\bigcup_{m>0} \mathbf{C}((q^{1/m}))$. Let $U_q'(\mathfrak{g})$ be the quantum affine algebra over the base field $\mathbf{k}$ associated with an affine Cartan matrix $\mathsf{A} = (a_{i,j})_{i,j\in I}$, and set $\mathscr{C}_{\mathfrak{g}}$ to be the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules.

Let $\mathsf{C} = (c_{i,j})_{i,j\in J}$ be a simply-laced finite Cartan matrix and R_{C} the symmetric quiver Hecke algebra associated with C. We denote by $K(R_{\mathsf{C}}\text{-gmod})$ the Grothendieck ring of the category R_{C}-gmod of finite-dimensional graded R_{C}-modules. Note that $K(R_{\mathsf{C}}\text{-gmod})$ is isomorphic to the quantum unipotent coordinate ring $A_q(\mathsf{C})_{\mathbf{Z}[q,q^{-1}]}$ ([19,25]) and the set of isomorphism classes of simple R_{C}-modules corresponds to the upper global basis of $A_q(\mathsf{C})_{\mathbf{Z}[q,q^{-1}]}$ ([26,28]). We set $K_{q=1}(R_{\mathsf{C}}\text{-gmod})$ to be the specialization of $K(R_{\mathsf{C}}\text{-gmod})$ at $q=1$.

We freely use new invariants Λ, Λ^∞ and $\mathfrak{d}$ for pairs of modules in $\mathscr{C}_\mathfrak{g}$ introduced in [14].

A simple $U_q'(\mathfrak{g})$-module M is *real* if $M \otimes M$ is simple. A *root module* is a real simple module L such that

$$\mathfrak{d}(L, \mathscr{D}^k(L)) = \delta(k = \pm 1) \quad \text{for any } k \in \mathbf{Z}. \tag{2.1}$$

Let $\mathcal{D} := \{L_i\}_{i\in J} \subset \mathscr{C}_\mathfrak{g}$ be a family of simple modules of $\mathscr{C}_\mathfrak{g}$. The family $\mathcal{D}$ is called a *duality datum* associated with C if it satisfies the following

(a) for each $i \in J$, L_i is a real simple module,
(b) for any $i, j \in J$ with $i \neq j$, $\mathfrak{d}(L_i, L_j) = -c_{i,j}$.

Then one can construct a monoidal functor

$$\mathcal{F}_\mathcal{D}: R_{\mathsf{C}}\text{-gmod} \longrightarrow \mathscr{C}_\mathfrak{g}$$

using the duality datum $\mathcal{D}$ (see [10,17]). Moreover, $\mathcal{F}_\mathcal{D}$ is an exact functor. The functor $\mathcal{F}_\mathcal{D}$ is called a *quantum affine Weyl-Schur duality* or shortly a *duality functor*.

A *strong duality datum* is a duality datum $\mathcal{D} = \{L_i\}_{i\in J}$ such that all L_i are root modules and

$$\mathfrak{d}(L_i, \mathscr{D}^k(L_j)) = -\delta(k = 0)c_{i,j} \tag{2.2}$$

for any $k \in \mathbf{Z}$ and $i, j \in J$ such that $i \neq j$.

Theorem 2.1. *Let $\mathcal{D} = \{L_i\}_{i\in J}$ be a* strong *duality datum associated with a simply-laced finite Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j\in J}$. Then we obtain the following*

(i) *The duality functor $\mathcal{F}_\mathcal{D}$ sends simple modules to simple modules.*
(ii) *The duality functor $\mathcal{F}_\mathcal{D}$ induces an injective ring homomorphism from $K_{q=1}(R_{\mathsf{C}}\text{-gmod})$ to $K(\mathscr{C}_\mathfrak{g})$.*

The duality functor also has compatibility with the new invariants.

Theorem 2.2. *Let $\mathcal{D} = \{L_i\}_{i\in J}$ be a strong duality datum. Then, for any simple modules M, N in R_{C}-gmod, we have*

(i) $\Lambda(M,N) = \Lambda(\mathcal{F}_\mathcal{D}(M), \mathcal{F}_\mathcal{D}(N))$,
(ii) $\mathfrak{d}(M,N) = \mathfrak{d}(\mathcal{F}_\mathcal{D}(M), \mathcal{F}_\mathcal{D}(N))$,
(iii) $(\mathrm{wt}\, M, \mathrm{wt}\, N) = -\Lambda^\infty(\mathcal{F}_\mathcal{D}(M), \mathcal{F}_\mathcal{D}(N))$,
(iv) $\mathfrak{d}(\mathscr{D}^k\mathcal{F}_\mathcal{D}(M), \mathcal{F}_\mathcal{D}(N)) = 0$ *for any* $k \neq 0, \pm 1$,
(v) $\widetilde{\Lambda}(M,N) = \mathfrak{d}(\mathscr{D}\mathcal{F}_\mathcal{D}(M), \mathcal{F}_\mathcal{D}(N)) = \mathfrak{d}(\mathcal{F}_\mathcal{D}(M), \mathscr{D}^{-1}\mathcal{F}_\mathcal{D}(N))$.

3. PBW theoretic approach. Let $\mathfrak{g}_0$ be a finite-dimensional simple Lie algebra of ADE type, and $\mathfrak{g}$ the *untwisted* affine Kac-Moody algebra associated with $\mathfrak{g}_0$. We denote by $\mathsf{A} = (a_{i,j})_{i,j\in I}$ the affine Cartan matrix of $\mathfrak{g}$ and by $I_0 \subset I$ the index set corresponding to $\mathfrak{g}_0$. For each $i \in I_0$ and $c \in \mathbf{k}^\times$, let $V(\varpi_i)_c$ be the i-th *fundamental representation.*

Following [7], we denote by $\mathscr{C}_\mathfrak{g}^0$ the smallest full subcategory of the category $\mathscr{C}_\mathfrak{g}$ such that

(a) it contains $\{V(\varpi_i)_{(-q)^p} \mid i \in I_0,\ p \equiv d(1,i) \bmod 2\}$,
(b) it is stable under taking subquotients, extensions, and tensor products.

Here 1 is an arbitrarily chosen element of I_0 and $d(1,i)$ is the distance between 1 and i in the Dynkin diagram of $\mathfrak{g}_0$.

A *Q-data* $\mathcal{Q} = (Q, \phi_Q)$ is a pair of a Dynkin quiver Q of $\mathfrak{g}_0$ and a function $\phi_Q(i): I_0 \to \mathbf{Z}$ such that $\phi_Q(1) \in 2\mathbf{Z}$ and $\phi_Q(i) = \phi_Q(j) + 1$ for any arrow $i \to j$ in Q. The function ϕ_Q is called a *height function* of Q. For any Q-data $\mathcal{Q}$, we denote by $\mathscr{C}_\mathcal{Q}$ the full monoidal subcategory of $\mathscr{C}_\mathfrak{g}^0$ introduced in [8]. For $i \in I_0$, let L_i be the fundamental representation in $\mathscr{C}_\mathcal{Q}$ corresponding to the simple root α_i in the Auslander-Reiten quiver of $\mathcal{Q}$. Then $\{L_i\}_{i\in I_0}$ forms a strong duality datum and it induces a duality functor

$$\mathcal{F}_\mathcal{Q}: R_{\mathfrak{g}_0}\text{-gmod} \longrightarrow \mathscr{C}_\mathcal{Q} \subset \mathscr{C}_\mathfrak{g}^0$$

([6,9]). Here $R_{\mathfrak{g}_0}$ is the symmetric quiver Hecke algebra associated with $\mathfrak{g}_0$. Note that the functor $\mathcal{F}_\mathcal{Q}$ is an equivalence of categories between $R_{\mathfrak{g}_0}$-gmod and $\mathscr{C}_\mathcal{Q}$ after forgetting grading ([6]).

Hence Theorem 2.2 implies the following result.

Theorem 3.1. *Let $\mathcal{Q}$ be a Q-data of $\mathfrak{g}_0$. For any simple modules M and N in $R_{\mathfrak{g}_0}$-gmod, we have*

(i) $\Lambda(M,N) = \Lambda(\mathcal{F}_\mathcal{Q}(M), \mathcal{F}_\mathcal{Q}(N))$,
(ii) $(\mathrm{wt}(M), \mathrm{wt}(N)) = -\Lambda^\infty(\mathcal{F}_\mathcal{Q}(M), \mathcal{F}_\mathcal{Q}(N))$.

Let Δ_0^+ be the set of positive roots of $\mathfrak{g}_0$ and let $\mathsf{W}_0 = \langle r_i \mid i \in I_0 \rangle$ be the Weyl group associated with $\mathfrak{g}_0$, where r_i is the i-th reflection. Let w_0 be the longest element of W_0, and ℓ denotes the length of w_0. We choose an arbitrary reduced expression $\underline{w_0} = r_{i_1} r_{i_2} \cdots r_{i_\ell}$ of the longest element w_0 of W_0. (We do not assume that $\underline{w_0}$ is Q-adapted.) Then we have $\Delta_0^+ = \{\beta_k \mid k = 1, \ldots, \ell\}$, where

$$\beta_k := r_{i_1} \cdots r_{i_{k-1}}(\alpha_{i_k}) \quad \text{for } k = 1, \ldots, \ell.$$

The reduced expression $\underline{w_0}$ gives the convex order on Δ_0^+ defined by $\beta_a > \beta_b$ for any $a > b$, and provides the PBW vectors $\{E(\beta_k)\}_{k=1,\dots,\ell}$ of the negative half $U^-_{\mathbf{Z}[q,q^{-1}]}(\mathfrak{g}_0)$ and its dual vectors $\{E^*(\beta_k)\}_{k=1,\dots,\ell}$ of the quantum unipotent coordinate ring $A_q(\mathfrak{n}_0)_{\mathbf{Z}[q,q^{-1}]}$, where $\mathfrak{n}_0$ is the maximal nilpotent subalgebra of $\mathfrak{g}_0$.

Let $\{V_k\}_{k=1,\dots,\ell} \subset R_{\mathfrak{g}_0}$-gmod be the *cuspidal modules* associated with the reduced expression $\underline{w_0}$ ([18,20]). Under the isomorphism between $K(R_{\mathfrak{g}_0}\text{-gmod})$ and $A_q(\mathfrak{n}_0)_{\mathbf{Z}[q,q^{-1}]}$, the cuspidal module V_k corresponds to the dual PBW vector $E^*(\beta_k)$ for $k = 1,\dots,\ell$. It is known that the modules

$$\{\mathrm{hd}(V_\ell^{\circ a_\ell} \circ \cdots \circ V_1^{\circ a_1}) \mid (a_1,\dots,a_\ell) \in \mathbf{Z}^\ell_{\geq 0}\}$$

gives a complete set of pairwise non-isomorphic simple graded $R_{\mathfrak{g}_0}$-modules ([18,20]). Here $\circ$ denotes the convolution product in $R_{\mathfrak{g}_0}$-gmod.

We now introduce the notion of cuspidal modules for quantum affine algebras.

Definition 3.2. We define a sequence of simple $U_q'(\mathfrak{g})$-modules $\{\mathsf{S}_k\}_{k\in\mathbf{Z}} \subset \mathscr{C}^0_{\mathfrak{g}}$ as follows:
(a) $\mathsf{S}_k = \mathcal{F}_{\mathcal{Q}}(V_k)$ for any $k = 1,\dots,\ell$, and we extend its definition to all $k \in \mathbf{Z}$ by
(b) $\mathsf{S}_{k+\ell} = \mathscr{D}(\mathsf{S}_k)$ for any $k \in \mathbf{Z}$.

The modules S_k $(k \in \mathbf{Z})$ are called the *cuspidal modules* corresponding to the Q-data $\mathcal{Q}$ and the reduced expression $\underline{w_0}$.

It is known that the cuspidal modules are fundamental representations if $\underline{w_0}$ is Q-adapted. However, they may not be fundamental in general.

We have the following property.

Proposition 3.3.
(i) S_k *is a root module for any* $k \in \mathbf{Z}$, *i.e.*, S_k *is a real simple module satisfying* (2.1).
(ii) $\mathfrak{d}(\mathsf{S}_j, \mathscr{D}(\mathsf{S}_k)) = 0$ *for any* $j < k$.

We define

$$\text{(3.1)}\quad \mathbf{Z} := \mathbf{Z}^{\oplus\mathbf{Z}}_{\geq 0} = \left\{(a_k)_{k\in\mathbf{Z}} \in \mathbf{Z}^{\mathbf{Z}}_{\geq 0} \mid \sum_{k\in\mathbf{Z}} a_k < \infty\right\}.$$

We denote by $\prec$ the bi-lexicographic order on $\mathbf{Z}$, i.e., for any $\mathbf{a} = (a_k)_{k\in\mathbf{Z}}$ and $\mathbf{a}' = (a'_k)_{k\in Z}$ in $\mathbf{Z}$, $\mathbf{a} \prec \mathbf{a}'$ if and only if the following conditions hold:

$$\text{(3.2)}\quad \begin{cases} \text{(a)} & \text{there exists } r \in \mathbf{Z} \text{ such that } a_k = a'_k \\ & \text{for any } k < r \text{ and } a_r < a'_r, \\ \text{(b)} & \text{there exists } s \in \mathbf{Z} \text{ such that } a_k = a'_k \\ & \text{for any } k > s \text{ and } a_s < a'_s. \end{cases}$$

For $\mathbf{a} = (a_k)_{k\in\mathbf{Z}} \in \mathbf{Z}$, we define

$$\begin{aligned}\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a}) &:= \bigotimes_{k=+\infty}^{-\infty} \mathsf{S}_k^{\otimes a_k} \\ &= \cdots \otimes \mathsf{S}_2^{\otimes a_2} \otimes \mathsf{S}_1^{\otimes a_1} \otimes \mathsf{S}_0^{\otimes a_0} \otimes \mathsf{S}_{-1}^{\otimes a_{-1}} \otimes \cdots.\end{aligned}$$

Here $\mathsf{P}_{Q,\underline{w_0}}(0)$ should be understood as the trivial module $\mathbf{1}$. We call the modules $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$ *standard modules* with respect to the cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbf{Z}}$.

Theorem 3.4.
(i) *For any* $\mathbf{a} \in \mathbf{Z}$, *the head of* $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$ *is simple. We denote the head by*

$$\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a}) := \mathrm{hd}(\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})).$$

(ii) *For any simple module* $V \in \mathscr{C}^0_{\mathfrak{g}}$, *there exists a unique* $\mathbf{a} \in \mathbf{Z}$ *such that*

$$V \simeq \mathsf{V}_{Q,\underline{w_0}}(\mathbf{a}).$$

Therefore the set $\{\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{Z}\}$ *is a complete and irredundant set of simple modules of* $\mathscr{C}^0_{\mathfrak{g}}$ *up to isomorphisms.*

Indeed, (i) follows from Proposition 3.3 ([14]).

The element $\mathbf{a} \in \mathbf{Z}$ associated with a simple module V in Theorem 3.4 (ii) is called the *cuspidal decomposition* of V with respect to the cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbf{Z}}$, and it is denoted by

$$\text{(3.3)}\quad \mathbf{a}_{Q,\underline{w_0}}(V) := \mathbf{a}.$$

Theorem 3.5. *Let* $\mathbf{a}$ *be an element of* $\mathbf{Z}$. *Then we have the following*
(i) *The simple module* $\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a})$ *appears only once in* $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$.
(ii) *If* V *is a simple subquotient of* $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$ *which is not isomorphic to* $\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a})$, *then we have*

$$\mathbf{a}_{Q,\underline{w_0}}(V) \prec \mathbf{a}.$$

(iii) *In the Grothendieck ring, we have*

$$[\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})] = [\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a})] + \sum_{\mathbf{a}'\prec\mathbf{a}} c(\mathbf{a}')[\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a}')],$$

for some $c(\mathbf{a}') \in \mathbf{Z}_{\geq 0}$.

Remark 3.6. In [11,16,23,24], the categories $\mathscr{C}_{\mathcal{Q}} \subset \mathscr{C}^0_{\mathfrak{g}}$ for untwisted non simply-laced affine types and twisted affine types are introduced, and a strong duality datum $\mathcal{D} = \{L_i\}_{i\in J}$ for each $\mathscr{C}_{\mathcal{Q}}$ is given. Then we can obtain the duality functor

$$\mathcal{F}_{\mathcal{D}}\colon R_{\mathsf{C}}\text{-gmod} \longrightarrow \mathscr{C}_{\mathcal{Q}} \subset \mathscr{C}^0_{\mathfrak{g}},$$

where C denotes the simply-laced finite Cartan matrix *determined* by the affine type of $\mathfrak{g}$ (cf. [15]).

Using the functors $\mathcal{F}_{\mathcal{D}}$, we can extend all the results in Section 3 to an arbitrary quantum affine algebra with an arbitrary choice of $\underline{w_0}$ in type of C.

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References

[1] T. Akasaka and M. Kashiwara, Finite-dimensional representations of quantum affine algebras, Publ. Res. Inst. Math. Sci. **33** (1997), no. 5, 839–867.

[2] V. Chari and A. Pressley, Quantum affine algebras, Comm. Math. Phys. **142** (1991), no. 2, 261–283.

[3] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.

[4] V. Chari and A. Pressley, Quantum affine algebras and their representations, in *Representations of groups* (*Banff, AB, 1994*), 59–78, CMS Conf. Proc., 16, Amer. Math. Soc., Providence, RI, 1995.

[5] V. Chari and A. Pressley, Twisted quantum affine algebras, Comm. Math. Phys. **196** (1998), no. 2, 461–476.

[6] R. Fujita, Geometric realization of Dynkin quiver type quantum affine Schur-Weyl duality, Int. Math. Res. Not. IMRN **2020**, no. 22, 8353–8386.

[7] D. Hernandez and B. Leclerc, Cluster algebras and quantum affine algebras, Duke Math. J. **154** (2010), no. 2, 265–341.

[8] D. Hernandez and B. Leclerc, Quantum Grothendieck rings and derived Hall algebras, J. Reine Angew. Math. **701** (2015), 77–126.

[9] S.-J. Kang, M. Kashiwara and M. Kim, Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras, II, Duke Math. J. **164** (2015), no. 8, 1549–1602.

[10] S.-J. Kang, M. Kashiwara and M. Kim, Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras, Invent. Math. **211** (2018), no. 2, 591–685.

[11] S.-J. Kang, M. Kashiwara, M. Kim and S.-j. Oh, Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras IV, Selecta Math. (N.S.) **22** (2016), no. 4, 1987–2015.

[12] S.-J. Kang, M. Kashiwara, M. Kim and S.-j. Oh, Monoidal categorification of cluster algebras, J. Amer. Math. Soc. **31** (2018), no. 2, 349–426.

[13] M. Kashiwara, On level-zero representations of quantized affine algebras, Duke Math. J. **112** (2002), no. 1, 117–175.

[14] M. Kashiwara, M. Kim, S.-j. Oh and E. Park, Monoidal categorification and quantum affine algebras, Compos. Math. **156** (2020), no. 5, 1039–1077.

[15] M. Kashiwara, M. Kim, S.-j. Oh and E. Park, Simply-laced root systems arising from quantum affine algebras, arXiv:2003.03265v2.

[16] M. Kashiwara and S.-j. Oh, Categorical relations between Langlands dual quantum affine algebras: doubly laced types, J. Algebraic Combin. **49** (2019), no. 4, 401–435.

[17] M. Kashiwara and E. Park, Affinizations and R-matrices for quiver Hecke algebras, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 5, 1161–1193.

[18] S. Kato, Poincaré-Birkhoff-Witt bases and Khovanov-Lauda-Rouquier algebras, Duke Math. J. **163** (2014), no. 3, 619–663.

[19] M. Khovanov and A. D. Lauda, A diagrammatic approach to categorification of quantum groups. I, Represent. Theory **13** (2009), 309–347.

[20] P. J. McNamara, Finite dimensional representations of Khovanov-Lauda-Rouquier algebras I: Finite type, J. Reine Angew. Math. **707** (2015), 103–124.

[21] H. Nakajima, Quiver varieties and finite-dimensional representations of quantum affine algebras, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238.

[22] H. Nakajima, Quiver varieties and t-analogs of q-characters of quantum affine algebras, Ann. of Math. (2) **160** (2004), no. 3, 1057–1097.

[23] S.-j. Oh and T. Scrimshaw, Categorical relations between Langlands dual quantum affine algebras: exceptional cases, Comm. Math. Phys. **368** (2019), no. 1, 295–367.

[24] U. R. Suh and S.-j. Oh, Twisted and folded Auslander-Reiten quivers and applications to the representation theory of quantum affine algebras, J. Algebra **535** (2019), 53–132.

[25] R. Rouquier, 2-Kac-Moody algebras, arXiv:0812.5023v1.

[26] R. Rouquier, Quiver Hecke algebras and 2-Lie algebras, Algebra Colloq. **19** (2012), no. 2, 359–410.

[27] M. Varagnolo and E. Vasserot, Standard modules of quantum affine algebras, Duke Math. J. **111** (2002), no. 3, 509–533.

[28] M. Varagnolo and E. Vasserot, Canonical bases and KLR-algebras, J. Reine Angew. Math. **659** (2011), 67–100.