Maximal L^1-regularity for parabolic boundary value problems with inhomogeneous data in the half-space

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Abstract: End-point maximal L^1-regularity for the parabolic initial-boundary value problem is considered in the half-space. For the inhomogeneous boundary data of both the Dirichlet and the Neumann type, maximal L^1-regularity for the initial-boundary value problem of parabolic equation is established in time end-point case upon the Besov space as well as the optimal trace estimates. We derive the almost orthogonal properties between the boundary potentials of the Dirichlet and the Neumann boundary data and the Littlewood-Paley dyadic decomposition of unity.

Key words: Maximal L^1-regularity; end-point estimate; initial-boundary value problem; the Dirichlet problem; the Neumann problem.

1. Introduction. In this article, we are concerned with maximal L^1-regularity for the initial-boundary value problem with inhomogeneous data of the parabolic equation in the half-space $\mathbf{R}^n_+$.

Let X be a proper Banach space and A be a closed linear operator in X with a dense domain $\mathcal{D}(A)$. For an initial data $u_0 \in X$ and an external force $f \in L^\rho(0,T;X)$ $(1 \le \rho \le \infty)$, let u be a solution to the abstract Cauchy problem:

$$\frac{d}{dt}u + Au = f,\ t>0, \quad u(0) = u_0. \tag{1}$$

Then A has maximal L^ρ-regularity if there exists a unique solution u of (1) such that $\frac{d}{dt}u,\ Au \in L^\rho(0,T;X)$ satisfy the estimate

$$\left\|\frac{d}{dt}u\right\|_{L^\rho(0,T;X)} + \|Au\|_{L^\rho(0,T;X)} \le C\Big(\|u_0\|_{(X,\mathcal{D}(A))_{1-\frac{1}{\rho},\rho}} + \|f\|_{L^\rho(0,T;X)}\Big),$$

under the restriction $u_0 \in (X,\mathcal{D}(A))_{1-\frac{1}{\rho},\rho}$, where $(X,\mathcal{D}(A))_{1-\frac{1}{\rho},\rho}$ denotes the real interpolation space between X and $\mathcal{D}(A)$, and C is a positive constant independent of u_0 and f. Maximal regularity for parabolic equations was first developed by Ladyzhenskaya-Solonnikov-Ural'tseva [12], then studied by Da Prato-Grisvard [6] and Dore-Venni [9], in the general framework on Banach spaces X that satisfy *the unconditional martingale differences* (called as UMD), well established especially by Amann [1], [2], Denk-Hieber-Prüss [7], [8], Weis [21]. On the other hand, maximal regularity on non-UMD Banach spaces, for instance non-reflexive Banach space such as L^1 or L^∞ requires independent arguments. For example, we have explicitly proved maximal regularity on the homogenous Banach spaces in [13]. Maximal L^1-regularity for the Cauchy problem is also shown by [4], [10], [11], [14] in various non-UMD spaces.

2. The Dirichlet boundary condition case. Let $I = (0,T)$ with $0 < T \le \infty$. Let u be a solution of the initial-boundary value problem of the second-order parabolic equation with variable coefficients and the inhomogeneous Dirichlet boundary condition in the half-space $\mathbf{R}^n_+ = \{x = (x', x_n);\ x' \in \mathbf{R}^{n-1}, x_n > 0\}$:

$$\begin{cases} \partial_t u - \displaystyle\sum_{i,j=1}^n a_{ij}(t,x)\partial_i\partial_j u = f, & t \in I,\ x \in \mathbf{R}^n_+, \\ u|_{x_n=0} = g, & t \in I,\ x' \in \mathbf{R}^{n-1}, \\ u|_{t=0} = u_0, & x \in \mathbf{R}^n_+, \end{cases} \tag{2}$$

where ∂_t and $\partial_i := \partial_{x_i}$ are partial derivatives with respect to t and x_i, $u = u(t,x)$ denotes the unknown function, $u_0 = u_0(x)$, $f = f(t,x)$ and $g = g(t,x')$ are given initial, external force and boundary data,

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respectively. We assume that the coefficients $\{a_{ij}(t,x)\}_{1\le i,j\le n}$ satisfy the following conditions:

Assumption: For $1\le i,j\le n$,

(a) $a_{ij}(t,x)=\delta_{ij}+b_{ij}(t,x)$,

(b) $b_{ij}(t,x)=b_{ji}(t,x)$,

(c) there exists a constant $c>0$ such that for any $\xi\in\mathbf{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \ge c|\xi|^2, \quad (t,x)\in I\times\mathbf{R}^n_+,$$

(d) $b_{ij}\in BUC(\mathbf{R}_+;\dot{B}^{\frac{n}{q}}_{q,1}(\mathbf{R}^n_+))$ for some $1\le q<\infty$, where BUC denotes a set of all bounded uniformly continuous functions.

For $1\le\rho\le\infty$ and a Banach space X, we denote the Bochner-Lebesgue space $L^\rho(I;X)$ and the inhomogeneous and homogeneous Bochner-Sobolev spaces as $W^{1,\rho}(I;X)$, $\dot{W}^{1,\rho}(I;X)$, respectively.

In this context, the following results have been obtained by Weidemaier [20] and Denk-Hieber-Prüss [8].

Proposition 1 (Dirichlet boundary condition, [8], [20]). *Let $1<p,\rho<\infty$, $I=(0,T)$ with $T<\infty$. Assume that the coefficients $\{a_{ij}\}_{1\le i,j\le n}$ satisfy Assumption. Then the problem* (2) *admits a unique solution*

$$u\in W^{1,\rho}(I;L^p(\mathbf{R}^n_+))\cap L^\rho(I;W^{2,p}(\mathbf{R}^n_+))$$

if and only if

$$f\in L^\rho(I;L^p(\mathbf{R}^n_+)),\quad u_0\in B^{2(1-1/\rho)}_{p,\rho}(\mathbf{R}^n_+),$$
$$g\in F^{1-1/2p}_{\rho,p}(I;L^p(\mathbf{R}^{n-1}))\cap L^\rho(I;B^{2-1/p}_{p,p}(\mathbf{R}^{n-1})),$$

if $1-1/(2p)>1/\rho$, then

$$u_0(x',x_n)|_{x_n=0}=g(t,x')|_{t=0}.$$

Besides there exists a constant $C_T>0$ depending on n, p, ρ, T such that the solution u is subject to the inequality:

$$\begin{aligned}&\|\partial_t u\|_{L^\rho(I;L^p(\mathbf{R}^n_+))}+\|\nabla^2 u\|_{L^\rho(I;L^p(\mathbf{R}^n_+))}\\ &\quad\le C_T(\|u_0\|_{B^{2(1-1/2p)}_{p,\rho}(\mathbf{R}^n_+)}+\|f\|_{L^\rho(I;L^p(\mathbf{R}^n_+))}\\ &\qquad+\|g\|_{F^{1-1/2p}_{\rho,p}(I;L^p(\mathbf{R}^{n-1}))}+\|g\|_{L^\rho(I;B^{2-1/p}_{p,p}(\mathbf{R}^{n-1}))}),\end{aligned}$$

where $|\nabla^2 u|=(\sum_{1\le i,j\le n}|\partial_i\partial_j u|^2)^{1/2}$, $L^\rho(I;X)$ denotes the ρ-th powered Bochner-Lebesgue space upon a Banach space X and $B^{2-1/p}_{p,p}(\mathbf{R}^{n-1})$ and $F^{1-1/2p}_{\rho,p}(I;X)$ denote the interpolation spaces of the Besov and Lizorkin-Triebel type, respectively.

Weidemaier [18] first obtained a trace theorem for functions in anisotropic Sobolev spaces. Then he extended his result to a boundary trace of a solution of parabolic equations in the Bochner space and obtained the optimal trace estimate ([19], [20]) with introducing the Lizorkin-Triebel space in the time variable. In the proof of the results, he employed an solution formula with respect to the time variable, and the proof is involved the maximal function for a test function and hence he imposed a restriction $\frac{n}{2}<p\le\rho<\infty$ for exponents. Denk-Hieber-Prüss [8] obtained the necessary and sufficient condition of maximal regularity to the initial-boundary value problem including higher order elliptic operators in a domain $\Omega\subset\mathbf{R}^n$ with a compact boundary. The proof in [8] is based on the vector valued version of Mikhlin's Fourier multiplier theorem, and accordingly the result is restricted in the cases $1<p,\rho<\infty$. Their result is essentially a time local estimate because the boundary conditions are limited in the inhomogeneous real interpolation spaces.

In this article, we show *time global maximal L^1-regularity* for the parabolic initial-boundary value problem (2). Danchin-Mucha [5] obtained time global maximal L^1-regularity result for the heat equation in the half-space with boundary data $g=0$. Therefore, it is essential to treat non-zero boundary data g. We show the results and an outline of the proof in this article. The full proof is given in a paper elsewhere.

Since the global estimate requires the base space for spatial variable x in the homogeneous Besov space, we introduce the homogeneous Besov space over $\mathbf{R}^n_+$ (see for details Peetre [15], Triebel [16]).

Definition (The Besov and Lizorkin-Triebel spaces). Let $s\in\mathbf{R}$, $1\le p,\sigma\le\infty$. Let $\{\phi_j\}_{j\in\mathbf{Z}}$ be the Littlewood-Paley dyadic decomposition of unity for $x\in\mathbf{R}^n$, namely $\hat\phi(\xi)$ is the Fourier transform of a smooth radial function $\phi(x)$ with $\hat\phi(\xi)\ge 0$, $\operatorname{supp}\hat\phi\subset\{\xi\in\mathbf{R}^n\mid 2^{-1}\le|\xi|\le 2\}$, and

$$\hat\phi_j(\xi)=\hat\phi(2^{-j}\xi),\quad \sum_{j\in\mathbf{Z}}\hat\phi_j(\xi)=1$$

for all $\xi\neq 0$. For $s\in\mathbf{R}$ and $1\le p,\sigma\le\infty$, $\dot{B}^s_{p,\sigma}(\mathbf{R}^n)$ be the homogeneous Besov space with norm

$$\|\tilde f\|_{\dot B^s_{p,\sigma}}\equiv\begin{cases}\left(\displaystyle\sum_{j\in\mathbf{Z}}2^{s\sigma j}\|\phi_j*\tilde f\|_p^\sigma\right)^{1/\sigma}, & 1\le\sigma<\infty,\\ \displaystyle\sup_{j\in\mathbf{Z}}2^{sj}\|\phi_j*\tilde f\|_p^\sigma, & \sigma=\infty.\end{cases}$$

$B^s_{p,\sigma}(\mathbf{R}^n)$ denotes the inhomogeneous Besov space with a usual norm. For $s \in \mathbf{R}$, $1 \le p < \infty$ and $1 \le \sigma \le \infty$, $\dot{F}^s_{p,\sigma}(\mathbf{R}^n)$ be the homogeneous Lizorkin-Triebel space with norm

$$\|\tilde{f}\|_{\dot{F}^s_{p,\sigma}} \equiv \begin{cases} \left\| \left(\sum_{j\in\mathbf{Z}} 2^{s\sigma j} |\phi_j * \tilde{f}(\cdot)|^\sigma \right)^{1/\sigma} \right\|_p, & 1 \le \sigma < \infty, \\ \left\| \sup_{j\in\mathbf{Z}} 2^{sj} |\phi_j * \tilde{f}(\cdot)| \right\|_p, & \sigma = \infty. \end{cases}$$

We define the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbf{R}^n_+)$ as the set of all measurable functions f in $\mathbf{R}^n_+$ satisfying

$$\|f\|_{\dot{B}^s_{p,\sigma}(\mathbf{R}^n_+)} \equiv \inf\left\{ \|\tilde{f}\|_{\dot{B}^s_{p,\sigma}(\mathbf{R}^n)} < \infty; \ \tilde{f} = \left\{ \begin{array}{ll} f(x', x_n) & (x_n > 0) \\ \text{any extension} & (x_n < 0) \end{array} \right\} \right\}.$$

Definition (The Bochner-Lizorkin-Triebel spaces). Let $s \in \mathbf{R}$, $1 \le p, \sigma \le \infty$ and $X(\mathbf{R}^n_+)$ be a Banach space on $\mathbf{R}^n_+$ with the norm $\|\cdot\|_X$. Let $\{\psi_k\}_{k\in\mathbf{Z}}$ be the Littlewood-Paley dyadic decomposition of unity for $t \in \mathbf{R}$. For $s \in \mathbf{R}$ and $1 \le p < \infty$, $\dot{F}^s_{p,\sigma}(\mathbf{R}; X)$ be the Bochner-Lizorkin-Triebel space with norm

$$\|\tilde{f}\|_{\dot{F}^s_{p,\sigma}(\mathbf{R};X)} \equiv \begin{cases} \left\| \left(\sum_{k\in\mathbf{Z}} 2^{s\sigma k} \|\psi_k * \tilde{f}(t,\cdot)\|_X^\sigma \right)^{1/\sigma} \right\|_{L^p(\mathbf{R}_t)}, & 1 \le \sigma < \infty, \\ \left\| \sup_{k\in\mathbf{Z}} 2^{sk} \|\psi_j * \tilde{f}(t,\cdot)\|_X \right\|_{L^p(\mathbf{R}_t)}, & \sigma = \infty. \end{cases}$$

Analogously above, we define the Bochner-Lizorkin-Triebel spaces $\dot{F}^s_{p,\sigma}(I; X)$ as the set of all measurable functions f on X satisfying

$$\|f\|_{\dot{F}^s_{p,\sigma}(I;X)} \equiv \inf\left\{ \|\tilde{f}\|_{\dot{F}^s_{p,\sigma}(\mathbf{R};X)} < \infty; \ \tilde{f} = \left\{ \begin{array}{ll} f(t, x) & (t \in I) \\ \text{any extension} & (t \in \mathbf{R} \setminus I) \end{array} \right\} \right\}.$$

We note that all the spaces of homogeneous type are understood as the Banach spaces by introducing the quotient spaces identifying all polynomial differences.

The next theorems are our main results maximal L^1-regularity for the problem (2).

Theorem 2 (The Dirichlet, constant coefficients case)**.** *Let $1 < p < \infty$ and assume that the coefficients $\{a_{ij}\}_{1\le i,j\le n}$ satisfy Assumption and be constants. Then the problem* (2) *admits a unique solution*

$$u \in \dot{W}^{1,1}(\mathbf{R}_+; \dot{B}^0_{p,1}(\mathbf{R}^n_+)) \cap L^1(\mathbf{R}_+; \dot{B}^2_{p,1}(\mathbf{R}^n_+))$$

if and only if the external, initial and boundary data in (2) *satisfies*

$$f \in L^1(\mathbf{R}_+; \dot{B}^0_{p,1}(\mathbf{R}^n_+)), \quad u_0 \in \dot{B}^0_{p,1}(\mathbf{R}^n_+),$$
$$g \in \dot{F}^{1-1/2p}_{1,1}(\mathbf{R}_+; \dot{B}^0_{p,1}(\mathbf{R}^{n-1})) \cap L^1(\mathbf{R}_+; \dot{B}^{2-1/p}_{p,1}(\mathbf{R}^{n-1})),$$

respectively. Besides the solution u satisfies the following estimate for some constant $C > 0$ depending only on p and n

$$\begin{aligned} &\|\partial_t u\|_{L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} + \|\nabla^2 u\|_{L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} \\ &\quad \le C(\|u_0\|_{\dot{B}^0_{p,1}(\mathbf{R}^n_+)} + \|f\|_{L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} \\ &\qquad + \|g\|_{\dot{F}^{1-1/2p}_{1,1}(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^{n-1}))} \\ &\qquad + \|g\|_{L^1(\mathbf{R}_+;\dot{B}^{2-1/p}_{p,1}(\mathbf{R}^{n-1}))}). \end{aligned}$$

Theorem 3 (The Dirichlet, variable coefficients case)**.** *Let $1 < p < \infty$. Assume that the coefficients $\{a_{ij}\}_{1\le i,j\le n}$ satisfy Assumption. For any $T < \infty$, let $I = (0, T)$. Then the problem* (2) *admits a unique solution*

$$u \in \dot{W}^{1,1}(I; \dot{B}^0_{p,1}(\mathbf{R}^n_+)) \cap L^1(I; \dot{B}^2_{p,1}(\mathbf{R}^n_+))$$

if and only if the external force, the initial data and the boundary data satisfy

$$f \in L^1(I; \dot{B}^0_{p,1}(\mathbf{R}^n_+)), \quad u_0 \in \dot{B}^0_{p,1}(\mathbf{R}^n_+),$$
$$g \in \dot{F}^{1-1/2p}_{1,1}(I; \dot{B}^0_{p,1}(\mathbf{R}^{n-1})) \cap L^1(I; \dot{B}^{2-1/p}_{p,1}(\mathbf{R}^{n-1})),$$

respectively. Besides the solution u satisfies the following estimate for some constant $C_M = C_M(n, p, q, a_{ij}) > 0$

$$\begin{aligned} &\|\partial_t u\|_{L^1(I;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} + \|\nabla^2 u\|_{L^1(I;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} \\ &\quad \le C_M \int_0^T e^{\mu(T-s)} \|f(s)\|_{\dot{B}^0_{p,1}} ds \\ &\qquad + C_M\Big(1 + \max_{1\le ij\le n} \|b_{ij}\|_{L^\infty(I;\dot{B}^{n/q}_{q,1})}\Big) e^{\mu T} \\ &\qquad\quad \times \big(\|u_0\|_{\dot{B}^0_{p,1}} + \|g\|_{\dot{F}^{1-1/2p}_{1,1}(I;\dot{B}^0_{p,1}(\mathbf{R}^{n-1}))} \\ &\qquad + \|g\|_{L^1(I;\dot{B}^{2-1/p}_{p,1}(\mathbf{R}^{n-1}))}\big), \end{aligned}$$

where $\mu = C_M^2 \log(1 + C_M)$.

Remarks.

(a) Since $1 - \frac{1}{2p} < 1$ for all $1 < p < \infty$, the compatibility condition

$$u_0(x',x_n)|_{x_n=0} = g(t,x')|_{t=0}$$

holds in the trace sense and does not necessarily hold in the point-wise sense.

(b) In Theorems 2 and 3, it holds that

$$u \in C_b([0,\infty); \dot{B}^0_{p,1}(\mathbf{R}^n_+)),$$
$$u \in C_b([0,T); \dot{B}^0_{p,1}(\mathbf{R}^n_+)),$$

respectively, where C_b is the set of all continuous and bounded functions.

(c) In Theorem 2, a similar estimate for the finite time interval $I=(0,T)$ with $T<\infty$ is also available. In such a case the class of u_0 is relaxed into $B^0_{p,1}(\mathbf{R}^n_+) \supset \dot{B}^0_{p,1}(\mathbf{R}^n_+)$ and the bound C can be estimated as $C = O(\log T)$.

3. Outline of the proof. First we decompose the problem (2) into the homogeneous and inhomogeneous problems on the boundary conditions. Let

$$u(t,x) = u_1(t,x)|_{x_n>0} + u_2(t,x) + u_3(t,x),$$

where each function solves the following decomposed equations:

$$(3)\begin{cases} \partial_t u_1 - \Delta u_1 = 0, \quad t>0, & x\in\mathbf{R}^n, \\ u_1|_{t=0} = \begin{cases} u_0(x',x_n), & x_n>0, \\ -u_0(x',-x_n), & x_n<0, \end{cases} & x\in\mathbf{R}^n, \end{cases}$$

$$(4)\begin{cases} \partial_t u_2 - \Delta u_2 = 0, & t>0, \quad x\in\mathbf{R}^n_+, \\ u_2|_{x_n=0} = g - u_1|_{x_n=0} \equiv h, & t>0, \quad x'\in\mathbf{R}^{n-1}, \\ u_2|_{t=0} = 0, & x\in\mathbf{R}^n_+, \end{cases}$$

$$(5)\begin{cases} \partial_t u_3 - \sum_{1\le i,j\le n} a_{ij}(t,x)\partial_i\partial_j u_3 \\ \quad = f + \sum_{1\le i,j\le n} b_{ij}(t,x)\partial_i\partial_j(u_1+u_2) \\ \qquad\qquad t>0, \quad x\in\mathbf{R}^n_+, \\ u_3|_{x_n=0} = 0, \qquad t>0, \quad x'\in\mathbf{R}^{n-1}, \\ u_3|_{t=0} = 0, \qquad x\in\mathbf{R}^n_+. \end{cases}$$

The problem (3) is considered by Chemin [3] and Danchin [4]. The third problem (5) is a simple extension by the odd reflection of the Cauchy problem in the whole space which is analyzed by Danchin [4] and authors [14]. Therefore we only consider the initial-boundary problem of the heat equation in the half-space (4) with non-zero Dirichlet boundary data h. Applying the Laplace transform with respect to time and the partial Fourier transform with respect to x', we obtain

$$\begin{cases} (\lambda + |\xi'|^2 - \partial_n^2)\mathcal{L}u(\lambda,\xi',x_n) = 0, \\ \mathcal{L}u(\lambda,\xi',0) = \mathcal{L}h(\lambda,\xi'). \end{cases}$$

Then it follows that

$$\mathcal{L}u(\lambda,\xi',x_n) = \mathcal{L}h(\lambda,\xi')e^{-\sqrt{\lambda+|\xi'|^2}x_n},$$

where we use a branch such that $\mathrm{Re}\sqrt{\lambda+|\xi'|^2}>0$. Hence the solution of (4) is expressed by

$$u(t,x) = \frac{c_{n-1}}{2\pi i} \times \int_\Gamma e^{\lambda t}\int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'}\mathcal{L}h(\lambda,\xi')e^{-\sqrt{\lambda+|\xi'|^2}x_n}d\xi' d\lambda,$$

where $c_{n-1} = (2\pi)^{-(n-1)/2}$. By taking a limit procedure, we take the path of integral $\Gamma = i\mathbf{R}$ and $\lambda = i\tau$. We set $x_n =: \eta$ like a parameter and

$$\Psi_D(t,x',\eta) = c_n\int_{\mathbf{R}}\int_{\mathbf{R}^{n-1}} e^{it\tau+ix'\cdot\xi'}\tau\eta^2 e^{-\sqrt{i\tau+|\xi'|^2}\eta}d\xi' d\tau.$$

Then it holds that

$$\begin{aligned}\partial_t u(t,x',\eta) &= \mathcal{L}^{-1}[i\tau e^{-\sqrt{i\tau+|\xi'|^2}\eta}] * h(t,x') \\ &= \eta^{-2}\Psi_D(t,x',\eta) * h(t,x').\end{aligned}$$

Here we introduce $\{\Phi_m(x)\}_{m\in\mathbf{Z}}$ as a Littlewood-Palay dyadic decomposition for $x=(x',x_n)$. For $m\in\mathbf{Z}$, let $\widehat{\zeta_m}(\xi_n)$ be a smooth function such that $\widehat{\zeta_m}(\xi_n) = 1$ for $0\le|\xi_n|\le 2^m$ and $=0$ for $2^{m+1}\le|\xi_n|$, and set

$$\widehat{\Phi_m}(\xi) := \widehat{\phi_m}(|\xi'|)\otimes\widehat{\zeta_{m-1}}(\xi_n) + \widehat{\zeta_m}(|\xi'|)\otimes\widehat{\phi_m}(\xi_n).$$

Then

$$\sum_{m\in\mathbf{Z}}\widehat{\Phi_m}(\xi) = 1, \quad \forall\xi=(\xi',\xi_n)\in\mathbf{R}^n_+\setminus\{0\}.$$

Using $\{\Phi_m(x)\}_{m\in\mathbf{Z}}$, we obtain

$$\|\partial_t u(t)\|_{\dot{B}^0_{p,1}(\mathbf{R}^n)} = \sum_{m\in\mathbf{Z}}\|\Phi_m * (\eta^{-2}\Psi_D * h)(t)\|_{L^p(\mathbf{R}^n)}.$$

Next we split the boundary data h as

$$\begin{aligned} h(t,x') &= \sum_{k\in\mathbf{Z}}\sum_{m\in\mathbf{Z}}\psi_k(t)\underset{(t)}{*}\phi_m(x')\underset{(x')}{*}h(t,x') \\ &= \sum_{k\in\mathbf{Z}}\Bigg(\sum_{2m\le k}+\sum_{2m>k}\Bigg)\psi_k(t)\underset{(t)}{*}\phi_m(x')\underset{(x')}{*}h(t,x') \\ &= h_T(t,x') + h_S(t,x'),\end{aligned}$$

where $\psi_k(t)$ and $\phi_m(x')$ are the Littlewood-Palay decomposition of unity for t and x' respectively. The term $h_T(t,x')$ where summation with respect to m runs $2m \le k$ is time dominated and the term $h_S(t,x')$ where summation with respect to m runs $2m > k$ is space dominated. The following almost orthogonality is the key to our argument.

Lemma 4 (Almost orthogonality)**.** *For any $\eta \in I_{-\ell} \equiv [2^{-\ell}, 2^{-\ell+1})$, there exists $C_n > 0$ such that*

$$\left\| \Psi_D(t,x',\eta) \underset{(t,x')}{*} \psi_k(t) \underset{(t)}{*} \phi_m(x') \right\|_{L^1_{x'}} \le \begin{cases} C_n 2^{k-2\ell}(1+2^{(n+2)(k-2\ell)}) e^{-2^{\frac{1}{2}(k-2\ell)}} \dfrac{2^k}{2^k\langle t\rangle^2}, \\ \hfill 2m \le k, \\ C_n 2^{2(m-\ell)}(1+2^{(n+2)2(m-\ell)}) e^{-2^{(m-\ell)}} \dfrac{2^k}{2^k\langle t\rangle^2}, \\ \hfill 2m \ge k. \end{cases}$$

Applying Lemma 4, we obtain

$$\left\| \sum_{m\in\mathbf{Z}} \|\Phi_m * (\eta^{-2}\Psi_D * h_T)\|_{L^p(\mathbf{R}^n)} \right\|_{L^1(\mathbf{R})} \le C\|h\|_{\dot{F}^{1-1/2p}_{1,1}(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^{n-1}))},$$

$$\left\| \sum_{m\in\mathbf{Z}} \|\Phi_m * (\eta^{-2}\Psi_D * h_S)\|_{L^p(\mathbf{R}^n)} \right\|_{L^1(\mathbf{R})} \le C\|h\|_{L^1(\mathbf{R}_+;\dot{B}^{2-1/p}_{p,1}(\mathbf{R}^{n-1}))},$$

where constant $C > 0$ is independent of h.

Finally, the necessity is obtained by the following optimality trace.

Lemma 5 (The Dirichlet boundary trace)**.** *Let $1 < p < \infty$. There exists a constant $C > 0$ depending only on p and n such that for all functions $u = u(t,x',x_n) \in \dot{W}^{1,1}(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+)) \cap L^1(\mathbf{R}_+;\dot{B}^2_{p,1}(\mathbf{R}^n_+))$ with $u(0,x) = 0$*

$$\sup_{x_n\in\mathbf{R}_+} (\|u(\cdot,\cdot,x_n)\|_{\dot{F}^{1-1/2p}_{1,1}(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^{n-1}))} + \|u(\cdot,\cdot,x_n)\|_{L^1(\mathbf{R}_+;\dot{B}^{2-1/p}_{p,1}(\mathbf{R}^{n-1}))}) \le C(\|\partial_t u\|_{L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} + \|\nabla^2 u\|_{L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+))}).$$

4. The Neumann boundary condition case. Similar to the initial-boundary value problem with the Dirichlet condition, we consider the initial-boundary value problem of the Neumann boundary condition:

$$(6)\quad \begin{cases} \partial_t u - \displaystyle\sum_{i,j=1}^n a_{ij}(t,x)\partial_i\partial_j u = f, & t\in I,\ x\in\mathbf{R}^n_+, \\ \partial_n u|_{x_n=0} = g, & t\in I,\ x'\in\mathbf{R}^{n-1}, \\ u|_{t=0} = u_0, & x\in\mathbf{R}^n_+. \end{cases}$$

For the case of Neumann boundary problem (6), we obtain maximal L^1-maximal regularity results in the similar manner as in the proof of Dirichlet boundary condition case.

Theorem 6 (The Neumann, constant coefficients case)**.** *Let $1 < p < \infty$ and assume that the coefficients $\{a_{ij}\}_{1\le i,j\le n}$ satisfy Assumption and be constants. Then the problem (6) admits a unique solution*

$$u \in \dot{W}^{1,1}(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+)) \cap L^1(\mathbf{R}_+;\dot{B}^2_{p,1}(\mathbf{R}^n_+))$$

if and only if

$$f \in L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+)),\quad u_0 \in \dot{B}^0_{p,1}(\mathbf{R}^n_+),$$
$$g \in \dot{F}^{1/2-1/2p}_{1,1}(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^{n-1})) \cap L^1(\mathbf{R}_+;\dot{B}^{1-1/p}_{p,1}(\mathbf{R}^{n-1}))$$

respectively. Besides the solution u satisfies the following estimate for some constant $C > 0$ depending only on p and n

$$\|\partial_t u\|_{L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} + \|\nabla^2 u\|_{L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} \le C\Big(\|u_0\|_{\dot{B}^0_{p,1}(\mathbf{R}^n_+)} + \|f\|_{L^1(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} + \|g\|_{\dot{F}^{1/2-1/2p}_{1,1}(\mathbf{R}_+;\dot{B}^0_{p,1}(\mathbf{R}^{n-1}))} + \|g\|_{L^1(\mathbf{R}_+;\dot{B}^{1-1/p}_{p,1}(\mathbf{R}^{n-1}))}\Big).$$

Theorem 7 (The Neumann, variable coefficients case)**.** *Let $1 < p < \infty$ and assume that the coefficients $\{a_{ij}\}_{1\le i,j\le n}$ satisfy Assumption. For any $T < \infty$, let $I = (0,T)$. Then the problem (6) admits a unique solution*

$$u \in \dot{W}^{1,1}(I;\dot{B}^0_{p,1}(\mathbf{R}^n_+)) \cap L^1(I;\dot{B}^2_{p,1}(\mathbf{R}^n_+))$$

if and only if the external force, the initial data and the boundary data satisfy

$$f \in L^1(I;\dot{B}^0_{p,1}(\mathbf{R}^n_+)),\quad u_0 \in \dot{B}^0_{p,1}(\mathbf{R}^n_+),$$
$$g \in \dot{F}^{1/2-1/2p}_{1,1}(I;\dot{B}^0_{p,1}(\mathbf{R}^{n-1})) \cap L^1(I;\dot{B}^{1-1/p}_{p,1}(\mathbf{R}^{n-1})),$$

respectively. Besides the solution u satisfies the following estimate for some constant $C_M = C_M(n,p,q,a_{ij}) > 0$

$$\begin{aligned}
&\|\partial_t u\|_{L^1(I;\dot{B}^0_{p,1}(\mathbf{R}^n_+))} + \|\nabla^2 u\|_{L^1(I;\dot{B}^0_{p,1}(\mathbf{R}^n_+))}\\
&\quad\le C_M \int_0^T e^{\mu(T-s)}\|f(s)\|_{\dot{B}^0_{p,1}}\,ds\\
&\qquad + C_M\Big(1+\max_{1\le ij\le n}\|b_{ij}\|_{L^\infty(I;\dot{B}^{n/q}_{q,1})}\Big)e^{\mu T}\\
&\qquad\times\Big(\|u_0\|_{\dot{B}^0_{p,1}} + \|g\|_{\dot{F}^{1/2-1/2p}_{1,1}(I;\dot{B}^0_{p,1}(\mathbf{R}^{n-1}))}\\
&\qquad + \|g\|_{L^1(I;\dot{B}^{1-1/p}_{p,1}(\mathbf{R}^{n-1}))}\Big),
\end{aligned}$$

where $\mu = C_M^2\log(1+C_M)$.

5. Further problems. It is possible to extend the results to more general domain Ω such as a bounded domain with smooth boundary if $\dot{B}^s_{p,\sigma}(\Omega)$ is the restriction of $\dot{B}^s_{p,\sigma}(\mathbf{R}^n)$ (cf. Triebel [17]).

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