Quasi traveling waves with quenching in a reaction-diffusion equation in the presence of negative powers nonlinearity

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Abstract: The quasi traveling waves with quenching of $u_t = u_{xx} + (1-u)^{-\alpha}$ for $\alpha \in 2\mathbf{N}$ are considered. The existence of quasi traveling waves with quenching and their quenching rates are studied by applying the Poincaré compactification.

Key words: Quasi traveling wave with quenching; Poincaré compactification.

1. Introduction. In this paper, we consider the quasi traveling waves with quenching (see Def. 3) of the following equation:

$$(1.1)\qquad u_t = u_{xx} + \frac{1}{(1-u)^{\alpha}},\ t>0,\ x\in\mathbf{R},\ \alpha\in\mathbf{N}.$$

First, we state the definition of "quenching" for the solution of (1.1).

Definition 1. We say that a solution $u(t,x)$ of (1.1) quenches at point (T, x_0) if

$$\lim_{t\uparrow T} u(t,x_0) = 1,\quad \lim_{t\uparrow T}\left|\frac{\partial u}{\partial t}(t,x_0)\right| = \infty.$$

In order to consider the traveling waves of (1.1), we introduce the following change of variables:

$$\phi(\xi) = 1 - u(t,x),\quad \xi = x - ct,\quad c>0.$$

We then seek the solution $\phi(\xi)$ of the following equation:

$$(1.2)\qquad c\phi' = -\phi'' + \phi^{-\alpha},\quad \xi\in\mathbf{R},\quad ' = \frac{d}{d\xi},$$

or

$$(1.3)\qquad \begin{cases} \phi' = \psi, \\ \psi' = -c\psi + \phi^{-\alpha}. \end{cases}$$

Second, we state the definition of quasi traveling waves and quasi traveling waves with quenching as follows:

Definition 2. We say that a function $u(t,x) \equiv 1 - \phi(\xi)$ is a quasi traveling wave of (1.1) if the function $\phi(\xi)$ is a solution of (1.2) on a finite interval or semi-infinite interval.

Definition 3. We say that a function $u(t,x) \equiv 1 - \phi(\xi)$ is a quasi traveling wave with quenching of (1.1) if the function $u(t,x)$ is a quasi traveling wave of (1.1) on a finite interval (resp. semi-infinite interval) such that $|\phi'|$ becomes infinite (namely, ϕ reaches 0) at both ends of the interval (resp. finite end point of the semi-infinite interval). More precisely, we have the following three cases:

(I) The function $\phi(\xi)$ is a solution of (1.2) on a semi-infinite interval $(-\infty,\xi_*)$ ($\phi(\xi)\in C^2(-\infty,\xi_*)\cap C^0(-\infty,\xi_*]$, $|\xi_*|<\infty$), and satisfies

$$\lim_{\xi\to\xi_*-0}\phi(\xi) = 0 \quad\text{and}\quad \lim_{\xi\to\xi_*-0}|\psi(\xi)| = \infty.$$

(II) The function $\phi(\xi)$ is a solution of (1.2) on a semi-infinite interval $(\xi_*,+\infty)$ ($\phi(\xi)\in C^2(\xi_*,+\infty)\cap C^0[\xi_*,+\infty)$, $|\xi_*|<\infty$), and satisfies

$$\lim_{\xi\to\xi_*+0}\phi(\xi) = 0 \quad\text{and}\quad \lim_{\xi\to\xi_*+0}|\psi(\xi)| = \infty.$$

(III) The function $\phi(\xi)$ is a solution of (1.2) on a finite interval (ξ_-,ξ_+) ($\phi(\xi)\in C^2(\xi_-,\xi_+)\cap C^0[\xi_-,\xi_+]$, $-\infty<\xi_-<\xi_+<+\infty$), and satisfies the followings

$$\lim_{\xi\to\xi_+-0}\phi(\xi) = 0,\qquad \lim_{\xi\to\xi_-+0}\phi(\xi) = 0,$$
$$\lim_{\xi\to\xi_+-0}|\psi(\xi)| = \infty,\quad \lim_{\xi\to\xi_-+0}|\psi(\xi)| = \infty.$$

Remark 1. The definition of quasi traveling wave (with quenching) implies that it satisfies (1.1) only on semi-infinite interval or finite interval. In this paper, we do not discuss the behavior of the solutions of (1.3) after ψ becomes infinity. It is

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necessary that more detailed (and hard) analysis in order to study the solutions after quenching (outside of the interval on that $\phi(\xi)$ satisfies (1.2)), and so we leave it open here.

In this setting, Matsue [4] proved the following theorem.

Theorem 1 (Theorem 4.21 of [4]). *Assume that $\alpha > 1$ with $\alpha \in \mathbf{N}$. Then, the quasi traveling waves with quenching for* (1.1) *are, if exist, characterized by trajectories whose initial data are on the stable manifold of a equilibrium at infinity $(\phi, \psi) = (0, +\infty)$ of* (1.3)*. The quenching rates, namely, the extinction rate of ϕ and blow-up rate of ψ, are*

$$\begin{cases} \phi(\xi) \sim C(\xi_* - \xi)^{\frac{2\alpha}{2\alpha^2-\alpha+1}}, \\ \psi(\xi) \sim C(\xi_* - \xi)^{\frac{1-\alpha}{2\alpha^2-\alpha+1}} \end{cases} \quad \text{as} \quad \xi \to \xi_*$$

with $|\xi_*| < \infty$ and $C \neq 0$.

The proof is given in [4].

Remark 2. We can obtain the equilibria at infinity (of (1.3)) not only $(\phi, \psi) = (0, +\infty)$ but also other equilibria by applying the Poincaré compactification (see [4] and Sec. 3 for the details).

We note that the existence of the quasi traveling waves has not been proved yet. In this paper, we give the proof of the existence of them by considering the restricted case of $\alpha \in 2\mathbf{N}$. The proof is based on Poincaré compactification (that is also used to prove Theorem 1 in [4]) and basic theory of dynamical systems. We then state the main theorem of this paper (see also Figure 1).

Theorem 2. *Assume that $\alpha \in 2\mathbf{N}$. Then, the Eq.* (1.1) *possesses a family of "quasi traveling waves with quenching on a finite interval". Moreover, each quasi traveling wave with quenching $u(t, x) = 1 - \phi(\xi)$ (which satisfies* (1.2) *on a finite interval (ξ_-, ξ_+)) satisfies the followings*

- $\begin{cases} \lim\limits_{\xi \to \xi_+ - 0} \phi(\xi) = 0, & \lim\limits_{\xi \to \xi_- + 0} \phi(\xi) = 0, \\ \lim\limits_{\xi \to \xi_+ - 0} \psi(\xi) = \infty, & \lim\limits_{\xi \to \xi_- + 0} \psi(\xi) = -\infty. \end{cases}$
- *$\phi(\xi) < 0$ holds for $\xi \in (\xi_-, \xi_+)$.*
- *There exists a constant $\xi_* \in (\xi_-, \xi_+)$ such that the following holds: $\psi(\xi) < 0$ for $\xi \in (\xi_-, \xi_*)$, $\psi(\xi_*) = 0$ and $\psi(\xi) > 0$ for $\xi \in (\xi_*, \xi_+)$.*

In addition, quenching rates are

$$\begin{cases} \phi(\xi) \sim -C(\xi_+ - \xi)^{\frac{2\alpha}{2\alpha^2-\alpha+1}} \\ \psi(\xi) \sim C(\xi_+ - \xi)^{\frac{1-\alpha}{2\alpha^2-\alpha+1}} \end{cases} \quad \text{as} \quad \xi \to \xi_+ - 0$$

and

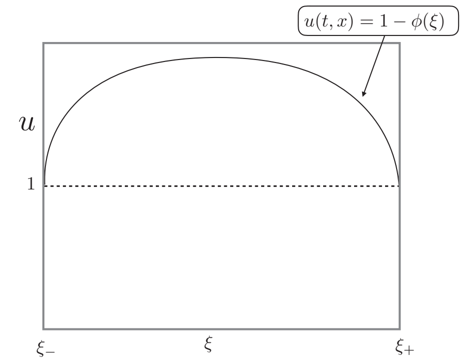

Fig. 1. Schematic picture of the quasi traveling wave with quenching on $\xi \in [\xi_-, \xi_+]$ obtained in Theorem 2.

$$\text{(1.4)} \quad \begin{cases} \phi(\xi) \sim -C(\xi - \xi_-)^{\frac{2\alpha}{2\alpha^2-\alpha+1}} \\ \psi(\xi) \sim -C(\xi - \xi_-)^{\frac{1-\alpha}{2\alpha^2-\alpha+1}} \end{cases} \quad \text{as} \quad \xi \to \xi_- + 0$$

with $C > 0$.

In order to prove Theorem 2, it is necessary to seek a family of orbits that connect $(\phi, \psi) = (0, -\infty)$ and $(0, +\infty)$ of (1.3) (see Sec. 4 for the details). As shown in [2,4], the Poincaré compactification is useful, and applicable for this problem. In the next section, we briefly introduce the Poincaré compactification for the convenience of readers.

2. Preparation. In this section, we briefly introduce the Poincaré compactification.

Let

$$X = P(\phi, \psi)\frac{\partial}{\partial \phi} + Q(\phi, \psi)\frac{\partial}{\partial \psi}$$

be a polynomial vector field on $\mathbf{R}^2$, or in other words

$$\begin{cases} \dot{\phi} = P(\phi, \psi), \\ \dot{\psi} = Q(\phi, \psi), \end{cases}$$

where $\dot{}$ denotes d/dt, and P, Q are polynomials of arbitrary degree in the variables ϕ and ψ.

First, we consider $\mathbf{R}^2$ as the plane in $\mathbf{R}^3$ defined by

$$(y_1, y_2, y_3) = (\phi, \psi, 1).$$

We consider the sphere

$$\mathbf{S}^2 = \{y \in \mathbf{R}^3 \mid y_1^2 + y_2^2 + y_3^2 = 1\}$$

which we call Poincaré sphere. We divide the sphere into

$$H_+ = \{y \in \mathbf{R}^3 \mid y_3 > 0\},$$
$$H_- = \{y \in \mathbf{R}^3 \mid y_3 < 0\}$$

and

$$\mathbf{S}^1 = \{y \in \mathbf{R}^3 \mid y_3 = 0\}.$$

Let us consider the projection of vector field X from $\mathbf{R}^2$ to $\mathbf{S}^2$ given by

$$f^+ : \mathbf{R}^2 \to \mathbf{S}^2 \quad \text{and} \quad f^- : \mathbf{R}^2 \to \mathbf{S}^2,$$

where

$$f^{\pm}(\phi, \psi) := \pm\left(\frac{\phi}{\Delta(\phi,\psi)}, \frac{\psi}{\Delta(\phi,\psi)}, \frac{1}{\Delta(\phi,\psi)}\right)$$

with $\Delta(\phi, \psi) = \sqrt{\phi^2 + \psi^2 + 1}$.

Second, we consider six local charts on $\mathbf{S}^2$ given by

$$U_k = \{y \in \mathbf{S}^2 \mid y_k > 0\} \quad \text{and} \quad V_k = \{y \in \mathbf{S}^2 \mid y_k < 0\}$$

for $k = 1, 2, 3$. Consider the local projection

$$g_k^+ : U_k \to \mathbf{R}^2 \quad \text{and} \quad g_k^- : V_k \to \mathbf{R}^2$$

defined as

$$g_k^+(y_1, y_2, y_3) = -g_k^-(y_1, y_2, y_3) = \left(\frac{y_m}{y_k}, \frac{y_n}{y_k}\right)$$

for $m < n$ and $m, n \neq k$. The projected vector fields are obtained as the vector fields on the planes

$$\overline{U}_k = \{y \in \mathbf{R}^3 \mid y_k = 1\}$$

and

$$\overline{V}_k = \{y \in \mathbf{R}^3 \mid y_k = -1\}$$

for each local chart U_k and V_k. We denote by (x, λ) the value of $g_k^{\pm}(y)$ for any k.

For instance, it follows that

$$(g_2^+ \circ f^+)(\phi, \psi) = \left(\frac{\phi}{\psi}, \frac{1}{\psi}\right) = (x, \lambda),$$

therefore, we can obtain the dynamics on the local chart $\overline{U}_2$ by the change of variables $\phi = x/\lambda$ and $\psi = 1/\lambda$. The locations of the Poincaré sphere, (ϕ, ψ)-plane and $\overline{U}_2$ are expressed as Figure 2. We refer to [2] and references therein for more details. We also refer to [5] and [4] for the Poincaré type compactification and its applications, respectively.

Throughout this paper, we follow the notations used here for the Poincaré compactification. It is sufficient to consider the dynamics on $H_+ \cup \mathbf{S}^1$, which is called Poincaré disk, to obtain our main result.

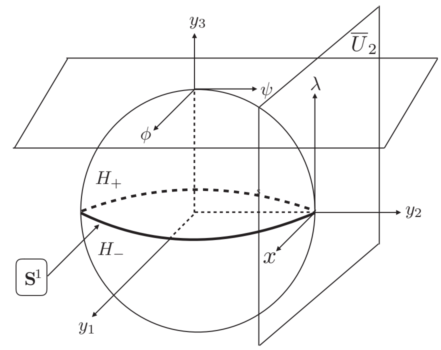

Fig. 2. Locations of the Poincaré sphere and chart $\overline{U}_2$.

3. Dynamics on the Poincaré disk of (1.3). In order to study the dynamics of (1.3) on the Poincaré disk, we desingularize it by the time-scale desingularization

$$ds/d\xi = \{\phi(\xi)\}^{-\alpha} \quad \text{for} \quad \alpha \in 2\mathbf{N}. \tag{3.1}$$

Since we assume that α is even, the direction of the time does not change via this desingularization. Then we have

$$\begin{cases} \dot{\phi} = \phi^\alpha \psi, \\ \dot{\psi} = -c\phi^\alpha \psi + 1. \end{cases} \quad \left(\dot{} = \frac{d}{ds}\right). \tag{3.2}$$

It should be noted that the time scale desingularization (3.1) is simply multiplying the vector field by ϕ^α. Then, with excepting the singularity $\{\phi = 0\}$, the solution curves of the system (vector field) remain the same but are parameterized differently. Still, we refer to Section 7.7 of [3] and references therein for the analytical treatments of desingularization with the time rescaling. In what follows, we use the similar time rescaling (re-parameterization of the solution curves) repeatedly to desingularize the vector fields.

Now we can consider the dynamics of (3.2) on the charts $\overline{U}_j$ and $\overline{V}_j$.

3.1. Dynamics on the chart $\overline{U}_2$. To obtain the dynamics on the chart $\overline{U}_2$, we introduce coordinates (λ, x) by the formulas

$$\phi(s) = x(s)/\lambda(s), \quad \psi(s) = 1/\lambda(s).$$

Then we have

$$\begin{cases} \dot{\lambda} = cx^\alpha \lambda^{1-\alpha} - \lambda^2, \\ \dot{x} = x(cx^\alpha \lambda^{-\alpha} - \lambda) + x^\alpha \lambda^{-\alpha}. \end{cases}$$

Time-scale desingularization $d\tau/ds = \lambda(s)^{-\alpha}$ yields

$$(3.3)\qquad \begin{cases} \lambda_\tau = cx^\alpha\lambda - \lambda^{2+\alpha}, \\ x_\tau = cx^{\alpha+1} - \lambda^{1+\alpha}x + x^\alpha, \end{cases}$$

where $\lambda_\tau = d\lambda/d\tau$ and $x_\tau = dx/d\tau$. The system (3.3) has the equilibria

$$p_0^+ : (\lambda, x) = (0,0) \quad \text{and} \quad p_c : (\lambda, x) = (0, -1/c).$$

The Jacobian matrices at these equilibria are

$$p_0^+ : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_c : \begin{pmatrix} c^{1-\alpha} & 0 \\ 0 & c^{1-\alpha} \end{pmatrix}.$$

Therefore, p_c is a source, and p_0^+ is not hyperbolic. In order to determine the dynamics near p_0^+, we desingularize p_0^+ by introducing the following blow-up coordinates:

$$\lambda = r^{\alpha-1}\bar{\lambda}, \quad x = r^{\alpha+1}\bar{x}$$

(see Sec. 3 of [2] for the desingularizations of vector fields by the blow-up). Since we are interested in the dynamics on the Poincaré disk, we consider the dynamics of blow-up vector fields on the charts $\{\bar{\lambda} = 1\}$ and $\{\bar{x} = \pm 1\}$.

Dynamics on the chart $\{\bar{\lambda} = 1\}$. By the change of coordinates $\lambda = r^{\alpha-1}$, $x = r^{\alpha+1}\bar{x}$, we have

$$\begin{cases} r_\tau = \dfrac{r}{\alpha-1}(c\bar{x}^\alpha r^{\alpha(\alpha+1)} - r^{\alpha^2-1}), \\ \bar{x}_\tau = \dfrac{2}{\alpha-1}(\bar{x}r^{\alpha^2-1} - c\bar{x}^{\alpha+1}r^{\alpha(\alpha+1)}) + \bar{x}^\alpha r^{\alpha^2-1}. \end{cases}$$

The time-rescaling $d\eta/d\tau = r(\tau)^{\alpha^2-1}$ yields

$$(3.4)\qquad \begin{cases} r_\eta = (\alpha-1)^{-1}(-r + c\bar{x}^\alpha r^{2+\alpha}), \\ \bar{x}_\eta = 2(\alpha-1)^{-1}(\bar{x} - c\bar{x}^{\alpha+1}r^{\alpha+1}) + \bar{x}^\alpha. \end{cases}$$

The equilibria of (3.4) on $\{r = 0\}$ are

$$\bar{p}_0^+ : (r, \bar{x}) = (0,0), \quad \bar{p}_\alpha^+ : (r, \bar{x}) = \left(0, \left(\frac{-2}{\alpha-1}\right)^{\frac{1}{\alpha-1}}\right).$$

The Jacobian matrices at these equilibria are

$$\bar{p}_0^+ : \begin{pmatrix} -\dfrac{1}{\alpha-1} & 0 \\ 0 & \dfrac{2}{\alpha-1} \end{pmatrix} \quad \text{and} \quad \bar{p}_\alpha^+ : \begin{pmatrix} -\dfrac{1}{\alpha-1} & 0 \\ 0 & -2 \end{pmatrix}.$$

Moreover, since $|1/(\alpha-1)| < 2$ holds, trajectories near $\bar{p}_\alpha^+$ are tangent to $\{\bar{x} = [-2/(\alpha-1)]^{\frac{1}{\alpha-1}}, r \geq 0\}$ as $\eta \to +\infty$. The solutions are approximated as

$$\begin{cases} r(\eta) \sim Ce^{\frac{-1}{\alpha-1}\eta}(1 + o(1)), \\ \bar{x}(\eta) \sim Ce^{-2\eta}(1 + o(1)) + \left(\dfrac{-2}{\alpha-1}\right)^{\frac{1}{\alpha-1}}. \end{cases}$$

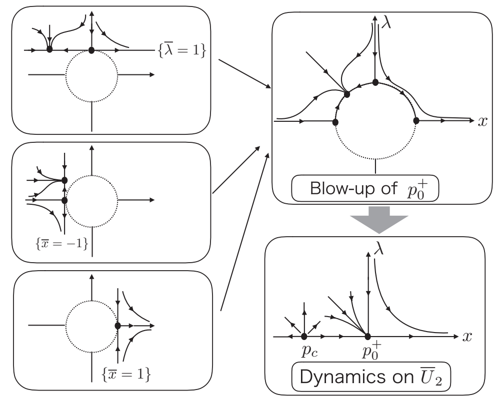

Fig. 3. Schematic pictures of the dynamics of the blow-up vector fields and $\overline{U}_2$.

Dynamics on the chart $\{\bar{x} = -1\}$. By the change of coordinates $\lambda = r^{\alpha-1}\bar{\lambda}$, $x = -r^{\alpha+1}$, and time-rescaling $d\eta/d\tau = r(\tau)^{\alpha^2-1}$, we have

$$\begin{cases} r_\eta = (\alpha+1)^{-1}(cr^{\alpha+2} - r\bar{\lambda}^{1+\alpha} - r), \\ \bar{\lambda}_\eta = -(\alpha+1)^{-1}(2\bar{\lambda}^{2+\alpha} - (\alpha-1)\bar{\lambda} - 2cr^{\alpha+1}\bar{\lambda}). \end{cases}$$

The equilibria on $\{r = 0\}$ are

$$(r, \bar{\lambda}) = (0,0), \quad (r, \bar{\lambda}) = (0, [(\alpha-1)/2]^{\frac{1}{\alpha+1}}).$$

By the further computations, we can see that $(0,0)$ is a saddle, and $(0, [(\alpha-1)/2]^{\frac{1}{\alpha+1}})$ is a sink.

Dynamics on the chart $\{\bar{x} = 1\}$. The change of coordinates $\lambda = r^{\alpha-1}\bar{\lambda}$, $x = r^{\alpha+1}$, and time-rescaling $d\eta/d\tau = r(\tau)^{\alpha^2-1}$ yield

$$\begin{cases} r_\eta = (\alpha+1)^{-1}(cr^{\alpha+2} - r\bar{\lambda}^{1+\alpha} + r), \\ \bar{\lambda}_\eta = -(\alpha+1)^{-1}(2\bar{\lambda}^{2+\alpha} + (\alpha-1)\bar{\lambda} - 2cr^{\alpha+1}\bar{\lambda}). \end{cases}$$

The equilibrium on $\{r = 0, \bar{\lambda} \geq 0\}$ is $(0,0)$. The linearized eigenvalues are $(\alpha+1)^{-1}$ and $-(\alpha-1)/(\alpha+1)$ with corresponding eigenvectors $(1,0)$ and $(0,1)$, respectively. Therefore, $(r, \bar{\lambda}) = (0,0)$ on the chart $\{\bar{x} = 1\}$ is a saddle.

Combining the dynamics on the charts $\{\bar{\lambda} = 1\}$ and $\{\bar{x} = \pm 1\}$, we obtain the dynamics on $\overline{U}_2$ (see Figure 3).

Still, we continue to study the dynamics on other charts in order to obtain the whole dynamics on the Poincaré disk.

3.2. Dynamics on the chart $\overline{V}_2$. The change of coordinates

$$\phi(s) = -x(s)/\lambda(s), \quad \psi(s) = -1/\lambda(s)$$

give the projected dynamics of (1.3) on the chart $\overline{V}_2$:

$$\text{(3.5)}\qquad \begin{cases} \lambda_\tau = cx^\alpha\lambda + \lambda^{2+\alpha}, \\ x_\tau = x^\alpha + cx^{\alpha+1} + \lambda^{1+\alpha}x, \end{cases}$$

where τ is the new time introduced by $d\tau/ds = \lambda(s)^{-\alpha}$. The system (3.5) can be transformed into (3.3) by the change of coordinates $(\lambda, x) \mapsto (-\lambda, x)$. Therefore, it is sufficient to consider the blow-up of singularity $p_0^- : (\lambda, x) = (0, 0)$ by the formulas

$$\lambda = r^{\alpha-1}\bar{\lambda}, \quad x = r^{\alpha+1}\bar{x} \quad \text{with} \quad \bar{\lambda} = 1.$$

Then we have

$$\text{(3.6)}\qquad \begin{cases} r_\eta = (\alpha-1)^{-1}(r + c\bar{x}^\alpha r^{\alpha+2}), \\ \bar{x}_\eta = \bar{x}^\alpha - 2(\alpha-1)^{-1}(\bar{x} + c\bar{x}^{\alpha+1}r^{\alpha+1}), \end{cases}$$

where η satisfies $d\eta/d\tau = \{r(\tau)\}^{\alpha^2-1}$. The equilibria of (3.6) on $\{r = 0\}$ are

$$\bar{p}_0^- : (r, \bar{x}) = (0, 0), \quad \bar{p}_\alpha^- : (r, \bar{x}) = \left(0, \left(\frac{2}{\alpha-1}\right)^{\frac{1}{\alpha-1}}\right).$$

The equilibrium $\bar{p}_0^-$ is a saddle with the eigenvalues $(\alpha-1)^{-1}$ and $-2(\alpha-1)^{-1}$ whose corresponding eigenvectors are $(1, 0)$ and $(0, 1)$, respectively. Further, $\bar{p}_\alpha^-$ is a source with the eigenvalues $(\alpha - 1)^{-1}$ and 2 whose corresponding eigenvectors are $(1, 0)$ and $(0, 1)$, respectively.

3.3. Dynamics on the chart $\overline{U}_1$. Let us study the dynamics on the chart $\overline{U}_1$. The transformations

$$\phi(s) = 1/\lambda(s), \quad \psi(s) = x(s)/\lambda(s)$$

yield

$$\text{(3.7)}\qquad \begin{cases} \lambda_\tau = -x\lambda, \\ x_\tau = -cx + \lambda^{1+\alpha} - x^2 \end{cases}$$

via time-rescaling $d\tau/ds = \{\lambda(s)\}^{-\alpha}$. The equilibria of (3.7) are $(0, 0)$ and $(0, -c)$ whose Jacobian matrices are

$$\begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix},$$

respectively. Then the center manifold theory is applicable to study the dynamics near $(0, 0)$ (for instance, see [1]). It implies that there exists a function $h(\lambda)$ satisfying

$$h(0) = \frac{dh}{d\lambda}(0) = 0$$

such that the center manifold of (3.7) is represented as $\{(\lambda, x) \mid x = h(\lambda)\}$ near $(0, 0)$. Differentiating it with respect to τ, we have

$$-\lambda h(\lambda)\frac{dh}{d\lambda}(\lambda) = -ch(\lambda) + \lambda^{1+\alpha} - \{h(\lambda)\}^2.$$

Then we can obtain the approximation of the (graph of) center manifold as follows:

$$\{(\lambda, x) \mid x = \lambda^{\alpha+1}/c + O(\lambda^{2\alpha+2})\}.$$

Therefore, the dynamics of (3.7) near $(0, 0)$ is topologically equivalent to the dynamics of the following equation:

$$\lambda_\tau = -\lambda^{\alpha+2}/c + O(\lambda^{2\alpha+3}).$$

These results give us the dynamics on the chart $\overline{U}_1$.

3.4. Dynamics on the chart $\overline{V}_1$. The transformations

$$\phi(s) = -1/\lambda(s), \quad \psi(s) = -x(s)/\lambda(s)$$

yield

$$\text{(3.8)}\qquad \begin{cases} \lambda_\tau = -x\lambda, \\ x_\tau = -cx - \lambda^{1+\alpha} - x^2 \end{cases}$$

via time-rescaling $d\tau/ds = \{\lambda(s)\}^{-\alpha}$. We can see that the system (3.8) can be transformed into the system (3.7) by the change of variables: $(\lambda, x) \mapsto (-\lambda, x)$. Therefore, the dynamics of (3.8) is equivalent to the reflected one of (3.7) with respect to $\{\lambda = 0\}$.

4. Proof of Theorem 2. Since the point $(y_1, y_2, y_3) = (0, 1, 0)$ on the Poincaré disk corresponds to p_0^+, we denote it by p_0^+ as well. Similarly, we denote by p_0^- the point $(y_1, y_2, y_3) = (0, -1, 0)$. In order to prove Theorem 2, it is necessary to find the orbits that connect p_0^- and p_0^+ on the Poincaré disk. The phase portrait on the Poincaré disk of (1.3) is shown in Figure 4 for the convenience of readers.

Proof. (I): For a given compact subset $W \subset H_+$, there are no equilibria or closed orbits in W. Therefore, by the Poincaré-Bendixson theorem, any trajectories starting from the points in W can not stay in W with increasing s. This implies that the trajectories in H_+ go to $\mathbf{S}^1$, which corresponds to $\{\|(\phi, \psi)\| = \infty\}$.

(II): The line $\{\phi = 0\}$ is invariant under the flow of (3.2). Therefore, any trajectories start from the points in $\{y \in H_+ \mid y_1 < 0\}$ can not go to $\{y \in H_+ \mid y_1 > 0\}$.

(III): Let $\overline{W}^s_{\bar{p}_\alpha^+}$ be a stable manifold of $\bar{p}_\alpha^+$ (which is the equilibrium of the system (3.4)). We denote by $W^s(\bar{p}_\alpha^+)$ the stable set, which corresponds to $\overline{W}^s_{\bar{p}_\alpha^+}$ on

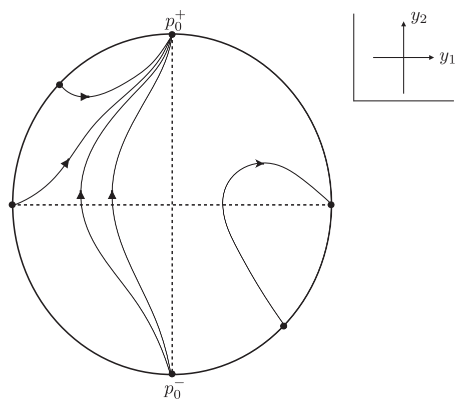

Fig. 4. Compactification of the system (1.3).

the blow-up vector filed (3.4), of the equilibrium p_0^+ of (3.3). Similarly, we denote by $W^u(\bar{p}_\alpha^-)$ the unstable set of p_0^-, corresponding to the unstable manifold of $\bar{p}_\alpha^-$ on the blow-up vector field (3.6). Consider the trajectories start from the points on $W^u(\bar{p}_\alpha^-) \subset \{y \in H_+ \mid y_1 < 0\}$. The trajectories can not stay in any compact subset on H_+, and can not go to $\{y \in H_+ \mid y_1 > 0\}$, therefore, they go to p_0^+ with lying on $W^s(\bar{p}_\alpha^+)$. This implies that the system (3.2) possesses the orbits that connect p_0^- and p_0^+ on the Poincaré disk. It is easy to see that $d\phi/d\psi$ takes the same values on the vector fields defined by (3.2) and (1.3) by excepting the singularity $\{\phi = 0\}$. Thus, there are orbits connecting $(\phi, \psi) = (0, -\infty)$ and $(0, +\infty)$ on the original vector field (1.3).

(IV): As shown in [4], we can obtain the quenching rates of $\phi(\xi)$ and $\psi(\xi)$. Indeed,

$$\frac{d\eta}{d\xi} = \frac{ds}{d\xi} \cdot \frac{d\tau}{ds} \cdot \frac{d\eta}{d\tau} = \phi^{-\alpha} \cdot \lambda^{-\alpha} \cdot r^{\alpha^2 - 1}$$

$$= r^{-\alpha-1} \cdot \bar{x}^{-\alpha} = C\, e^{\left(\frac{\alpha+1}{\alpha-1} + 2\alpha\right)\eta}$$

holds with a constant C. This yields

$$\xi(\eta) = C\, e^{-\left(\frac{\alpha+1}{\alpha-1} + 2\alpha\right)\eta} + \tilde{C}, \quad (\tilde{C} \in \mathbf{R}).$$

Set $\xi_+ = \lim_{\eta \to +\infty} \xi(\eta)$, then we have

$$\xi_+ = C \int_0^{+\infty} e^{-(\frac{\alpha+1}{\alpha-1} + 2\alpha)\eta}\, d\eta < \infty.$$

Therefore,

$$\xi_+ - \xi \sim C\, e^{-(\frac{\alpha+1}{\alpha-1} + 2\alpha)\eta}$$

holds. Finally, we obtain

$$\phi(\xi) = r^2(\eta) \cdot \bar{x}(\eta) \sim -C\, e^{\frac{-2\alpha}{\alpha-1}\eta}$$

$$= -C\, (\xi_+ - \xi)^{\frac{2\alpha}{2\alpha^2 - \alpha + 1}}$$

with $C > 0$. Similarly, we can obtain the quenching rates for $\psi(\xi)$ as $\xi \to \xi_+$ and (1.4).

This completes the proof. □

References

[1] J. Carr, *Applications of centre manifold theory*, Applied Mathematical Sciences, 35, Springer-Verlag, New York, 1981.

[2] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, Berlin, 2006.

[3] C. Kuehn, *Multiple time scale dynamics*, Applied Mathematical Sciences, 191, Springer, Cham, 2015.

[4] K. Matsue, Geometric treatments and a common mechanism in finite-time singularities for autonomous ODEs, J. Differential Equations **267** (2019), no. 12, 7313–7368.

[5] K. Matsue, On blow-up solutions of differential equations with Poincaré-type compactifications, SIAM J. Appl. Dyn. Syst. **17** (2018), no. 3, 2249–2288.