Absolute multiple sine functions

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(Communicated by Masaki KASHIWARA, M.J.A., April 12, 2019)

Abstract: In this paper we formulate a unified theory of multiple sine functions by using a view point of absolute zeta functions and absolute automorphic forms.

Key words: Absolute multiple sine function; primitive multiple sine function; regularized multiple sine function; absolute zeta function; absolute automorphic form.

1. Introduction. We recall that there are two kinds of multiple sine functions. The first one is the primitive multiple sine function defined as

$$\mathcal{S}_r(s) = \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} P_r\left(\frac{s}{n}\right)^{n^{r-1}} \times \begin{cases} 2\pi s & \text{if } r=1, \\ \exp\left(\dfrac{s^{r-1}}{r-1}\right) & \text{if } r>1, \end{cases}$$

where

$$P_r(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right).$$

For example,

$$\mathcal{S}_1(s) = 2\pi s \prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right) = 2\sin(\pi s)$$

is Euler's sine function [2], and

$$\mathcal{S}_2(s) = e^s \prod_{n=1}^{\infty}\left\{\left(\frac{1-\frac{s}{n}}{1+\frac{s}{n}}\right)^n e^{2s}\right\}$$

is Hölder's double sine function [3]. The triple sine function

$$\mathcal{S}_3(s) = e^{\frac{s^2}{2}} \prod_{n=1}^{\infty}\left\{\left(1 - \frac{s^2}{n^2}\right)^{n^2} e^{s^2}\right\}$$

and the general $\mathcal{S}_r(s)$ was discovered in [4] (see also [6], [7]). By construction $\mathcal{S}_r(s)$ is a meromorphic function in $s \in \mathbf{C}$ and when $r \geq 2$ it has the following expression

$$\mathcal{S}_r(s) = \exp\left(\int_0^s \pi u^{r-1}\cot(\pi u)du\right),$$

where $\int_0^s \subset \mathbf{C} - \{\pm 1, \pm 2, \ldots\}$.

The second kind of multiple sine function is the regularized multiple sine function constructed in Shintani [12] ($r = 2$) and [5] (general r):

$$S_r(s, (\omega_1, \ldots, \omega_r)) = \Gamma_r(s, (\omega_1, \ldots, \omega_r))^{-1} \times \Gamma_r(\omega_1 + \cdots + \omega_r - s, (\omega_1, \ldots, \omega_r))^{(-1)^r},$$

where $\Gamma_r(s, (\omega_1, \ldots, \omega_r))$ is the regularized version of the multiple gamma function introduced by Barnes [1]. Here we explain the construction when $\mathrm{Re}(\omega_1), \ldots, \mathrm{Re}(\omega_r) > 0$ and we put $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ for simplicity. First, the multiple Hurwitz zeta function $\zeta_r(w, s, \boldsymbol{\omega})$ is defined as

$$\zeta_r(w, s, \boldsymbol{\omega}) = \sum_{n_1, \ldots, n_r \geq 0} (s + n_1\omega_1 + \cdots + n_r\omega_r)^{-w}$$

for $\mathrm{Re}(w) > r$. It has an analytic continuation to all $w \in \mathbf{C}$ and it is holomorphic at $w = 0$. Then we obtain the regularized multiple gamma function

$$\Gamma_r(s, \boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial w}\zeta_r(w, s, \boldsymbol{\omega})\Big|_{w=0}\right).$$

It is a meromorphic function in $s \in \mathbf{C}$.

A defect of this construction is the difficulty to treat the general case $\boldsymbol{\omega} \in (\mathbf{C} - \{0\})^r$. For example

$$\zeta_2(w, s, (1, -1)) = \sum_{n_1, n_2 \geq 0} (s + n_1 - n_2)^{-w}$$

is meaningless, so we do not have $\Gamma_2(s, (1, -1))$ nor $S_2(s, (1, -1))$ in this way.

In this paper we construct multiple sine functions from the absolute automorphic form

$$f_{\boldsymbol{\omega}}(x) = \prod_{k=1}^{r}(1 - x^{-\omega_k})^{-1},$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ and $x > 0$. We recall that a function $f(x)$ on $\mathbf{R}_{>0}$ is an absolute automorphic form when it satisfies the absolute automorphy

2010 Mathematics Subject Classification. Primary 14G10.

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doi: 10.3792/pjaa.95.41

$$f\left(\frac{1}{x}\right) = Cx^{-D} f(x)$$

for constants C and D. For example, $f_{\boldsymbol{\omega}}(x)$ is an absolute automorphic form satisfying

$$f_{\boldsymbol{\omega}}\left(\frac{1}{x}\right) = (-1)^r x^{-|\boldsymbol{\omega}|} f_{\boldsymbol{\omega}}(x)$$

where $|\boldsymbol{\omega}| = \omega_1 + \cdots + \omega_r$. We notice in passing that it is natural to regard as the absolute tensor product

$$f_{(\omega_1,\ldots,\omega_r)} = f_{\omega_1} \underset{\mathbf{F}_1}{\otimes \cdots \otimes} f_{\omega_r}.$$

From an absolute automorphic form $f(x)$ we define the absolute zeta function $\zeta_f(s)$ and the absolute ε-function $\varepsilon_f(s)$ by

$$\zeta_f(s) = \exp\left(\frac{\partial}{\partial w} Z_f(w,s)\Big|_{w=0}\right)$$

and

$$\varepsilon_f(s) = \frac{\zeta_{f^*}(-s)}{\zeta_f(s)}$$

respectively, where

$$Z_f(w,s) = \frac{1}{\Gamma(w)} \int_1^\infty f(x) x^{-s-1} (\log x)^{w-1} dx$$

and

$$f^*(x) = f\left(\frac{1}{x}\right).$$

We refer to [9], [10] for a general theory of $\zeta_f(s)$ and $\varepsilon_f(s)$ (see also [8], [11]).

Now, for $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) \in (\mathbf{C} - \sqrt{-1}\mathbf{R})^r$ we define the absolute multiple gamma function

$$\boldsymbol{\Gamma}_r(s, \boldsymbol{\omega}) = \zeta_{f_{\boldsymbol{\omega}}}(s)$$

and the absolute multiple sine function

$$\mathbf{S}_r(s, \boldsymbol{\omega}) = \varepsilon_{f_{\boldsymbol{\omega}}}(s).$$

Then we have the following results.

Theorem 1. *For* $\boldsymbol{\omega} \in (\mathbf{C} - \sqrt{-1}\mathbf{R})^r$, $\boldsymbol{\Gamma}_r(s, \boldsymbol{\omega})$ *and* $\mathbf{S}_r(s, \boldsymbol{\omega})$ *are meromorphic functions in* $s \in \mathbf{C}$.

Theorem 2. *When* $\mathrm{Re}(\omega_k) > 0$ $(k = 1, \ldots, r)$, *we have*

$$\boldsymbol{\Gamma}_r(s, \boldsymbol{\omega}) = \Gamma_r(s, \boldsymbol{\omega})$$

and

$$\mathbf{S}_r(s, \boldsymbol{\omega}) = S_r(s, \boldsymbol{\omega}).$$

Theorem 3. *Let*

$$\mathbf{S}_{2n}(s, \pm) = \mathbf{S}_{2n}(s, (\overbrace{1, \ldots, 1}^{n}, \overbrace{-1, \ldots, -1}^{n})).$$

Then

$$\mathbf{S}_{2n}(s, \pm) = \exp\left(\frac{(-1)^{n-1}}{(2n-1)!} \int_0^s \prod_{k=1}^{n-1} (u^2 - k^2) \times \pi u \cot(\pi u) du\right).$$

Theorem 4. *For* $n \geq 1$, *let*

$$\prod_{k=0}^{n-1} (X - k^2) = \sum_{m=1}^{n} c(n,m) X^m \in \mathbf{Z}[X] :$$

$c(n,n) = 1$ *and* $c(n,1) = (-1)^{n-1}((n-1)!)^2$. *Then*:

$$\mathbf{S}_{2n}(s, \pm)^{(-1)^{n-1}(2n-1)!} = \prod_{m=1}^{n} \mathcal{S}_{2m}(s)^{c(n,m)}.$$

Moreover,

$$\mathcal{S}_{2n}(s) = \prod_{m=1}^{n} \mathbf{S}_{2m}(s, \pm)^{a(n,m)}$$

where $a(n,m) \in \mathbf{Z}$ *is defined as*

$$X^n = \sum_{m=1}^{n} a(n,m) \frac{(-1)^{m-1}}{(2m-1)!} \prod_{k=0}^{m-1} (X - k^2).$$

Example.

(1)

$$\mathbf{S}_2(s, \pm) = \exp\left(\int_0^s \pi u \cot(\pi u) du\right) = \mathcal{S}_2(s).$$

(2)

$$\mathbf{S}_4(s, \pm)^{-3!} = \exp\left(\int_0^s (u^2 - 1) \pi u \cot(\pi u) du\right) = \mathcal{S}_4(s) \mathcal{S}_2(s)^{-1}.$$

(3)

$$\mathbf{S}_6(s, \pm)^{5!} = \exp\left(\int_0^s (u^2 - 1)(u^2 - 4) \pi u \cot(\pi u) du\right) = \mathcal{S}_6(s) \mathcal{S}_4(s)^{-5} \mathcal{S}_2(s)^4.$$

(4)

$$\mathbf{S}_8(s, \pm)^{-7!} = \exp\left(\int_0^s (u^2 - 1)(u^2 - 4)(u^2 - 9) \pi u \cot(\pi u) du\right) = \mathcal{S}_8(s) \mathcal{S}_6(s)^{-14} \mathcal{S}_4(s)^{49} \mathcal{S}_2(s)^{-36}.$$

(5)

$$\begin{aligned}\mathbf{S}_{10}(s,\pm)^{9!} &= \exp\left(\int_0^s (u^2-1)(u^2-4)(u^2-9)(u^2-16) \times \pi u\cot(\pi u)du\right) \\ &= \mathcal{S}_{10}(s)\mathcal{S}_8(s)^{-30}\mathcal{S}_6(s)^{273}\mathcal{S}_4(s)^{-820}\mathcal{S}_2(s)^{576}.\end{aligned}$$

Conversly:

(1*)

$$\mathcal{S}_2(s) = \mathbf{S}_2(s,\pm).$$

(2*)

$$\mathcal{S}_4(s) = \mathbf{S}_4(s,\pm)^{-6}\mathbf{S}_2(s,\pm).$$

(3*)

$$\mathcal{S}_6(s) = \mathbf{S}_6(s,\pm)^{120}\mathbf{S}_4(s,\pm)^{-30}\mathbf{S}_2(s,\pm).$$

(4*)

$$\mathcal{S}_8(s) = \mathbf{S}_8(s,\pm)^{-5040}\mathbf{S}_6(s,\pm)^{1680} \times \mathbf{S}_4(s,\pm)^{-126}\mathbf{S}_2(s,\pm).$$

(5*)

$$\mathcal{S}_{10}(s) = \mathbf{S}_{10}(s,\pm)^{362880}\mathbf{S}_8(s,\pm)^{-151200}\mathbf{S}_6(s,\pm)^{17640} \times \mathbf{S}_4(s,\pm)^{-510}\mathbf{S}_2(s,\pm).$$

Theorem 5. *Let $n \geq 1$. Put*

$$\mathbf{S}_{2n+1}(s,\pm) = \frac{\mathbf{S}_{2n+1}(s,(\overbrace{1,\dots,1}^{n},\overbrace{-1,\dots,-1}^{n+1}))}{\mathbf{S}_{2n+1}(s,(\underbrace{1,\dots,1}_{n+1},\underbrace{-1,\dots,-1}_{n}))}.$$

For $n \geq m \geq 1$ let $c(n,m)$ and $a(n,m)$ be as in Theorem 4. Then:

$$\begin{aligned}\mathbf{S}_{2n+1}(s,\pm)^{(-1)^{n-1}\frac{(2n)!}{2}} &= \exp\left(-2\sum_{m=1}^{n} c(n,m)\zeta'(-2m)\right) \\ &\quad \times \exp\left(\int_0^s \prod_{k=0}^{n-1}(u^2-k^2)\cdot\pi\cot(\pi u)du\right) \\ &= \prod_{m=1}^{n}\{\exp(-2\zeta'(-2m))\mathcal{S}_{2m+1}(s)\}^{c(n,m)} \\ &= \prod_{m=1}^{n}\tilde{\mathcal{S}}_{2m+1}(s)^{c(n,m)},\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathcal{S}}_{2m+1}(s) &= \exp(-2\zeta'(-2m))\mathcal{S}_{2m+1}(s) \\ &= \exp\left((-1)^{m-1}\frac{\zeta(2m+1)(2m)!}{2^{2m}\pi^{2m}}\right)\mathcal{S}_{2m+1}(s).\end{aligned}$$

Moreover,

$$\mathcal{S}_{2n+1}(s) = \exp(2\zeta'(-2n))\prod_{m=1}^{n}\mathbf{S}_{2m+1}(s,\pm)^{a(n,m)m}.$$

Example.

(1)

$$\begin{aligned}\mathbf{S}_3(s,\pm) &= \frac{\mathbf{S}_3(s,(1,-1,-1))}{\mathbf{S}_3(s,(1,1,-1))} \\ &= \exp(-2\zeta'(-2))\exp\left(\int_0^s u^2\pi\cot(\pi u)du\right) \\ &= \exp(-2\zeta'(-2))\mathcal{S}_3(s) \\ &= \exp\left(\frac{\zeta(3)}{2\pi^2}\right)\mathcal{S}_3(s) \\ &= \tilde{\mathcal{S}}_3(s).\end{aligned}$$

(2)

$$\begin{aligned}\mathbf{S}_5(s,\pm)^{-12} &= \left(\frac{\mathbf{S}_5(s,(1,1,-1,-1,-1))}{\mathbf{S}_5(s,(1,1,1,-1,-1))}\right)^{-12} \\ &= \exp(-2\zeta'(-4)+2\zeta'(-2)) \\ &\quad \times \exp\left(\int_0^s u^2(u^2-1)\pi\cot(\pi u)du\right) \\ &= \exp\left(-\frac{3\zeta(5)}{2\pi^4}-\frac{\zeta(3)}{2\pi^2}\right)\mathcal{S}_5(s)\mathcal{S}_3(s)^{-1} \\ &= \tilde{\mathcal{S}}_5(s)\tilde{\mathcal{S}}_3(s)^{-1}.\end{aligned}$$

(3)

$$\begin{aligned}\mathbf{S}_7(s,\pm)^{360} &= \left(\frac{\mathbf{S}_7(s,(1,1,1,-1,-1,-1,-1))}{\mathbf{S}_7(s,(1,1,1,1,-1,-1,-1))}\right)^{360} \\ &= \exp(-2\zeta'(-6)+10\zeta'(-4)-8\zeta'(-2)) \\ &\quad \times \exp\left(\int_0^s u^2(u^2-1)(u^2-4)\pi\cot(\pi u)du\right) \\ &= \exp\left(\frac{45\zeta(7)}{4\pi^6}+\frac{15\zeta(5)}{2\pi^4}+\frac{2\zeta(3)}{\pi^2}\right) \\ &\quad \times \mathcal{S}_7(s)\mathcal{S}_5(s)^{-5}\mathcal{S}_3(s)^4 \\ &= \tilde{\mathcal{S}}_7(s)\tilde{\mathcal{S}}_5(s)^{-5}\tilde{\mathcal{S}}_3(s)^4.\end{aligned}$$

Conversly:

(1*)

$$\tilde{\mathcal{S}}_3(s) = \mathbf{S}_3(s,\pm).$$

(2*)

$$\tilde{\mathcal{S}}_5(s) = \mathbf{S}_5(s,\pm)^{-12}\mathbf{S}_3(s,\pm).$$

(3*)

$$\tilde{\mathcal{S}}_7(s) = \mathbf{S}_7(s,\pm)^{360}\mathbf{S}_5(s,\pm)^{-60}\mathbf{S}_3(s,\pm).$$

2. Proofs of Theorems 1 and 2. First we show Theorem 2. Let $\mathrm{Re}(\omega_k) > 0$ for $k = 1,\dots,r$. Then, for $\mathrm{Re}(\omega) > r$

$$\begin{aligned} Z_{f_\omega}(w,s) &= \frac{1}{\Gamma(w)}\int_1^\infty f_{\boldsymbol{\omega}}(x)x^{-s-1}(\log x)^{w-1}dx \\ &= \zeta_r(w,s,\boldsymbol{\omega}) \end{aligned}$$

since the condition $\mathrm{Re}(\omega_k) > 0$ for $k = 1,\dots,r$ implies

$$f_{\boldsymbol{\omega}}(x) = \sum_{\boldsymbol{n}\geq\mathbf{0}} x^{-\boldsymbol{n}\cdot\boldsymbol{\omega}}$$

for $x > 1$, where

$$\boldsymbol{n}\cdot\boldsymbol{\omega} = n_1\omega_1 + \cdots + n_r\omega_r.$$

Hence, we have

$$\begin{aligned} \boldsymbol{\Gamma}_r(s,\boldsymbol{\omega}) &= \exp\left(\frac{\partial}{\partial w}\zeta_r(w,s,\boldsymbol{\omega})\Big|_{w=0}\right) \\ &= \Gamma_r(s,\boldsymbol{\omega}) \end{aligned}$$

and

$$\mathbf{S}_r(s,\boldsymbol{\omega}) = S_r(s,\boldsymbol{\omega}).$$

Next, we prove Theorem 1. For $k = 1,\dots,r$, let

$$\omega_k' = \mathrm{sgn}(\mathrm{Re}(\omega_k))\omega_k.$$

Notice that $\mathrm{Re}(\omega_k') > 0$. Put $\boldsymbol{\omega}' = (\omega_1',\dots,\omega_r')$. Then we easily see that

$$f_{\boldsymbol{\omega}}(x) = (-1)^l f_{\omega'}(x)x^\Delta,$$

where $l = |\{k|\mathrm{Re}(\omega_k) < 0\}|$, $\Delta = \sum_{\mathrm{Re}(\omega_k)<0}\omega_k$. This implies that

$$\zeta_{f_\omega}(s) = \zeta_{f_{\omega'}}(s-\Delta)^{(-1)^l}.$$

Hence

$$\boldsymbol{\Gamma}_r(s,\boldsymbol{\omega}) = \boldsymbol{\Gamma}_r(s-\Delta,\boldsymbol{\omega}')^{(-1)^l}.$$

Then the above proof of Theorem 2 gives

$$\boldsymbol{\Gamma}_r(s,\boldsymbol{\omega}) = \Gamma_r(s-\Delta,\boldsymbol{\omega}')^{(-1)^l}$$

and

$$\mathbf{S}_r(s,\boldsymbol{\omega}) = S_r(s-\Delta,\boldsymbol{\omega}')^{(-1)^l}.$$

Thus, $\boldsymbol{\Gamma}_r(s,\boldsymbol{\omega})$ and $\mathbf{S}_r(s,\boldsymbol{\omega})$ are meromorphic in $s \in \mathbf{C}$.

3. Proof of Theorem 3. The absolute automorphic form

$$\begin{aligned} f_{2n}(x) &= (1-x^{-1})^{-n}(1-x)^{-n} \\ &= (-1)^n x^{-n}(1-x^{-1})^{-2n} \end{aligned}$$

gives

$$\begin{aligned} \boldsymbol{\Gamma}_{2n}(s,\pm) &= \zeta_{f_{2n}}(s) \\ &= \boldsymbol{\Gamma}_{2n}(s+n,(1,\dots,1))^{(-1)^n} \\ &= \Gamma_{2n}(s+n,(1,\dots,1))^{(-1)^n} \end{aligned}$$

and

$$\mathbf{S}_{2n}(s,\pm) = S_{2n}(s+n,(1,\dots,1))^{(-1)^n}.$$

Here, using

$$\mathbf{S}_{2n}(0,\pm) = S_{2n}(n,(1,\dots,1))^{(-1)^n} = 1$$

and

$$\begin{aligned} \frac{\mathbf{S}_{2n}'}{\mathbf{S}_{2n}}(s,\pm) &= (-1)^n\frac{S_{2n}'}{S_{2n}}(s+n,(1,\dots,1)) \\ &= \frac{(-1)^{n-1}}{(2n-1)!}\prod_{k=1}^{n-1}(s^2-k^2)\pi s\cot(\pi s) \end{aligned}$$

we have

$$\begin{aligned} &\mathbf{S}_{2n}(s,\pm) \\ &= \exp\left(\frac{(-1)^{n-1}}{(2n-1)!}\int_0^s\prod_{k=1}^{n-1}(u^2-k^2)\pi u\cot(\pi u)du\right). \end{aligned}$$

We notice that we used the following differential equation for $S_r(s,(1,\dots,1))$:

$$\begin{aligned} &S_r'(s,(1,\dots,1)) \\ &= S_r(s,(1,\dots,1))\binom{s-1}{r-1}(-1)^{r-1}\pi\cot(\pi s). \end{aligned}$$

4. Proof of Theorem 4 and Examples. To prove each identity

$$F_1(s) = F_2(s)$$

belonging to Theorem 4 and Examples it is sufficient to show

$$\begin{cases} F_1(0) = F_2(0) = 1, \\ \dfrac{F_1'}{F_1}(s) = \dfrac{F_2'}{F_2}(s). \end{cases}$$

For example, let

$$F_1(s) = \mathbf{S}_2(s,\pm)$$

and

$$F_2(s) = \mathcal{S}_2(s).$$

Then

$$\begin{cases} F_1(0) = 1 = F_2(0), \\ \dfrac{F_1'}{F_1}(s) = \pi s \cot(\pi s) = \dfrac{F_2'}{F_2}(s). \end{cases}$$

Hence

$$\mathbf{S}_2(s, \pm) = \mathcal{S}_2(s).$$

Proof of Theorem 4 and their examples are exactly similar by using

$$\prod_{k=1}^{n-1}(s^2 - k^2) = \sum_{m=1}^{n} c(n,m) s^{2m-2}$$

and

$$s^{2n-2} = \sum_{m=1}^{n} a(n,m) \frac{(-1)^{m-1}}{(2m-1)!} \prod_{k=1}^{m-1}(s^2 - k^2).$$

5. Proof of Theorem 5 and Examples. For simplicity, put

$$\begin{aligned} \mathbf{S}_{2n+1,k}(s) &= \mathbf{S}_{2n+1}(s, (\overbrace{1, \ldots, 1}^{2n+1-k}, \overbrace{-1, \ldots, -1}^{k})) \\ &= S_{2n+1}(s+k, (1, \ldots, 1))^{(-1)^k}. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{\mathbf{S}_{2n+1,k}'(s)}{\mathbf{S}_{2n+1,k}(s)} &= (-1)^k \frac{S_{2n+1}'(s+k, (1, \ldots, 1))}{S_{2n+1}(s+k, (1, \ldots, 1))} \\ &= \frac{(-1)^k}{(2n)!}(s+k-1) \cdots (s+k-2n) \\ &\quad \times \pi \cot(\pi s). \end{aligned}$$

We obtain the equality

$$\left(\log \frac{\mathbf{S}_{2n+1,n+1}(s)}{\mathbf{S}_{2n+1,n}(s)}\right)' = \frac{2(-1)^{n+1}}{(2n)!} \prod_{k=0}^{n-1}(s^2 - k^2) \pi \cot(\pi s)$$

from

$$\frac{\mathbf{S}_{2n+1,n+1}'(s)}{\mathbf{S}_{2n+1,n+1}(s)} = \frac{(-1)^{n+1}}{(2n)!}(s+n)s \prod_{k=0}^{n-1}(s^2 - k^2) \pi \cot(\pi s)$$

and

$$\frac{\mathbf{S}_{2n+1,n}'(s)}{\mathbf{S}_{2n+1,n}(s)} = \frac{(-1)^{n}}{(2n)!}(s-n)s \prod_{k=0}^{n-1}(s^2 - k^2) \pi \cot(\pi s).$$

Hence

$$\begin{aligned} \mathbf{S}_{2n+1}(s, \pm) &= \frac{\mathbf{S}_{2n+1,n+1}(s)}{\mathbf{S}_{2n+1,n}(s)} \\ &= C_{2n+1} \exp\left(\frac{2(-1)^{n+1}}{(2n)!} \int_0^s \prod_{k=0}^{n-1}(u^2 - k^2) \right. \\ &\quad \left. \times \pi \cot(\pi u) du \right), \end{aligned}$$

where

$$\begin{aligned} C_{2n+1} &= \mathbf{S}_{2n+1}(0, \pm) \\ &= \frac{\mathbf{S}_{2n+1,n+1}(0)}{\mathbf{S}_{2n+1,n}(0)} \\ &= \frac{S_{2n+1}(n+1, (1, \ldots, 1))^{(-1)^{n+1}}}{S_{2n+1}(n, (1, \ldots, 1))^{(-1)^n}} \\ &= S_{2n+1}(n, (1, \ldots, 1))^{2(-1)^{n+1}} \end{aligned}$$

since

$$\begin{aligned} S_{2n+1}(n, (1, \ldots, 1)) &= \Gamma_{2n+1}(n, (1, \ldots, 1))^{-1} \\ &\quad \times \Gamma_{2n+1}(n+1, (1, \ldots, 1))^{-1} \\ &= S_{2n+1}(n+1, (1, \ldots, 1)). \end{aligned}$$

Now, for $k \geq 1$

$$\Gamma_{2n+1}(k, (1, \ldots, 1)) = \exp(\varphi_{2n+1,k}'(0))$$

with

$$\begin{aligned} \varphi_{2n+1,k}(s) &= \sum_{l=0}^{\infty} \frac{(l+1) \cdots (l+2n)}{(2n)!} (l+k)^{-s} \\ &= \sum_{l=1}^{\infty} \frac{(l-k+1) \cdots (l-k+2n)}{(2n)!} l^{-s}. \end{aligned}$$

Hence

$$S_{2n+1}(n, (1, \ldots, 1)) = \exp(-\varphi_{2n+1}'(0)),$$

where

$$\begin{aligned} \varphi_{2n+1}(s) &= \varphi_{2n+1,n}(s) + \varphi_{2n+1,n+1}(s) \\ &= \frac{2}{(2n)!} \sum_{l=1}^{\infty} \left(\prod_{k=0}^{n-1}(l^2 - k^2) \right) l^{-s} \\ &= \frac{2}{(2n)!} \sum_{m=1}^{n} c(n,m) \zeta(s - 2m). \end{aligned}$$

Thus

$$\begin{aligned} S_{2n+1}(n, (1, \ldots, 1)) &= \exp\left(-\frac{2}{(2n)!} \sum_{m=1}^{n} c(n,m) \right. \\ &\quad \left. \times \zeta'(-2m) \right). \end{aligned}$$

These calculations imply Theorem 5 and Examples.

6. Views from absolute automorphic forms. Concluding the paper we explain fundamental absolute automorphic forms especially for Theorems 3–5. For $n \geq 2$ let

$$f_n(x) = \begin{cases} \frac{1}{(1-x)^m(1-x^{-1})^m} \\ \text{if } n = 2m \text{ is even,} \\ \frac{1}{(1-x)^{m+1}(1-x^{-1})^m} - \frac{1}{(1-x)^m(1-x^{-1})^{m+1}} \\ \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

Then $f_n(x)$ is an absolute automorphic form satisfying

$$f_n\left(\frac{1}{x}\right) = (-1)^n f_n(x).$$

Moreover, our construction of the absolute multiple sine function $\mathbf{S}_n(s, \pm)$ shows that it is quite simply given as

$$\mathbf{S}_n(s, \pm) = \varepsilon_{f_n}(s).$$

We expect future studies on multiple sine functions via absolute automorphic forms.

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