Modular forms of weight $3m$ and elliptic modular surfaces

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Abstract: We prove that the graded ring of modular forms of weight divisible by 3 is naturally isomorphic to a certain log canonical ring of the associated elliptic modular surface. This extends the Shioda correspondence between weight 3 cusp forms and holomorphic 2-forms.

Key words: Modular forms; elliptic modular surfaces; pluricanonical forms.

Let Γ be a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of finite index which does not contain -1. In [3], Shioda introduced the elliptic modular surface $\pi : S_\Gamma \to X_\Gamma$ over the (compactified) modular curve X_Γ associated to Γ, and proved, among other things, that the space of cusp forms of weight 3 with respect to Γ is canonically isomorphic to that of holomorphic 2-forms on the surface S_Γ ([3] Theorem 6.1). Our purpose is to extend this correspondence to that between modular forms of weight $3m$ and certain rational m-pluricanonical forms on S_Γ. This is parallel to the situation for weight $2m$ where modular forms of weight $2m$ correspond to m-pluricanonical forms on the curve X_Γ.

To state the result, let $\Delta \subset X_\Gamma$ be the (reduced) cusp divisor. A cusp is called *irregular* if its stabilizer in Γ contains an element conjugate to $\begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbf{Z})$, and *regular* otherwise. We write $\Delta = \Delta_{reg} + \Delta_{irr}$ for the corresponding decomposition. The singular fibres over Δ are divided accordingly, which we write $\pi^*\Delta = D_{reg} + D_{irr}$. By [3], D_{reg} consists of type I_n fibres, and D_{irr} consists of type I_n^* fibres (n depends on the cusps). We consider the $\mathbf{Q}$-divisor

$$D = D_{reg} + \frac{1}{2} D_{irr}.$$

We write $M_k(\Gamma)$ for the space of Γ-modular forms of weight k (cf. [1]).

Our main result is the following

Theorem 0.1. *We have a natural isomorphism of graded rings*

$$(0.1) \qquad \bigoplus_{m\geq 0} M_{3m}(\Gamma) \simeq \bigoplus_{m\geq 0} H^0(K_{S_\Gamma}^{\otimes m}(mD)).$$

Here $K_{S_\Gamma}^{\otimes m}(mD)$ should be understood as $K_{S_\Gamma}^{\otimes m}(mD_{reg} + [m/2]D_{irr})$. Explicitly, the isomorphism (0.1) is given by associating to $f(\tau) \in M_{3m}(\Gamma)$ the m-canonical form $f(\tau)(d\tau \wedge dz)^{\otimes m}$ where z is the uniformizing coordinate on the smooth fibres $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ of π.

Independently of our proof of Theorem 0.1, we can check that the m-th components of both sides of (0.1) indeed have the same dimension, which is given by

$$(0.2) \quad (3m-1)(g-1) + m\varepsilon_3 + \frac{3m}{2}\varepsilon_{reg} + \left[\frac{3m}{2}\right]\varepsilon_{irr}.$$

Here g is the genus of X_Γ, and $\varepsilon_3, \varepsilon_{reg}, \varepsilon_{irr}$ are the number of elliptic points, regular cusps and irregular cusps respectively. All elliptic points are of order 3 because $-1 \notin \Gamma$. It is classical that $\dim M_{3m}(\Gamma)$ is given by (0.2) (see [1] §3.5 and §3.6). On the other hand, $h^0(K_{S_\Gamma}^{\otimes m}(mD))$ can be calculated by expressing $K_{S_\Gamma}^{\otimes m}(mD_{reg} + [m/2]D_{irr})$ as the pullback of a line bundle L_m on X_Γ by the canonical bundle formula, and then computing $h^0(L_m)$ by the Riemann-Roch formula on X_Γ. This also gives (0.2). Anyway, our proof of Theorem 0.1 is conceptual and does not use this equality.

By imposing the vanishing condition mD on both sides of (0.1), we also obtain an isomorphism between the canonical ring $\oplus_{m\geq 0} H^0(K_{S_\Gamma}^{\otimes m})$ of S_Γ and the following subring of $\oplus_m M_{3m}(\Gamma)$:

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$$\bigoplus_{m\geq 0}\{f\in M_{3m}(\Gamma)\mid \mathrm{ord}_s(f)\geq m \text{ at every cusp } s\}.$$

Here we measure the vanishing order of f at a cusp $s=\gamma(i\infty)$, $\gamma\in \mathrm{SL}_2(\mathbf{Z})$, by the parameter $e^{2\pi i\tau/N}$ where N is the smallest positive integer such that $\begin{pmatrix}1 & N\\ 0 & 1\end{pmatrix}$ is contained in $\gamma^{-1}\Gamma\gamma$. (For irregular cusps, this is square root of local coordinate.) When $m=1$, this is the Shioda isomorphism $H^0(K_{S_\Gamma})\simeq S_3(\Gamma)$.

The isomorphism (0.1) looks analogous to the classical correspondence between modular forms of even weight and pluricanonical forms on X_Γ (see [1] §3.5), but some coefficients are different. Specifically, there is no contribution from the elliptic points to the pole condition, and the contributions from the regular and irregular cusps have different weights. The main reason is that around the singular fibres over the elliptic points and irregular cusps, after contracting some (-2)-curves, the projection from the universal family is unramified in codimension 1.

The proof of Theorem 0.1 is given in §1. In §2 we explain the interpretation of the Hecke operators on $M_{3m}(\Gamma)$ and the Petersson inner product on $S_3(\Gamma)$ in terms of the pluricanonical forms.

1. Proof. Before proceeding to the proof, let us first explain why modular forms and pluricanonical forms on elliptic modular surfaces correspond, at the level of period domain. We prefer to view the upper half plane as the open set of $\mathbf{P}^1$

$$\mathcal{D}=\{[\omega]\in\mathbf{P}^1(\mathbf{C})\mid \sqrt{-1}(\omega,\bar{\omega})>0\},$$

where $(\,,)$ is the symplectic form $((x_1,y_1),(x_2,y_2))=-x_1y_2+y_1x_2$ on $\mathbf{C}^2$. Let $\mathcal{L}=\mathcal{O}_{\mathbf{P}^1}(-1)|_{\mathcal{D}}$ be the tautological bundle over $\mathcal{D}$, endowed with the natural action of the group $\mathrm{GL}_2^+(\mathbf{R})$ of 2×2 matrices of determinant >0. Modular forms of weight k for $\Gamma<\mathrm{SL}_2(\mathbf{Z})$ are Γ-invariant sections of $\mathcal{L}^{\otimes k}$ with cusp condition.

Remark 1.1. The relation with the more traditional definition ([1]) is as follows. Pick a primitive vector $l\in\mathbf{Z}^2$, which corresponds to the rational boundary point $[l]\in\mathbf{P}^1(\mathbf{Q})$ of $\mathcal{D}$. Let s_l be the rational section of $\mathcal{O}_{\mathbf{P}^1}(-1)$ defined by the equation $(s_l([\omega]),l)=1$, where $s_l([\omega])\in\mathbf{C}\omega\subset\mathbf{C}^2$. Then s_l has a pole of order 1 at $[l]$. A choice of a vector $m\in\mathbf{Z}^2$ with $(m,l)=1$ induces an isomorphism $\iota_l:\mathbf{H}\to s_l(\mathcal{D})\simeq\mathcal{D}$ defined by $\tau\mapsto\tau l+m$. By the frame $s_l^{\otimes k}$ of $\mathcal{L}^{\otimes k}$ and this coordinate τ, we can identify sections of $\mathcal{L}^{\otimes k}$ with functions on $\mathbf{H}$. This gives the Fourier expansion of modular forms of weight k at the cusp $[l]$, and defines the condition of holomorphicity there.

For example, when $l=(1,0)$ and $m=(0,1)$, we have $\iota_l(\tau)=(\tau,1)$, so the action of $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ on $\mathcal{D}$ coincides with $\gamma(\tau)=(a\tau+b)/(c\tau+d)$ on $\mathbf{H}$ via ι_l, and the factor of automorphy of s_l by γ is $c\tau+d$.

Let $\underline{\mathbf{C}^2}=\mathbf{C}^2\times\mathcal{D}$ and $\underline{\mathbf{Z}^2}=\mathbf{Z}^2\times\mathcal{D}$ be the local systems over $\mathcal{D}$. Then $\mathcal{L}$ is naturally a sub line bundle of the vector bundle $\mathcal{E}=\mathcal{O}_{\mathcal{D}}^{\oplus 2}$ corresponding to $\underline{\mathbf{C}^2}$. The universal marked elliptic curve $\tilde{\pi}:\mathcal{S}\to\mathcal{D}$ over $\mathcal{D}$ is the quotient

$$\mathcal{S}=\mathcal{E}/(\mathcal{L}+\underline{\mathbf{Z}^2})\simeq\mathcal{L}^\vee/\underline{\mathbf{Z}^2}^\vee.$$

The second isomorphism is induced by the symplectic form $(\,,)$. If $\omega=(\tau,1)$, the fibre $(\mathbf{C}\omega)^\vee/(\mathbf{Z}^2)^\vee$ over $[\omega]$ is isomorphic to $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$ by the pairing with ω. The local systems $R^1\tilde{\pi}_*\mathbf{Z}$, $R^1\tilde{\pi}_*\mathbf{C}$ are identified with $\underline{\mathbf{Z}^2}$, $\underline{\mathbf{C}^2}$ respectively, and the Hodge bundle of $\mathcal{S}\to\mathcal{D}$ is identified with $\mathcal{L}$. Let $K_{\tilde{\pi}}$ be the relative canonical bundle of $\mathcal{S}/\mathcal{D}$. Since $K_{\tilde{\pi}}|_F\simeq K_F$ is trivial at each fibre F, $\tilde{\pi}_*K_{\tilde{\pi}}$ is an invertible sheaf on $\mathcal{D}$, and the natural homomorphism $(\tilde{\pi}_*K_{\tilde{\pi}})^{\otimes m}\to\tilde{\pi}_*(K_{\tilde{\pi}}^{\otimes m})$ is an isomorphism for any m.

We have two fundamental $\mathrm{SL}_2(\mathbf{Z})$-equivariant isomorphisms:

$$K_{\mathcal{D}}\simeq\mathcal{L}^{\otimes 2},\quad \tilde{\pi}_*K_{\tilde{\pi}}\simeq\mathcal{L}. \tag{1.1}$$

The first isomorphism is just $K_{\mathcal{D}}=K_{\mathbf{P}^1}|_{\mathcal{D}}\simeq\mathcal{O}_{\mathbf{P}^1}(-2)|_{\mathcal{D}}$. The second isomorphism is given by the period integral: integration of the 1-forms in $H^0(K_{\tilde{\pi}}|_F)=H^0(K_F)$ along the 1-cycles in each fibre F gives $H^0(K_F)\simeq H^{1,0}(F)$. Combining these isomorphisms, we obtain

$$\begin{aligned}\mathcal{L}^{\otimes 3m}&\simeq(\tilde{\pi}_*K_{\tilde{\pi}})^{\otimes m}\otimes K_{\mathcal{D}}^{\otimes m}\\ &\simeq\tilde{\pi}_*(K_{\tilde{\pi}}^{\otimes m}\otimes\tilde{\pi}^*K_{\mathcal{D}}^{\otimes m})\simeq\tilde{\pi}_*(K_{\mathcal{S}}^{\otimes m}).\end{aligned} \tag{1.2}$$

This is the source of (0.1). To summarize, modular forms correspond to two types of differential forms: (local) weight 2 forms to (local) base differentials, and (local) weight 1 forms to fibre differentials. Then (0.1) comes from the combination of these two correspondences.

Now let $\Gamma<\mathrm{SL}_2(\mathbf{Z})$ be a finite-index subgroup not containing -1. Taking the quotient of $\mathcal{S}\to\mathcal{D}$ by Γ and resolving the A_2-singularities arising from the fixed points, we obtain an elliptic fibration over $Y_\Gamma = X_\Gamma - \Delta$. The elliptic modular surface $\pi : S_\Gamma \to X_\Gamma$ is the nonsingular, relatively minimal extension of this fibration over X_Γ. (See [3] §4 for the construction of S_Γ.) Let us abbreviate $Y_\Gamma, X_\Gamma, S_\Gamma$ as Y, X, S respectively. We shall prove Theorem 0.1 in two steps.

1.1. Semi-stable case. We first consider the case where Γ has no elliptic point nor irregular cusp. In this case Γ acts on $\mathcal{D}$ freely and π has only singular fibres of type I_n over the regular cusps ([3] p. 35). For instance, $\Gamma(N)$ with $N > 2$ and more generally neat subgroups of $\mathrm{SL}_2(\mathbf{Z})$ satisfy this condition.

The Γ-equivariant bundles $K_{\mathcal{D}}$, $\tilde{\pi}_* K_{\tilde{\pi}}$ and $\mathcal{L}$ descend to line bundles on Y. We extend them over X as follows. The first one, K_Y, is extended to K_X. The second one is extended to $\pi_* K_\pi$ where K_π is the relative canonical bundle of π. Recall that local sections of $K_\pi|_F$ at a singular fibre F are identified with local 1-forms on $F \backslash \mathrm{Sing}(F)$ that have a pole of order ≤ 1 at each node and for which the sum of its residues is zero. Hence $K_\pi|_F \simeq \mathcal{O}_F$, so that $\pi_* K_\pi$ is still invertible and we have $(\pi_* K_\pi)^{\otimes m} \simeq \pi_*(K_\pi^{\otimes m})$.

The third one, the descent of $\mathcal{L}$, is extended as follows. Let l be a primitive vector of $\mathbf{Z}^2$ and s_l be the frame of $\mathcal{L}$ as in Remark 1.1. Since s_l is invariant under the stabilizer of l in Γ, it descends to a local frame of the descent of $\mathcal{L}$ near the cusp $[l]$. The extension over the cusp is defined so that this punctured local frame extends as a local frame. We write L for the extended line bundle on X. By construction, a local Γ-invariant section s of $\mathcal{L}$ extends holomorphically over the cusps as a local section of L if and only if $(s([\omega]), l)$ does not diverge as $[\omega] \to [l]$. By Remark 1.1, this coincides with the usual cusp condition for modular forms. Thus we have $M_k(\Gamma) = H^0(L^{\otimes k})$.

Now the isomorphisms (1.1) descend to Y as

$$K_Y \simeq L^{\otimes 2}|_Y, \quad \pi_* K_\pi|_Y \simeq L|_Y.$$

These isomorphisms extend over X to

$$K_X \simeq L^{\otimes 2}(-\Delta), \quad \pi_* K_\pi \simeq L.$$

Indeed, let $l \in \mathbf{Z}^2$ be a primitive vector and F be the singular fibre over the cusp $[l]$. The first isomorphism holds because the frame $s_l^{\otimes 2}$ of $\mathcal{L}^{\otimes 2}$ corresponds to the frame $d\tau$ of $K_{\mathbf{H}}$ up to constant via ι_l (both have a pole of order 2 at $[l]$), and we have $d\tau = q^{-1} dq$ up to constant for the local parameter $q = \exp(2\pi i \tau / N)$ around the cusp $[l]$. Next, for the second isomorphism, note that in the period map around $[l]$, the vanishing cycle near a node of F corresponds to the vector $l \in \mathbf{Z}^2$ (up to ± 1) because both are invariant under the stabilizer of l in Γ (= local monodromy around the cusp). The integral of a generator of $H^0(K_\pi|_F)$ along the vanishing cycle is equal to its residue at the node, whence nonzero. Therefore a local frame of $\pi_* K_\pi$ corresponds to a local frame of L under the isomorphism $\pi_* K_\pi|_Y \simeq L|_Y$.

To sum up, we obtain

$$\begin{aligned} L^{\otimes 3m} &\simeq (\pi_* K_\pi)^{\otimes m} \otimes K_X^{\otimes m}(m\Delta) \\ &\simeq \pi_*(K_\pi^{\otimes m} \otimes \pi^* K_X^{\otimes m}(mD)) \simeq \pi_*(K_S^{\otimes m}(mD)). \end{aligned}$$

Here recall that $D = D_{reg} = \pi^* \Delta$. Taking global sections give (0.1). Compatibility of the multiplications is obvious.

1.2. General case. We next study the general case. Let $\Gamma < \mathrm{SL}_2(\mathbf{Z})$ be a finite-index subgroup not containing -1. We choose a normal subgroup $\Gamma' \lhd \Gamma$ of finite index that has no elliptic point nor irregular cusp. We will abbreviate the elliptic modular surfaces $S_\Gamma \to X_\Gamma$ and $S_{\Gamma'} \to X_{\Gamma'}$ as $\pi: S \to X$ and $\pi': S' \to X'$ respectively. The quotient group $\bar{\Gamma} = \Gamma / \Gamma'$ acts on S' biregularly. S'/X' is the nonsingular, relatively minimal elliptic surface birational to the base change $X' \times_X S$ of S/X by the projection $f: X' \to X$. We observe this process of birational transformation around each singular fibre of π, with emphasis on the relation between the canonical divisors.

Let $p \in X$ be either an elliptic point or a cusp, and $F = \pi^* p$ be the singular fibre over p. Choose an arbitrary point $p' \in f^{-1}(p)$ and let $F' = (\pi')^* p'$ be the fibre over p'. Take a small neighborhood $V \subset X$ of p and let $V' \subset X'$ be the connected component of $f^{-1}(V)$ that contains p'. We write $U = \pi^{-1}(V)$ and $U' = (\pi')^{-1}(V')$. The stabilizer $G \subset \bar{\Gamma}$ of p' is cyclic since it is embedded in $\mathrm{GL}(T_{p'} X') \simeq \mathbf{C}^\times$. We have $V \simeq V'/G$. Write $d = |G|$.

(1) When p is a regular cusp, F is a type I_n fibre ([3] p. 35). The fibre product $U'' = V' \times_V U$ is normal and has A_{d-1}-singularities ([2] §7.5) at the n nodes of the central fibre F'' of $U'' \to V'$ (which is isomorphic to F). Then U' is the minimal resolution of those A_{d-1}-points, and so F' is of type I_{dn}. Let $\varphi: U' \to U''$ be the resolution map and $f: U'' \to U$ be the base change map. We have $K_{U'}(F') \simeq \varphi^*(K_{U''}(F''))$ because $F' = \varphi^* F''$ and $K_{U'} \simeq \varphi^* K_{U''}$,

where the second holds since A_{d-1}-points are canonical singularities ([2] §7.5). On the other hand, we have $K_{U''}(F'') \simeq f^*(K_U(F))$ by the ramification formula for f (see [2] §6.1). It follows that

$$
\begin{aligned}
(1.3)\quad H^0(U', K_{U'}^{\otimes m}(mF'))^G &= H^0(U'', K_{U''}^{\otimes m}(mF''))^G \\
&= H^0(U'', f^*(K_U^{\otimes m}(mF)))^G \\
&= H^0(U, K_U^{\otimes m}(mF)).
\end{aligned}
$$

The last isomorphism holds because f is the quotient map by G.

(2) When p is an irregular cusp, F is a type I_n^* fibre ([3] p. 35). Then d is an even number, say $d = 2d'$, because the generator $\gamma \sim \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$ of the stabilizer of p in Γ must satisfy $\gamma^d \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. (Here $\sim$ means conjugation in $\mathrm{SL}_2(\mathbf{Z})$.) Let G' be the subgroup of G of order d' and let $G'' = G/G' \simeq \mathbf{Z}/2$. We set $V'' = V'/G'$. Below we will consider the following factorization:

$$
\begin{tikzcd}
U' \arrow[r] \arrow[dd] & U'' \arrow[d] \arrow[dr, "f''"] & U \arrow[d, "\varphi"] \\
& V'' \times_V \bar{U} \arrow[r, "g"'] \arrow[d] & \bar{U} \arrow[d] \\
V' \arrow[r, "/G'"] & V'' \arrow[r, "/G''"] & V
\end{tikzcd}
$$

Let $\varphi\colon U \to \bar{U}$ be the contraction of the four components of F of multiplicity 1. The central fibre of $\bar{U} \to V$ is a multiple fibre of multiplicity 2, say $2\bar{F}$. Here $\bar{F}$ is a Weil divisor with $2\bar{F}$ Cartier. We take the fibre product $\bar{U}'' = V'' \times_V \bar{U}$ and write $g\colon \bar{U}'' \to \bar{U}$ for the natural map. Let U'' be the normalization of $\bar{U}''$.

Claim 1.2. U'' is nonsingular, the composition $f''\colon U'' \to \bar{U}'' \to \bar{U}$ is a double covering unramified outside the four A_1-points, and the central fibre F'' of $U'' \to V''$ is of type I_{2n}. In particular, U''/V'' is the nonsingular relatively minimal model of $\bar{U}''/V''$.

Proof. Let p be a point in the central fibre of $\bar{U}''$. When $g(p)$ is a nonsingular point of $\bar{F}$, the local equation of $\bar{U}''$ around p is given by $t^2 = x^2$ in $\mathbf{C}^3_{(x,y,t)}$. So $\bar{U}''$ consists of two smooth branches around p, which are separated by the normalization. The same picture holds in case $g(p)$ is a node of $\bar{F}$ but is nonsingular at $\bar{U}$, where the local equation of $\bar{U}''$ is given by $t^2 = x^2y^2$ in $\mathbf{C}^3_{(x,y,t)}$. Finally, when $g(p)$ is an A_1-point of $\bar{U}$, the germ of $(\bar{U}, g(p))$ is defined by $x^2 = ys$ in $\mathbf{C}^3_{(x,y,s)}$ with $\bar{U} \to V$ given by $(x,y,s) \to s$. So the germ of $(\bar{U}'', p)$ is defined by $x^2 = yt^2$ in $\mathbf{C}^3_{(x,y,t)}$. Its normalization is given by $\mathbf{C}^2 \to \mathbf{C}^3$, $(u,v) \mapsto (uv, u^2, v)$. The claim follows from these calculation of normalization. □

Now by the same argument as in case (1), we obtain

$$
K_{U''} + F'' = (f'')^*(K_{\bar{U}} + \bar{F}),
$$
$$
\varphi^*(K_{\bar{U}} + \bar{F}) = K_U + \frac{1}{2}F.
$$

Here $(f'')^*\bar{F}$ and $\varphi^*\bar{F}$ mean pullback of $\mathbf{Q}$-Cartier divisor. Since f'' is the quotient map by G'', this gives

$$
\begin{aligned}
H^0(U'', K_{U''}^{\otimes m}(mF''))^{G''} &= H^0(\bar{U}, K_{\bar{U}}^{\otimes m}(m\bar{F})) \\
&= H^0(U, K_U^{\otimes m}((m/2)F)).
\end{aligned}
$$

On the other hand, the relation of U'/V' and U''/V'' is the same as described in case (1). Hence we have a natural isomorphism

$$
(1.4)\quad H^0(U', K_{U'}^{\otimes m}(mF'))^G = H^0(U, K_U^{\otimes m}((m/2)F)).
$$

(3) When p is an elliptic point, F is of type IV^*, F' is smooth of j-invariant 0, and $d = 3$ (see [3] p. 35). In this case it is more convenient to look from F'. The group $G \simeq \mathbf{Z}/3$ acts on U' with three isolated fixed points in F'. The action around the fixed points are locally given by $(\tau, z) \to (\zeta_3^2\tau, \zeta_3 z)$ where $\zeta_3 = \exp(2\pi i/3)$. So the quotient $U'' = U'/G$ has three A_2-singularities ([2] §7.5), and the central fibre of $U'' \to V$ has multiplicity 3. Resolving these A_2-points, we obtain U. Let $f\colon U' \to U''$ be the quotient map by G and $\varphi\colon U \to U''$ be the resolution map. Since f is unramified in codimension 1, we have $K_{U'} \simeq f^*K_{U''}$. We also have $\varphi^*K_{U''} \simeq K_U$ because A_2-points are canonical singularities ([2] §7.5). As in case (1), these isomorphisms give

$$
\begin{aligned}
(1.5)\quad H^0(U', K_{U'}^{\otimes m})^G &= H^0(U', f^*K_{U''}^{\otimes m})^G \\
&= H^0(U'', K_{U''}^{\otimes m}) = H^0(U, K_U^{\otimes m}).
\end{aligned}
$$

Now we can complete the proof of Theorem 0.1. Let D' be the sum of the singular fibres of π'. By §1.1 we have an isomorphism

$$
\bigoplus_m M_{3m}(\Gamma') \simeq \bigoplus_m H^0(K_{S'}^{\otimes m}(mD')),
$$

which is $\bar{\Gamma}$-equivariant by construction. We take the $\bar{\Gamma}$-invariant part. On the one hand, we have

$$
M_{3m}(\Gamma')^{\bar{\Gamma}} = M_{3m}(\Gamma).
$$

On the other hand, by the local analysis (1.3), (1.4), (1.5), we see that

$$H^0(K_{S'}^{\otimes m}(mD'))^{\bar{\Gamma}} \simeq H^0(K_S^{\otimes m}(mD)).$$

This gives (0.1).

2. Remarks. Let Γ be a finite-index subgroup of $\mathrm{SL}_2(\mathbf{Z})$ not containing -1. The Hecke operators on $M_{3m}(\Gamma)$ (for Γ congruence subgroup) and the Petersson scalar product on $S_3(\Gamma)$ have natural interpretations in terms of the pluricanonical forms. The weight 3 case must be well-known, but we have included it here since we could not find a suitable reference.

2.1. Hecke operators. Assume that Γ is a congruence subgroup, and let α be a 2×2 matrix with integral coefficients and $\det(\alpha) > 0$. Putting $\Gamma_\alpha = \Gamma \cap \alpha\Gamma\alpha^{-1}$ and $\Gamma^\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$, we have the self correspondence

$$X_\Gamma \xleftarrow{\pi_1} X_{\Gamma_\alpha} \xleftarrow{\alpha} X_{\Gamma^\alpha} \xrightarrow{\pi_2} X_\Gamma \tag{2.1}$$

of X_Γ, where π_i are the projections and α is the isomorphism induced from the α-action on $\mathcal{D}$. This induces the Hecke operator (see [1] §5.1)

$$[\Gamma\alpha\Gamma]_k := \det(\alpha)^{k-1}\pi_{2*}\circ\alpha^*\circ\pi_1^*$$

on $M_k(\Gamma)$. Here α^* is the pullback by the α-action on $\mathcal{L}^{\otimes k}$, namely the k-th power of the α-action on $\mathcal{L}$ induced from the natural action of α on $\mathbf{C}^2$.

On the other hand, the curve correspondence (2.1) lifts to the rational correspondence of the elliptic modular surfaces

$$S_\Gamma \overset{\hat{\pi}_1}{\dashleftarrow} S_{\Gamma_\alpha} \overset{\hat{\alpha}}{\dashleftarrow} S_{\Gamma^\alpha} \overset{\hat{\pi}_2}{\dashrightarrow} S_\Gamma. \tag{2.2}$$

Here $\hat{\pi}_i$ are the base change maps; $\hat{\alpha}$ is induced from the natural α-action on the bundle $\underline{\mathbf{C}^2}$ and gives an isogeny of degree $\det(\alpha)$ at each smooth fibre. The indeterminacy points and ramification divisors of $\hat{\alpha}, \hat{\pi}_1, \hat{\pi}_2$ are contained in the singular fibres and their preimage. The correspondence (2.2) induces the endomorphism $\hat{\pi}_{2*}\circ\hat{\alpha}^*\circ\hat{\pi}_1{}^*$ on rational pluricanonical forms on S_Γ. Here $\hat{\pi}_{2*}$ takes the fibre sum of rational pluricanonical forms on S_{Γ^α} by the finite dominant rational map $\hat{\pi}_2$.

Proposition 2.1. *Via the isomorphism* (0.1)*, the weight* $3m$ *Hecke operator* $[\Gamma\alpha\Gamma]_{3m}$ *agrees with the endomorphism* $\det(\alpha)^{2m-1}\hat{\pi}_{2*}\circ\hat{\alpha}^*\circ\hat{\pi}_1{}^*$ *of* $H^0(K_{S_\Gamma}^{\otimes m}(mD))$.

Proof. It is clear that the pullbacks $\pi_1^*, \hat{\pi}_1{}^*$ agree. Also π_{2*} and $\hat{\pi}_{2*}$ agree because the pushforward from elliptic modular surfaces to modular curves (cf. (1.2)) translates the trace map $\hat{\pi}_{2*}$ for surfaces to the trace map π_{2*} for curves. We verify that the pullback $\hat{\alpha}^*$ of rational m-canonical forms by $\hat{\alpha}$ corresponds to the $\det(\alpha)^m$ multiple of the pullback α^* of weight $3m$ modular forms by α. Since both operators are locally defined and compatible with multiplication, we only have to check this for $m = 1$ and at the level of period domain $\mathcal{D}$. Under the isomorphism $\mathcal{L}^{\otimes 2} \simeq K_{\mathcal{D}}$, the pullback of weight 2 modular forms agrees with the pullback of 1-forms on $\mathcal{D}$. On the other hand, under $\mathcal{L} \simeq \tilde{\pi}_* K_{\tilde{\pi}}$, the pullback of weight 1 modular forms agrees with the $\det(\alpha)^{-1}$ multiple of the pullback of 1-forms on the fibres by the isogeny $\alpha : \mathcal{S}_{[\omega]} \to \mathcal{S}_{[\alpha\omega]}$, because α multiplies the symplectic form by $\det(\alpha)$. □

2.2. Petersson inner product in weight 3. Let $(\,,)_{\mathcal{L}}$ be the $\mathrm{SL}_2(\mathbf{R})$-invariant Hermitian metric on $\mathcal{L}$ that corresponds to (half) the Hodge metric

$$(\omega_1, \omega_2) = \frac{\sqrt{-1}}{2}\int_F \omega_1 \wedge \bar{\omega_2}, \quad \omega_i \in H^0(K_F)$$

in each fibre $\mathcal{L}_{[\omega]} = H^0(K_F)$ where $F = \mathcal{S}_{[\omega]}$. Let 2Ω be the Kähler form of the metric induced on $\mathcal{L}^{\otimes -2} \simeq T_{\mathcal{D}}$. In the upper half plane model, Ω is expressed as $y^{-2}dx\wedge dy$. The Petersson inner product on $S_3(\Gamma)$ is defined by

$$(f, g) = \int_{Y_\Gamma} (f, g)_{\mathcal{L}^{\otimes 3}}\Omega, \quad f, g \in S_3(\Gamma).$$

Via the trivialization of $\mathcal{L}$ given by the frame $\iota_{(1,0)} : \mathbf{H} \to \mathcal{L}$ (see Remark 1.1), this agrees with the classical definition ([1] §5.4)

$$\int_{Y_\Gamma} f(\tau)\overline{g(\tau)}y dx\wedge dy.$$

Proposition 2.2. *Via the Shioda isomorphism* $S_3(\Gamma) \simeq H^0(K_{S_\Gamma})$*, the Petersson inner product agrees with the* (1/4*-scaled*) *Hodge metric on* $H^0(K_{S_\Gamma})$

$$(\omega, \eta) = \frac{1}{4}\int_{S_\Gamma} \omega\wedge\bar{\eta}, \quad \omega, \eta \in H^0(K_{S_\Gamma}). \tag{2.3}$$

Proof. We have $(\eta_1, \eta_2)_{\mathcal{L}^{\otimes 2}} 2\Omega = \sqrt{-1}\eta_1\wedge\bar{\eta}_2$ for 1-forms η_i on $\mathcal{D}$. If $f, g \in S_3(\Gamma)$ correspond to 2-forms which locally can be written in the form $\tilde{\pi}^*\eta_1\otimes\omega_1$ and $\tilde{\pi}^*\eta_2\otimes\omega_2$, we locally have the equality of $(1,1)$-forms

$$(f,g)_{\mathcal{L}^{\otimes 3}}\Omega = -\frac{1}{4}\left(\int_F \omega_1 \wedge \bar{\omega_2}\right)\eta_1 \wedge \bar{\eta_2}.$$

Integration of the right-hand side over Y_Γ gives (2.3). (Here the minus sign cancels when exchanging $\bar{\omega_2}$ and η_1.) □

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