A generalization of the Tutte polynomials

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Abstract: In this paper, we introduce the concept of the Tutte polynomials of genus g and discuss some of its properties. We note that the Tutte polynomials of genus one are well-known Tutte polynomials. The Tutte polynomials are matroid invariants, and we claim that the Tutte polynomials of genus g are also matroid invariants. The main result of this paper and the forthcoming paper declares that the Tutte polynomials of genus g are complete matroid invariants.

Key words: Matroid; Tutte polynomial.

1. Introduction. This is an announcement paper.

Let E be a set. A *matroid* M on $E = E(M)$ is a pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a non-empty family of subsets of E with the properties

$$\begin{cases} \text{(i)} & \text{if } I \in \mathcal{I} \text{ and } J \subset I, \text{ then } J \in \mathcal{I}; \\ \text{(ii)} & \text{if } I_1, I_2 \in \mathcal{I} \text{ and } |I_1| < |I_2|, \\ & \text{then there exists } e \in I_2 \setminus I_1 \\ & \text{such that } I_1 \cup \{e\} \in \mathcal{I}. \end{cases}$$

Each element of the set $\mathcal{I}$ is called an *independent set*. A matroid $(E, \mathcal{I})$ is *isomorphic* to another matroid $(E', \mathcal{I}')$ if there is a bijection φ from E to E' such that $\varphi(I) \in \mathcal{I}'$ holds for each member $I \in \mathcal{I}$, and $\varphi^{-1}(I') \in \mathcal{I}$ holds for each member $I' \in \mathcal{I}'$.

It follows from the second axiom that all maximal independent sets in a matroid M take the same cardinality, called the *rank* of M. These maximal independent sets $\mathcal{B}(M)$ are called the *bases* of M. The *rank* $\rho(A)$ of an arbitrary subset A of E is the cardinality of the largest independent set contained in A.

We provide examples below.

Example 1.1. Let A be a $k \times n$ matrix over a finite field $\mathbf{F}_q$. This offers a matroid M on the set

$$E = \{z \in \mathbf{Z} \mid 1 \leq z \leq n\}$$

in which a set I is independent if and only if the family of columns of A whose indices belong to I is linearly independent. Such a matroid is called a vector matroid.

Example 1.2. Let

$$E = \{z \in \mathbf{Z} \mid 1 \leq z \leq n\}$$

and r a natural number with $1 \leq r \leq n$. We define a matroid on E by taking every r-element subset of E to be a basis. This is known as the uniform matroid $U_{r,n}$.

The classification of the matroids is one of the most important problems in the theory of matroids. One tool to classify the matroids is the Tutte polynomials. Let M be a matroid on the set E having a rank function ρ. The Tutte polynomial of M is defined as follows [2–4]:

$$T(M) := T(M; x, y) := \sum_{A \subset E} (x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)}.$$

For example, the Tutte polynomial of the uniform matroid $U_{r,n}$ is

$$T(U_{r,n}; x, y) = \sum_{i=0}^{r} \binom{n}{i} (x-1)^{r-i} + \sum_{i=r+1}^{n} \binom{n}{i} (y-1)^{i-r}.$$

It is easy to demonstrate that $T(M; x, y)$ is a matroid invariant. Two matroids are called *T-equivalent* if their Tutte polynomials are equivalent.

It is well known that there exist two inequivalent matroids, which are T-equivalent [5, p. 269]. We provide more examples below. Let

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$$E := \{z \in \mathbf{Z} \mid 1 \leq z \leq n\}.$$

Let us define the subsets X_1, X_2, X_3 of E as follows:

$$\begin{cases} X_1 := \{z \in \mathbf{Z} \mid 1 \leq z \leq r\}; \\ X_2 := \{z \in \mathbf{Z} \mid r+1 \leq z \leq 2r\}; \\ X_3 := \{z \in \mathbf{Z} \mid r \leq z \leq 2r-1\}. \end{cases}$$

Let $R_{r,n}$ denote the matroid on E such that

$$\mathcal{B}(R_{r,n}) = \mathcal{B}(U_{r,n}) \setminus \{X_1, X_2\},$$

$Q_{r,2n}$ denote the matroid on E such that

$$\mathcal{B}(Q_{r,n}) = \mathcal{B}(U_{r,n}) \setminus \{X_1, X_3\}.$$

Then, for $2r \leq n$, $r \geq 3$, $R_{r,n}$ and $Q_{r,n}$ are matroids. Both matroids have exactly two dependent sets of size r, namely $\{X_1, X_2\}$ of $R_{r,n}$ and $\{X_1, X_3\}$ of $Q_{r,n}$. Therefore, if $R_{r,n}$ and $Q_{r,n}$ are isomorphic, there exists φ such that

$$\varphi(\{X_1, X_2\}) = \{X_1, X_3\}.$$

This is a contradiction since φ is bijective, and we know that $R_{r,n}$ and $Q_{r,n}$ are non-isomorophic matroids.

On the other hand,

$$T(R_{r,n}) = T(Q_{r,n}).$$

In fact, the difference between

$$T(U_{r,n}) - T(R_{r,n})$$

and

$$T(U_{r,n}) - T(Q_{r,n})$$

are zero since $R_{r,n}$ and $Q_{r,n}$ are obtained from $U_{r,n}$ after deleting the two maximal independent sets.

This gives rise to a natural question: is there a generalization of the Tutte polynomial which identifies such T-equivalent but inequivalent matroids? This paper aims to provide a candidate generalization that answers this. In Section 2, we provide the concept of the Tutte polynomial of genus g. In Section 3, we provide the main results. The details of the proofs will be presented in our forthcoming paper [1].

2. Tutte polynomials of genus *g*. We now present the concept of the Tutte polynomial of genus g.

Definition 2.1. Let $M := (E, \mathcal{I})$ be a matroid. Let

$$\Lambda_1 := \{z \in \mathbf{Z} \mid 1 \leq z \leq g\}$$

and let

$$\Lambda_2 := \binom{\Lambda_1}{2}.$$

For every element $\lambda \in \Lambda_2$, and $A_i \subset E$, let us denote

$$A_{\cap(\lambda)} := \cap_{i\in\lambda} A_i \text{ and } A_{\cup(\lambda)} := \cup_{i\in\lambda} A_i.$$

Let g be a natural number. Then, the genus g of the Tutte polynomial $T^{(g)}(M)$ of the matroid M will be defined as follows:

$$\begin{aligned} T^{(g)}(M) &:= T^{(g)}(M; x_{\lambda_1}, y_{\lambda_1}, x_{\cap\lambda_2}, y_{\cap\lambda_2}, \\ &\qquad x_{\cup\lambda_2}, y_{\cup\lambda_2} : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2) \\ &:= \sum_{A_1,\dots,A_g \subseteq E} \prod_{\lambda\in\Lambda_1} (x_\lambda - 1)^{\rho(M)-\rho(A_\lambda)} \\ &\qquad (y_\lambda - 1)^{|A_\lambda|-\rho(A_\lambda)} \\ &\qquad \prod_{\lambda\in\Lambda_2} (x_{\cap(\lambda)} - 1)^{\rho(M)-\rho(A_{\cap(\lambda)})} \\ &\qquad (y_{\cap(\lambda)} - 1)^{|A_{\cap(\lambda)}|-\rho(A_{\cap(\lambda)})} \\ &\qquad \prod_{\lambda\in\Lambda_2} (x_{\cup(\lambda)} - 1)^{\rho(M)-\rho(A_{\cup(\lambda)})} \\ &\qquad (y_{\cup(\lambda)} - 1)^{|A_{\cup(\lambda)}|-\rho(A_{\cup(\lambda)})}. \end{aligned}$$

It is easy to demonstrate that $T^{(g)}(M)$ is matroid invariant and if two matroids have the same Tutte polynomial for genus g, we call them $T^{(g)}$-equivalent. For example, the genus 2 for the Tutte polynomial $T^{(2)}(M)$ of the matroid M is as follows:

$$\begin{aligned} &T^{(2)}(M; x_1, x_2, y_1, y_2, x_{\cap\{12\}}, y_{\cap\{12\}}, x_{\cup\{12\}}, y_{\cup\{12\}}) \\ &= \sum_{A_1, A_2 \subset E} (x_1 - 1)^{\rho(E)-\rho(A_1)} \\ &\qquad (x_2 - 1)^{\rho(E)-\rho(A_2)} \\ &\qquad (y_1 - 1)^{|A_1|-\rho(A_1)} \\ &\qquad (y_2 - 1)^{|A_2|-\rho(A_2)} \\ &\qquad (x_{\cap\{1,2\}} - 1)^{\rho(E)-\rho(A_1\cap A_2)} \\ &\qquad (y_{\cap\{1,2\}} - 1)^{|A_1\cap A_2|-\rho(A_1\cap A_2)} \\ &\qquad (x_{\cup\{1,2\}} - 1)^{\rho(E)-\rho(A_1\cup A_2)} \\ &\qquad (y_{\cup\{1,2\}} - 1)^{|A_1\cup A_2|-\rho(A_1\cup A_2)}. \end{aligned}$$

We remark that for $g \in \mathbf{N}_{\geq 2}$, the specialization of $T^{(g)}(M)$ is $T^{(g-1)}(M)$. For example,

$$T^{(2)}(M; x_1, 2, y_1, 2, 2, 2, 2, 2) = 2^{|E|} T^{(1)}(M; x_1, y_1).$$

Therefore, if

$$T^{(g)}(M) = T^{(g)}(M')$$

then

$$T^{(i)}(M) = T^{(i)}(M'),$$

for all $1 \leq i \leq g$.

3. Main results. The main result of the present paper is as follows:

Theorem 3.1.

(a) *The Tutte polynomial of genus g* $\{T^{(g)}\}_{g=1}^{\infty}$ *is a complete invariant for matroids.*

(b) *For* $2r \leq n$, $r \geq 3$,

$$T^{(2)}(R_{r,n}) \neq T^{(2)}(Q_{r,n}).$$

(c) *Let*

$$E := \{z \in \mathbf{Z} \mid 1 \leq z \leq 4n\},$$

with $n \geq 3$. *Let us define the subsets* Y_1, Y_2 *of* 2^E *as follows*:

$$\begin{aligned} Y_1 := \{&\{1,2,3\},\{3,4,5\},\ldots,\\ &\{2n-3,2n-2,2n-1\},\\ &\{2n-1,2n,1\},\\ &\{2n+1,2n+2,2n+3\},\\ &\{2n+3,2n+4,2n+5\},\ldots,\\ &\{4n-3,4n-2,4n-1\},\\ &\{4n-1,4n,2n+1\}\},\\ Y_2 := \{&\{1,2,3\},\{3,4,5\},\ldots,\\ &\{2n-3,2n-2,2n-1\},\\ &\{2n-1,2n,2n+1\},\\ &\{2n+1,2n+2,2n+3\},\\ &\{2n+3,2n+4,2n+5\},\ldots,\\ &\{4n-3,4n-2,4n-1\},\\ &\{4n-1,4n,1\}\}. \end{aligned}$$

Let S_{4n} *denote the independence system on* E *such that*

$$\mathcal{B}(S_{4n}) = \mathcal{B}(U_{3,4n}) \setminus Y_1,$$

S'_{4n} *denote the independence system on* E *such that*

$$\mathcal{B}(S'_{4n}) = \mathcal{B}(U_{3,4n}) \setminus Y_2.$$

Then, S_{4n} *and* S'_{4n} *are matroids. Let*

$$m_1 = \left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor \text{ and } m_2 = 2\lceil \sqrt{n} \rceil.$$

We have

$$\begin{cases} T^{(m_1)}(S_{4n}) = T^{(m_1)}(S'_{4n}); \\ T^{(m_2)}(S_{4n}) \neq T^{(m_2)}(S'_{4n}). \end{cases}$$

In particular, for a matroid M, $T^{(|\mathcal{B}(M)|)}(M)$ determines M. For detailed explanation, see [1].

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