Classification of irreducible symmetric spaces which admit standard compact Clifford–Klein forms

By Koichi TOJO

Graduate School of Mathematical Science, The University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

(Communicated by Masaki KASHIWARA, M.J.A., Jan. 15, 2019)

Abstract: We give a classification of irreducible symmetric spaces which admit standard compact Clifford–Klein forms. The method uses representation theory over the real number field and the criterion for properness and cocompactness of the action on homogeneous spaces due to T. Kobayashi.

Key words: Clifford–Klein form; tangential homogeneous space; symmetric space; properness criterion; discontinuous group; real representation.

1. Introduction and main theorem. Let G be a Lie group, H a closed subgroup of G and Γ a discrete subgroup of G. If Γ acts on a homogeneous space G/H properly discontinuously and freely, then the double coset space $\Gamma\backslash G/H$ has a natural manifold structure. In this case, the double coset space $\Gamma\backslash G/H$ with the manifold structure is called a Clifford–Klein form of G/H and Γ is called a discontinuous group for G/H.

In the late 1980s, a systematic study of Clifford–Klein forms for non-Riemannian homogeneous spaces was initiated by T. Kobayashi. One of important problems in the field is the following:

Problem 1.1 ([Ko89], [Ko01, Problem B]). Which homogeneous space G/H admits a compact Clifford–Klein form?

Our interest is in the case where G/H is of *reductive type*, namely, where $G \supset H$ are both real reductive linear Lie groups. In this case, G/H carries naturally a pseudo-Riemannian structure for which G acts isometrically.

Not every homogeneous space G/H carries a large discontinuous group when H is noncompact because the action of an isometric discrete group is not necessarily properly discontinuous. A useful method to construct large discontinuous groups is to use continuous analogue of discontinuous groups.

Definition 1.2 (standard Clifford–Klein form [KK16, Definition 1.4]). Let G/H be a homogeneous space of reductive type and Γ a discontinuous group for G/H. A Clifford–Klein form $\Gamma\backslash G/H$ is called *standard* if there exists a reductive subgroup L containing Γ and acting on G/H properly.

Some homogeneous spaces admit compact Clifford–Klein forms by using continuous analogue as follows:

Fact 1.3 ([Ko89]). Let G/H be a homogeneous space of reductive type. If there exists a reductive subgroup L of G acting on G/H properly and cocompactly, then G/H admits a standard compact Clifford–Klein form $\Gamma\backslash G/H$ by taking any torsion-free uniform lattice Γ of L.

Thus Fact 1.3 gives an affirmative answer to Problem 1.1 for G/H admitting such a reductive subgroup L. Conversely, the following conjecture was proposed by T. Kobayashi.

Conjecture 1.4 ([Ko01, Conjecture 4.3], [KY05, Conjecture 3.3.10]). Let G/H be a homogeneous space of reductive type. If G/H admits a compact Clifford–Klein form, then G/H admits a standard compact Clifford–Klein form.

No counterexample to Conjecture 1.4 has been known so far. (We remark that not all compact Clifford–Klein forms are standard: it may happen that a deformation of a standard compact Clifford–Klein form yields a nonstandard compact Clifford–Klein form [G85], [Ko98].) On the other hand, many evidences supporting the conjecture have been obtained by T. Kobayashi, K. Ono, R. J. Zimmer, R. Lipsman, Y. Benoist, F. Labourie, K. Corlette, S. Mozes, G. A. Margulis, H. Oh, D. Witte,

2010 Mathematics Subject Classification. Primary 22F30, 57S30; Secondary 22E46, 53C30, 53C35.

doi: 10.3792/pjaa.95.11

Table I. Symmetric spaces G/H that admit proper and cocompact actions of reductive subgroups L

	G/H	L
1	$SU(2,2n)/Sp(1,n)$	$U(1,2n)$
2	$SU(2,2n)/U(1,2n)$	$Sp(1,n)$
3	$SO(2,2n)/U(1,n)$	$SO(1,2n)$
4	$SO(2,2n)/SO(1,2n)$	$U(1,n)$
5	$SO(4,4n)/SO(3,4n)$	$Sp(1,n)$
6	$SO(4,4)/SO(4,1)\times SO(3)$	$Spin(4,3)$
7	$SO(4,3)/SO(4,1)\times SO(2)$	$G_{2(2)}$
8	$SO(8,8)/SO(7,8)$	$Spin(1,8)$
9	$SO(8,\mathbf{C})/SO(7,\mathbf{C})$	$Spin(1,7)$
10	$SO(8,\mathbf{C})/SO(7,1)$	$Spin(7,\mathbf{C})$
11	$SO^*(8)/U(3,1)$	$Spin(1,6)$
12	$SO^*(8)/SO^*(6)\times SO^*(2)$	$Spin(1,6)$

T. Yoshino, T. Okuda, N. Tholozan and Y. Morita among others, see [Ko89], [KO90], [Ko92a], [Ko92b], [Z94], [Li95], [Bn96], [LMZ95], [C94], [Ma97], [OW00], [KY05], [Ok13], [Th], [Mo15].

If Conjecture 1.4 were proved to be true, the answer to the following question would give a solution to Problem 1.1.

Question 1.5. Classify homogeneous spaces G/H of reductive type which admit standard compact Clifford–Klein forms.

The goal of this paper is to give an answer to Question 1.5 for irreducible symmetric spaces G/H.

In the case where G/H is a symmetric space, T. Kobayashi discovered that 5 series and 7 sporadic types of non-Riemannian symmetric spaces admit compact Clifford–Klein forms by using Fact 1.3.

Fact 1.6 ([KY05, Corollary 3.3.7])**.** Symmetric spaces G/H in Table I admit standard compact Clifford–Klein forms. Here $n = 1, 2, \cdots$.

From now on, we assume that G is a linear noncompact semisimple Lie group. For a Cartan involution θ of G, we write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G and set $K := G^\theta = \{g \in G : \theta(g) = g\}$. Then K is a maximal compact subgroup of G and G/K is a Riemannian symmetric space.

Theorem 1.7. *Let G be a linear noncompact semisimple Lie group and G/H an irreducible symmetric space. If G/H admits a standard compact Clifford–Klein form, then G/H is locally isomorphic to one of the following*:

- *a Riemannian symmetric space G/K,*
- *a group manifold $G' \times G'/\operatorname{diag} G'$,*
- *one of the homogeneous spaces in Table I.*

2. Outline of proof of Theorem 1.7. Among all irreducible symmetric spaces [Br57], we prove that there are not many candidates for irreducible symmetric spaces G/H that admit standard compact Clifford–Klein forms other than those listed in Theorem 1.7. This statement is formulated as follows:

Proposition 2.1. *Let G be a simple Lie group and G/H a symmetric space with noncompact H. If G/H admits a standard compact Clifford–Klein form, then G/H is locally isomorphic to one of the homogeneous spaces listed in Table I or in the following symmetric spaces*:

- $SO(p,q+1)/SO(p,q)$ $(1 \le q < HR(p))$,
- $SO(p,q+1)/SO(p,1)\times SO(q)$ $(1 \le q < HR(p))$,
- $SU(2p,2q)/Sp(p,q)$ $(1 \le q \le p)$,
- $E_{6(-14)}/F_{4(-20)}$.

Here $HR(p)$ denotes the Hurwitz–Radon number defined by

$$HR(p) := 8\alpha + 2^\beta,$$

where $\alpha \in \mathbf{Z}_{\ge 0}$ and $\beta \in \{0,1,2,3\}$ are determined by the following equation

$$p = 2^{4\alpha+\beta}\cdot(\text{odd number}).$$

The proof of Proposition 2.1 uses Facts 2.2 and 2.3 below.

Fact 2.2 ([Ko89, Example 4.11])**.** The symmetric space $Sp(2n,\mathbf{R})/Sp(n,\mathbf{C})$ does not admit compact Clifford–Klein forms for any positive integer n.

Fact 2.3 ([KY05])**.** Let G/H be a homogeneous space of reductive type. If G/H admits a standard compact Clifford–Klein form, then its tangential homogeneous space G_θ/H_θ admits a compact Clifford–Klein form. Here G_θ and H_θ are the Cartan motion groups of G and H, respectively, defined by using a Cartan involution θ of G such that $\theta(H) = H$ as follows:

$$G_\theta := K \ltimes \mathfrak{p},$$
$$H_\theta := (K\cap H) \ltimes (\mathfrak{p}\cap\mathfrak{h}).$$

For tangential homogeneous spaces, a criterion for the existence of compact Clifford–Klein forms was obtained in [KY05].

For the three families of symmetric spaces in Proposition 2.1 except for $E_{6(-14)}/F_{4(-20)}$, the method of our proof for Theorem 1.7 is the following:

(A) Kobayashi's criterion for a reductive subgroup L to act properly on G/H [Ko89],
(B) Kobayashi's criterion for a reductive subgroup L to act cocompactly on G/H [Ko89],
(C) an upper bound of the dimension of representations of the "primary simple factor" of L,
(D) a generalization of Iwahori's criterion for finite dimensional representations of real semisimple Lie algebras to admit certain structures (see Proposition 2.9).

The step (A) concerns the criterion for properness of the action in the setting that G is a linear reductive Lie group and H, L are reductive subgroups of G.

Fact 2.4 (properness criterion, [Ko89, Theorem 4.1])**.** We fix a Cartan involution of G and take a maximally split abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$. Take Cartan involutions θ_1 and θ_2 of G such that $\theta_1(H) = H$ and $\theta_2(L) = L$. Take maximal abelian subspaces $\mathfrak{a}'_H \subset \mathfrak{h}^{-\theta_1}$ and $\mathfrak{a}'_L \subset \mathfrak{l}^{-\theta_2}$. Then we can and do take $S_1, S_2 \in \mathrm{Int}(\mathfrak{g})$ such that $\mathfrak{a}_H := S_1(\mathfrak{a}'_H), \mathfrak{a}_L := S_2(\mathfrak{a}'_L) \subset \mathfrak{a}$. Then the following two conditions on the triple G, H and L are equivalent:

(i) the natural action of L on G/H is proper,
(ii) $\mathfrak{a}_H \cap \mathfrak{a}_L = \{0\}$ modulo W-actions.

Here $W = W(\mathfrak{g}, \mathfrak{a})$ is the Weyl group coming from the restricted root system of $\mathfrak{g}$ with the maximally split abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$.

Remark 2.5 ([Ko89, Corollary 4.2])**.** If the L-action on G/H is proper, then the following inequality holds:

$$\mathrm{rank}_{\mathbf{R}}\, L \leq \mathrm{rank}_{\mathbf{R}}\, G - \mathrm{rank}_{\mathbf{R}}\, H.$$

For the step (B), to state the criterion of the cocompactness, let us recall the noncompact dimension $d(G)$ of a linear reductive Lie group G, which is defined by

$$d(G) := \dim G/K = \dim \mathfrak{p}.$$

Fact 2.6 (cocompactness criterion, [Ko89, Theorem 4.7])**.** In the same setting, under the assumption that the L-action on G/H is proper, the following two conditions on the triple G, H and L are equivalent:

(i) $L\backslash G/H$ is compact,
(ii) $d(G) = d(L) + d(H)$.

Owing to Facts 2.4 and 2.6, Question 1.5 is reduced to classifying G/H that admits a reductive subgroup L of G such that

$$\begin{cases} \mathfrak{a}_L \cap \mathfrak{a}_H = \{0\} \text{ modulo } W\text{-actions} \\ d(L) = d(G) - d(H). \end{cases}$$

Suppose there exists such a subgroup L. Let n be the dimension of the natural representation of G. We write $\rho : \mathfrak{l} \to \mathfrak{sl}(n, \mathbf{C})$ for the differential of the composition $L \to G \subset GL(n, \mathbf{C})$. We shall find a numerical necessary condition for the pair $(\mathfrak{l}, \rho)$, namely, find an upper bound of the dimension of representations of the "primary simple factor" of L.

Definition 2.7. Let $\mathfrak{l}$ be a reductive Lie algebra and $\mathfrak{l} = \mathfrak{z} \oplus \mathfrak{l}^{ss}$ a Levi decomposition, where $\mathfrak{z}$ is the center of $\mathfrak{l}$ and $\mathfrak{l}^{ss}$ is the semisimple Lie subalgebra. Suppose

$$\mathfrak{l}^{ss} := \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_k$$

is the decomposition into noncompact simple ideals labeled as follows:

$$\frac{d(L_i)}{\mathrm{rank}_{\mathbf{R}}\, L_i} \geq \frac{d(L_{i+1})}{\mathrm{rank}_{\mathbf{R}}\, L_{i+1}} \ \ (i = 1, \cdots, k-1).$$

We call $\mathfrak{l}_1$ the *primary simple factor* of $\mathfrak{l}$.

We use the primary simple factor $\mathfrak{l}_1$, when $\mathrm{rank}_{\mathbf{R}}\, G - \mathrm{rank}_{\mathbf{R}}\, H \geq 2$ because $\mathfrak{l}$ is not necessarily simple in this case. Then we also use the following inequality:

Lemma 2.8.

$$\frac{d(L^{ss})}{\mathrm{rank}_{\mathbf{R}}\, L^{ss}} \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbf{R}}\, L_1}.$$

The step (C) concerns an inequality for the primary simple factor $\mathfrak{l}_1$ and any irreducible component π of the restriction $\rho|_{\mathfrak{l}_1}$. We illustrate the case $G/H = SU(2p, 2q)/Sp(p, q)$. In this case, we get the following inequalities.

$$\dim \pi \leq \dim \rho \leq \frac{d(L^{ss})}{\mathrm{rank}_{\mathbf{R}}\, L^{ss}} \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbf{R}}\, L_1}.$$

These inequalities give strong constraints about possible pair $(\pi, \mathfrak{l}_1)$ (see Table II), which is useful for the classification of the triple G, H and L in Question 1.5.

For the last step (D), we prepare some notations. Let $\mathfrak{g}$ be a real semisimple Lie algebra. We denote by $\mathrm{Irr}(\mathfrak{g})$ the set of equivalence classes of irreducible finite dimensional representations of $\mathfrak{g}$. Then we define a subset $\mathrm{Irr}^c(\mathfrak{g})$ of $\mathrm{Irr}(\mathfrak{g})$ by

$$\mathrm{Irr}^c(\mathfrak{g}) := \{\rho \in \mathrm{Irr}(\mathfrak{g}) : \overline{\rho} \simeq \rho\},$$

where $\overline{\rho}$ is the complex conjugate representation of

Table II. Possible pairs of a simple Lie algebra $\mathfrak{l}_1$ and its irreducible representation π satisfying the above inequality. Here, π is the standard representation of each $\mathfrak{l}_1$.

$\mathfrak{l}_1$	$\dim \pi$	$\frac{d(L_1)}{\operatorname{rank}_{\mathbf{R}} L_1}$
$\mathfrak{sl}(n, \mathbf{C})$ $(n \geq 2)$	n	$n+1$
$\mathfrak{su}^*(2n)$ $(n \geq 2)$	$2n$	$2n+1$
$\mathfrak{su}(k, \ell)$ $(2 \leq k \geq \ell \geq 1)$	$k+\ell$	$2k$
$\mathfrak{so}(2n+1, \mathbf{C})$ $(n \geq 2)$	$2n+1$	$2n+1$
$\mathfrak{so}^*(4n+2)$ $(n \geq 1)$	$4n+2$	$4n+2$
$\mathfrak{sp}(n, \mathbf{C})$ $(n \geq 2)$	$2n$	$2n+1$
$\mathfrak{sp}(k, \ell)$ $(k \geq \ell \geq 1)$	$2(k+\ell)$	$4k$
$\mathfrak{g}_2^{\mathbf{C}}$	7	7

ρ. For a representation $\rho : \mathfrak{g} \to \mathfrak{sl}(n, \mathbf{C})$ and $\pi \in \operatorname{Irr}(\mathfrak{g})$, we write $m_\pi := \dim \operatorname{Hom}_{\mathfrak{g}}(\pi, \rho)$. We use the following map given in [I59, §9]:

$$\operatorname{index} : \operatorname{Irr}^c(\mathfrak{g}) \to \{\pm 1\}.$$

We apply these notations to real semisimple Lie algebras $\mathfrak{g}_{\mathbf{C}}^\sigma$ instead of $\mathfrak{g}$, where $\mathfrak{g}_{\mathbf{C}}$ is a complex semisimple Lie algebra and σ is a real structure on $\mathfrak{g}_{\mathbf{C}}$ and write $\operatorname{index}_\sigma \pi \in \{\pm 1\}$ for $\pi \in \operatorname{Irr}^c(\mathfrak{g}_{\mathbf{C}}^\sigma)$.

Proposition 2.9. *Let $\mathfrak{g}_{\mathbf{C}}$ be a semisimple Lie algebra over $\mathbf{C}$, τ a real structure on $\mathfrak{g}_{\mathbf{C}}$ and $\rho : \mathfrak{g}_{\mathbf{C}}^\tau \to \mathfrak{sl}(n, \mathbf{C})$ a representation. Let $\mathfrak{m}$ be one of the following Lie algebras $\mathfrak{sl}(n, \mathbf{R})$, $\mathfrak{su}^*(2m)$, $\mathfrak{so}(n, \mathbf{C})$ and $\mathfrak{sp}(m, \mathbf{C})$, where $m := \frac{1}{2}n$ for n even. Then the following two conditions on $\mathfrak{g}_{\mathbf{C}}$, τ and ρ are equivalent*:

(i) *there exists $\alpha \in \operatorname{Int}(\mathfrak{sl}(n, \mathbf{C}))$ such that $\alpha(\rho(\mathfrak{g}_{\mathbf{C}}^\tau)) \subset \mathfrak{m}$,*

(ii) *$\rho \simeq \overline{\rho}$ as a representation of $\mathfrak{g}_{\mathbf{C}}^\sigma$ and $(\varepsilon \operatorname{index}_\sigma \pi)^{m_\pi} = 1$ for any $\pi \in \operatorname{Irr}^c(\mathfrak{g}_{\mathbf{C}}^\sigma)$.*

Here, $\varepsilon \in \{\pm 1\}$ and σ are given as follows:

$\mathfrak{m}$	ε	σ
$\mathfrak{sl}(n, \mathbf{R})$	$+1$	τ
$\mathfrak{su}^*(2m)$	-1	τ
$\mathfrak{so}(n, \mathbf{C})$	$+1$	θ
$\mathfrak{sp}(m, \mathbf{C})$	-1	θ

Here θ is a compact real structure on $\mathfrak{g}_{\mathbf{C}}$.

The last step by using Proposition 2.9 gives further constraints on possible triples G, H and L in Question 1.5.

Detailed proof will appear elsewhere.

Acknowledgements. The author would like to thank his supervisor Dr. Taro Yoshino for many constructive comments. The author would like to thank Dr. Takayuki Okuda for helpful suggestions on exceptional Lie algebras. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

References

[Bn96] Y. Benoist, Actions propres sur les espaces homogènes réductifs, Ann. of Math. (2) **144** (1996), no. 2, 315–347.

[Br57] M. Berger, Les espaces symétriques noncompacts, Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177.

[C94] K. Corlette, Harmonic maps, rigidity, and Hodge theory, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2* (*Zürich, 1994*), 465–471, Birkhäuser, Basel, 1995.

[G85] W. M. Goldman, Nonstandard Lorentz space forms, J. Differential Geom. **21** (1985), no. 2, 301–308.

[I59] N. Iwahori, On real irreducible representations of Lie algebras, Nagoya Math. J. **14** (1959), 59–83.

[KK16] F. Kassel and T. Kobayashi, Poincaré series for non-Riemannian locally symmetric spaces, Adv. Math. **287** (2016), 123–236.

[Ko89] T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann. **285** (1989), no. 2, 249–263.

[Ko92a] T. Kobayashi, A necessary condition for the existence of compact Clifford–Klein forms of homogeneous spaces of reductive type, Duke Math. J. **67** (1992), no. 3, 653–664.

[Ko92b] T. Kobayashi, Discontinuous groups acting on homogeneous spaces of reductive type, in *Representation theory of Lie groups and Lie algebras* (*Fuji-Kawaguchiko, 1990*), 59–75, World Sci. Publ., River Edge, NJ, 1992.

[Ko98] T. Kobayashi, Deformation of compact Clifford–Klein forms of indefinite-Riemannian homogeneous manifolds, Math. Ann. **310** (1998), no. 3, 395–409.

[Ko01] T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces, in *Mathematics unlimited—2001 and beyond*, 723–747, Springer, Berlin, 2001.

[KO90] T. Kobayashi and K. Ono, Note on Hirzebruch's proportionality principle, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37** (1990), no. 1, 71–87.

[KY05] T. Kobayashi and T. Yoshino, Compact Clifford–Klein forms of symmetric spaces—revisited, Pure Appl. Math. Q. **1** (2005), no. 3, Special Issue: In memory of Armand Borel. Part 2, 591–663.

[LMZ95] F. Labourie, S. Mozes and R. J. Zimmer, On manifolds locally modelled on non-Riemannian homogeneous spaces, Geom. Funct. Anal. **5** (1995), no. 6, 955–965.

[Li95] R. L. Lipsman, Proper actions and a compactness condition, J. Lie Theory **5** (1995), no. 1, 25–39.

[Ma97] G. Margulis, Existence of compact quotients of homogeneous spaces, measurably proper actions, and decay of matrix coefficients, Bull. Soc. Math. France **125** (1997), no. 3, 447–456.

[Mo15] Y. Morita, A topological necessary condition for the existence of compact Clifford–Klein forms, J. Differential Geom. **100** (2015), no. 3, 533–545.

[OW00] H. Oh and D. Witte, New examples of compact Clifford–Klein forms of homogeneous spaces of $SO(2,n)$, Internat. Math. Res. Notices **2000** (2000), no. 5, 235–251.

[Ok13] T. Okuda, Classification of semisimple symmetric spaces with proper $SL(2,\mathbf{R})$-actions, J. Differential Geom. **94** (2013), no. 2, 301–342.

[Th] N. Tholozan, Volume and non-existence of compact Clifford–Klein forms, arXiv:1511.09448v2.

[Z94] R. J. Zimmer, Discrete groups and non-Riemannian homogeneous spaces, J. Amer. Math. Soc. **7** (1994), no. 1, 159–168.