The number of orientable small covers over a product of simplices

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Abstract: In this paper, we give a formula for the number of orientable small covers over a product of simplices up to D-J equivalence. We also give an approximate value for the ratio between the number of small covers and the number of orientable small covers over a product of equidimensional simplices up to D-J equivalence.

Key words: Orientable small cover; polytope; acyclic digraph.

1. Introduction. In 1991, Davis and Januszkiewicz introduced the notion of small cover as a topological analogue of real toric manifolds in [5]. A small cover is a smooth closed manifold M^n which admits a locally standard $\mathbf{Z}_2^n$-action whose orbit space is a simple convex polytope.

There have been several studies on the number of small covers over a specific polytope. In [6], Garrison and Scott use a computer program to find the number of small covers over a dodecahedron. In [2], Choi constructs a bijection between the set of Davis-Januszkiewicz equivalence classes of small covers over an n-cube and the set of acyclic digraphs with n-labeled nodes. He also gives a formula for the number of small covers over a product of simplices up to Davis-Januszkiewicz equivalence in terms of acyclic digraphs with labeled nodes. However, for the orientable small covers very little is known. In [3], Choi gave a formula for the number of orientable small covers over an n-cube, by using a simple criterion found by Nakayama and Nishimura [7] for a small cover to be orientable. In [1], Chen and Wang calculated the number of the equivariant homeomorphism classes of orientable small covers over a product of at most three simplices.

In this paper, we obtain a formula for the number of orientable small covers over a product of simplices up to Davis-Januszkiewicz equivalence by using Choi's idea. It turns out that when all the simplices have even dimensions, there is no orientable small cover over the product. Then we restrict our attention to the case where the simplices are equidimensional. We estimate the ratio O_n^m/R_n^m, where R_n^m and O_n^m are the number of small covers and the number of the orientable small covers over a product of n-copy of m-simplices, respectively.

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2. Preliminaries. A small cover is a smooth closed n-dimensional manifold M^n with a locally standard $\mathbf{Z}_2^n$-action such that the orbit space is a simple convex polytope P. Two small covers M_1 and M_2 over P are said to be Davis-Januszkiewicz equivalent (or simply D-J equivalent) if there is a weakly $\mathbf{Z}_2^n$-homeomorphism $f: M_1 \to M_2$ covering the identity on P.

Let P be an n-dimensional simple convex polytope, and $\mathcal{F}(P) = \{F_1, \dots, F_m\}$ be the set of facets of P. A map $\lambda: \mathcal{F}(P) \to \mathbf{Z}_2^n$ is called a characteristic function if it satisfies the non-singularity condition : whenever the intersection $\lambda(F_{i_1}) \cap \dots \cap \lambda(F_{i_n})$ is non-empty, the set $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\}$ forms a basis for $\mathbf{Z}_2^n$. For a given point $p \in P$, let $\mathbf{Z}_2^n(p)$ be the subgroup of $\mathbf{Z}_2^n$ generated by $\lambda(F_{i_1}), \dots, \lambda(F_{i_k})$ where the intersection $\bigcap_{j=1}^{k} F_{i_j}$ is the minimal face containing p in its relative interior. Then the manifold $M(\lambda) = (P \times \mathbf{Z}_2^n)/\sim$, where $(p, g) \sim (q, h)$ if $p = q$ and $g^{-1}h \in \mathbf{Z}_2^n(p)$, is a small cover over P. On the other hand, every small cover over P can be distinguished by a characteristic function λ by the following theorem.

Theorem 2.1 ([5]). *For every small cover M over P, there is a characteristic function λ with $\mathbf{Z}_2^n$-homeomorphism $M(\lambda) \to M$ covering the identity on P.*

In 2005, Nakayama and Nishimura give the following simple criterion for the orientable small covers.

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Theorem 2.2 ([7]). *For a basis $\{e_1,\cdots,e_n\}$ of $\mathbf{Z}_2^n$, let the homomorphism $\epsilon:\mathbf{Z}_2^n\to\mathbf{Z}_2$ be defined by $\epsilon(e_i)=1$ for $i=1,\cdots,n$. Then a small cover $M(\lambda)$ over a simple convex polytope P of dimension n is orientable if and only if there exists a basis $\{e_1,\cdots,e_n\}$ of $\mathbf{Z}_2^n$ such that the image of the composition $\epsilon\circ\lambda$ is $\{1\}$.*

Let $\Lambda(P)$ be the set of characteristic functions over P. There is a left action of $GL(n,\mathbf{Z}_2)$ on $\Lambda(P)$ by $\sigma\cdot\lambda:=\sigma\circ\lambda$ for any $\sigma\in GL(n,\mathbf{Z}_2)$ and $\lambda\in\Lambda(P)$. Two small covers M_1 and M_2 over P are D-J equivalent if and only if $\lambda_1=\sigma\cdot\lambda_2$ for some $\sigma\in GL(n,\mathbf{Z}_2)$, where λ_1 and λ_2 are the characteristic functions corresponding to M_1 and M_2, respectively. This means that the number of D-J equivalence classes does not depend on the choose of a basis. Moreover the action of $GL(n,\mathbf{Z}_2)$ on $\Lambda(P)$ is free and hence the number of D-J equivalence classes of small covers over P is $|GL(n,\mathbf{Z}_2)\backslash\Lambda(P)|$.

To a given characteristic function λ over P, we can assign an $n\times m$ $\mathbf{Z}_2$-matrix Λ by ordering the facets and choosing a basis $\{e_1,\cdots,e_n\}$ of $\mathbf{Z}_2^n$ by

$$\Lambda=(\lambda(F_1)\cdots\lambda(F_m)).$$

Assuming that the first n facets intersect at a vertex, we can choose a representative $\Lambda=(I_n|\Lambda_*)$ for the D-J equivalence class of λ, where I_n is the $n\times n$ identity matrix and Λ_* is an $n\times(m-n)$ matrix. Then the D-J equivalence classes of small covers over P are distinguished by Λ_*. The matrix Λ_* is called the reduced submatrix of λ. In this setting we may rephrase the Nakayama and Nishimura's criterion for orientability as follows:

($*$) *$M(\lambda)$ is orientable if and only if the sum of entries of the i-th column of Λ_* is odd for all $i=1,\cdots,m-n$.*

3. Orientable small covers over product of simplices. In this section, we consider the case where $P=\Delta^{n_1}\times\cdots\times\Delta^{n_l}$ with $n=\sum_{n=1}^{l}n_i$. Here Δ^k is the standard k-simplex. A facet of P is a product of a facet of a simplex Δ^{n_i} and the remaining simplices. Therefore if $\{f_0^i,\cdots,f_{n_i}^i\}$ is the set of facets of Δ^{n_i} then the set of facets of P is

$$\{F_j^i=\Delta^{n_1}\times\cdots\times\Delta^{n_{i-1}}\times f_j^i\times\Delta^{n_{i+1}}\cdots\times\Delta^{n_l}|\ 1\le i\le l,\ 0\le j\le n_i\}.$$

Let $\lambda:\mathcal{F}(P)\to\mathbf{Z}_2^n$ be a characteristic function. Since $\{F_1^1,\cdots,F_{n_1}^1,\cdots,F_1^l,\cdots,F_{n_l}^l\}$ intersects at a vertex, by choosing them to be the first n-facets, one can assume that

$$\lambda(F_j^i)=e_{n_1+\cdots+n_{i-1}+j}$$

for $1\le j\le n_i$. Then

$$\lambda(F_0^i)=\sum_{j=1}^{n}a_{ji}e_j$$

where $a_{ji}=1$ for $n_1+\cdots+n_{i-1}+1\le j\le n_1+\cdots+n_i$. Therefore λ corresponds to an $(n\times l)$-matrix $\Lambda_*=[a_{ij}]$ up to D-J equivalence. The matrix Λ_* can be viewed as an $(l\times l)$-vector matrix $[\mathbf{v_{ij}}]$ whose entries in the i-th row are vectors in $\mathbf{Z}_2^{n_i}$. We refer reader to [4] for details. Note that the diagonal $\mathbf{v_{ii}}$ is the vector $[1,\cdots,1]^T\in\mathbf{Z}_2^{n_i}$.

Given a vector matrix $\Lambda_*=[\mathbf{v_{ij}}]$ corresponding to a characteristic function λ, let $B(\Lambda_*):=(b_{ij})$ be the $(l\times l)$-matrix over $\mathbf{Z}_2$ with $b_{ij}=1$ if $\mathbf{v_{ij}}$ is nonzero, and $b_{ij}=0$ otherwise. Then $B(\Lambda_*)-I_l$ is an adjacency matrix of an acyclic digraph with l nodes (see [2]). Here, a digraph is a graph with at most one edge directed from vertex v_i to vertex v_j. Given a directed graph G with vertices $v_1,\cdots,v_l$, the adjacency matrix of G is an $(l\times l)$-matrix $A(G)=[a_{ij}]$ where $a_{ij}=1$ if there is an edge directed from v_i to v_j, and $a_{ij}=0$ otherwise. A directed graph is said to be acyclic if there is no directed cycle. The outdegree $\mathrm{outdeg}(v)$ (the indegree $\mathrm{indeg}(v)$) of a vertex v is the number of edges directed from (to) v.

Let $\mathcal{G}_l$ be the set of acyclic digraphs with labeled vertex set $V(G)=\{v_1,\dots,v_l\}$. By counting the pre-images of the function $\Psi:GL(n,\mathbf{Z}_2)\backslash\Lambda(P)\to\mathcal{G}_l$ defined by $\Psi([\lambda])=G$ where G is the acyclic graph whose adjacency matrix is $B(\Lambda_*)-I_l$, Choi [2] obtains the following formula for the number of small covers over P.

Theorem 3.1 (Theorem 2.8, [2]). *The number of D-J equivalence classes of small covers over $P=\Delta^{n_1}\times\cdots\times\Delta^{n_l}$ with $\sum_{i=1}^{l}n_i=n$ is*

$$\sum_{G\in\mathcal{G}_l}\prod_{v_i\in V(G)}(2^{n_i}-1)^{\mathrm{outdeg}(v_i)}.$$

Using the above correspondence and ($*$), we have the following result.

Theorem 3.2. *Let $p\ge0$ and $P=\Delta^{n_1}\times\cdots\times\Delta^{n_l}$ such that n_i is odd for $i\le p$ and even otherwise. Then the number of orientable small covers over P up to D-J equivalence is*

$$\sum_{k=1}^{p}(-1)^{k+1}\left(\sum_{1\le i_1<\cdots<i_k\le p} 2^{(n_{i_1}+\cdots+n_{i_k}-1)(l-k)} \cdot\left(\sum_{G\in\mathcal{G}_l(i_1,\cdots,i_k)}\prod_{v_i\in V(G)}(2^{n_i}-1)^{\mathrm{outdeg}(v_i)}\right)\right)$$

where $\mathcal{G}_l(i_1,\cdots,i_k)$ is the set of acyclic digraphs with $(l-k)$-labelled nodes whose labelled vertex set is $\{v_1,\cdots,v_l\}\backslash\{v_{i_1},\cdots,v_{i_k}\}$.

Proof. Let M be a small cover over P with vector matrix $\Lambda_*=[\mathbf{v_{ij}}]$. Since every acyclic digraph has a vertex of indegree 0, $B(\Lambda_*-I_l)$ has a zero column, say i-th one. This means that the sum of the entries of the i-th column of Λ_* is n_i. If $p=0$ then the small cover corresponding to λ is not orientable. Therefore there is no orientable small cover over P.

Now suppose that $p\ge 1$. Without loss of generality, we may assume that $B(\Lambda_*)$ is of the following form

$$\begin{pmatrix} I_k & S\\ 0 & T\end{pmatrix}. \tag{1}$$

Here, $B(\Lambda_*)$ corresponds to a characteristic function if and only if $T-I_{n-k}$ is an adjacency matrix. Moreover, M is an orientable small cover then $n_1,\cdots,n_k$ must be odd. For the remaining columns of Λ_*, it suffices to control the first row of S. Indeed, if the sum of the entries greater than n_1 of i-th column of Λ_*, with $i>k$, is even, then the sum of the entries of $\mathbf{v_{1i}}$ must be odd or vice a versa. In both case, we have 2^{n_1-1} choices for $\mathbf{v_{1i}}$. For $\mathbf{v_{ji}}$ with $2\le j\le k$, we have 2^{n_j}-many choices. Therefore, the number of orientable small covers corresponding to (1) up to D-J equivalence is $2^{(n_1+\cdots+n_k-1)(l-k)}\cdot(\sum_{G\in\mathcal{G}_l}\prod_{v_i\in V(G)}(2^{n_{i+k}}-1)^{\mathrm{outdeg}(v_i)})$. By applying the Principle of Inclusion-Exclusion and taking the cases where the set of vertices with 0-indegree is not necessarily $v_1,\cdots,v_k$ into an account, we obtain the required formula. □

Example 3.3. Let $P=\Delta^{n_1}\times\Delta^{n_2}\times\Delta^{n_3}$ with n_1 and n_2 are odd and n_3 is even. There are three cases: $k=1$ and $i_1=1$, $k=1$ and $i_1=2$, and $k=2$, $(i_1,i_2)=(1,2)$. Therefore the number of D-J equivalence classes of orientable small covers over P is

$$2^{2(n_1-1)}(2^{n_2}+2^{n_3}-1) + 2^{2(n_2-1)}(2^{n_1}+2^{n_3}-1)-2^{n_1+n_2-1}.$$

This is the first case of Theorem 5.4 in [1].

4. Equidimensional case. In this section, we consider the case where all the simplices are equidimensional, that is, $P=\underbrace{\Delta^m\times\cdots\times\Delta^m}_{n\text{ times}}$. Let R_n^m be the number of all small covers over P and O_n^m the numbers of all orientable small covers over P up to D-J equivalence. The sum of the outdegree of the vertices of a graph is $|E(G)|$ where $E(G)$ denotes the number of edges in the graph G. Therefore, by Theorem 3.1, we have

$$R_n^m=\sum_{G\in\mathcal{G}_n}(2^m-1)^{|E(G)|}=\sum_{t\ge 0}(2^m-1)^t R_{n,t}$$

where $R_{n,t}$ is the number of acyclic digraphs with n-labeled vertices and t edges.

Proposition 4.1.

$$R_n^m=\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}2^{m(n-j)j}R_{n-j}^m.$$

Proof. By (4) in [8],

$$R_{n,t}=\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\sum_{k=0}^{t}\binom{(n-j)j}{t-k}R_{n-j,k}.$$

Therefore, we have

$$\begin{aligned} R_n^m&=\sum_{t\ge0}(2^m-1)^t\left(\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\sum_{k=0}^{t}\binom{(n-j)j}{t-k}R_{n-j,k}\right)\\ &=\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\underbrace{\left[\sum_{t\ge0}(2^m-1)^t\sum_{k=0}^{t}\binom{(n-j)j}{t-k}R_{n-j,k}\right]}_{(*)}. \end{aligned}$$

Now, changing the order of the summation in $(*)$ we get,

$$\begin{aligned} (*)&=\sum_{k\ge0}(2^m-1)^k\left(\sum_{t\ge k}(2^m-1)^{t-k}\binom{(n-j)j}{t-k}\right)R_{n-j,k}\\ &=\sum_{k\ge0}(2^m-1)^k\left(\sum_{i\ge0}(2^m-1)^{i}\binom{(n-j)j}{i}\right)R_{n-j,k}\\ &=\sum_{k\ge0}(2^m-1)^k(2^m)^{(n-j)j}R_{n-j,k}\\ &=2^{m(n-j)j}\underbrace{\left(\sum_{k\ge0}(2^m-1)^kR_{n-j,k}\right)}_{R_{n-j}^m}. \end{aligned}$$

Hence, $R_n^m=\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}2^{m(n-j)j}R_{n-j}^m.$ □

In the previous section we have shown that if m is an even number, then $O_n^m=0$. From now on we assume that m is odd.

Proposition 4.2.

$$O_n^m = \sum_{k=1}^{p}(-1)^{k+1}\binom{n}{k}(2^{km-1})^{(m-k)}R_{n-k}^m.$$

Proof. Since all the simplices have the same dimension, the different choices for $i_1, \cdots, i_k$ for fix k will make the same contribution to the sum given in Theorem 3.2. Therefore we have

$$\begin{aligned} O_n^m &= \sum_{k=1}^{p}(-1)^{k+1}\binom{n}{k}(2^{km-1})^{(m-k)} \\ &\quad\cdot\left(\sum_{G\in\mathcal{G}_{m-k}}\prod_{v\in V(G)}(2^m-1)^{\mathrm{outdeg}(v)}\right) \\ &= \sum_{k=1}^{p}(-1)^{k+1}\binom{n}{k}(2^{km-1})^{(m-k)}R_{n-k}^m. \end{aligned}$$

□

In the following tables, we give a few values of R_n^m and O_n^m, respectively.

Table I. Some values of R_n^m

n	1	2	3	4	5
R_n^1	1	3	25	543	29281
R_n^2	1	7	289	63487	69711361
R_n^3	1	15	2689	5140479	98267258881
R_n^4	1	31	23041	365330431	115851037900801
R_n^5	1	63	190465	24568397823	126764735665274881

Table II. Some values of O_n^m

n	1	2	3	4	5
O_n^1	1	1	4	43	1156
O_n^3	1	7	625	597247	5708501761
O_n^5	1	31	46849	3021553663	7795103914721281

Given a sequence A_n, we define the family of chromatic generating functions $\{A^m(x)\}_{m\in\mathbf{N}}$ with respect to A_n as

$$A^m(x) = \sum_{n=0}^{\infty} A_n \frac{x^n}{n!2^{m\binom{n}{2}}}.$$

Obviously, $C^m(x) = A^m(x)B^m(x)$, where $A^m(x)$, $B^m(x)$ and $C^m(x)$ are the m^{th} chromatic generating functions with respect to the sequences A_n, B_n and C_n, respectively, if and only if $C_n = \sum_{k=0}^n A_k B_{n-k}\binom{n}{k}2^{mk(n-k)}$. Then by Proposition 4.1, we have $R^m(x)F^m(-x) = 1$ where $R^m(x)$ and $F^m(x)$ are the m^{th} chromatic generating functions for R_n^m and the constant sequence $F_n^m = 1$. An easy computation shows that

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}2^{m(n-j)j}\frac{R_{n-j}^m}{2^{n-j}} + O_n^m = \frac{R_n^m}{2^n}.$$

Therefore, we have

$$R^m\left(\frac{x}{2}\right)F^m(-x) + O^m(x) = R^m\left(\frac{x}{2}\right).$$

Since $R^m(\frac{x}{2}) = \frac{1}{F^m(-\frac{x}{2})}$, we get

Corollary 4.3. $O^m(x) = \dfrac{1-F^m(-x)}{F^m(-\frac{x}{2})}$.

Following Choi [3], let $G^m(x) = \frac{F^m(\frac{x}{2})}{1-F^m(x)}$. Using Rouche's theorem, it can be easily shown that $F^m(x)$ has a unique zero α_m satisfying $|\alpha_m| \leq 2$ (See Table III for the approximate values of α_m for odd $m \leq 7$). Then $G^m(x)$ has an isolated zero at $2\alpha_m$. Standard techniques implies the asymptotic formula

$$G^m(x) \sim (G^m)'(2\alpha_m)(x - 2\alpha_m).$$

So we obtain

$$O^m(x) = \frac{1}{G^m(-x)} \sim \frac{1}{(G^m)'(2\alpha_m)(-x-2\alpha_m)}.$$

Table III. Zeroes of $F^k(x)$

k	1	3	5	7	9
$\alpha_k \approx$	-1.488	-1.071	-1.016	-1.004	-1.001

Note that $(F^m)'(x) = F^m(\frac{x}{2^m})$. Since

$$\frac{1}{(G^m)'(2\alpha_m)} = \frac{2(1-F^m(2\alpha_m))}{(F^m)'(\alpha_m)} \text{ and}$$

$$\frac{1}{-x-2\alpha_m} = -\frac{1}{2\alpha_m}\sum_{n=0}^{\infty}\left(-\frac{x}{2\alpha_m}\right)^n,$$

we have the following asymptotic formula

$$O^m(x) \sim -\frac{1-F^m(2\alpha_m)}{\alpha_m F^m(\frac{\alpha_m}{2^m})}\sum_{n=0}^{\infty}\left(-\frac{x}{2\alpha_m}\right)^n.$$

Therefore

$$O_n^m \sim -\frac{1-F^m(2\alpha_m)}{\alpha_m F^m(\frac{\alpha_m}{2^m})}\left(-\frac{1}{2\alpha_m}\right)^n 2^{\binom{n}{2}}n!. \tag{2}$$

On the other hand, since $R^m(x)F^m(-x) = 1$ and $F^m(x)$ has an isolated zero at α_m, we have

$$R^m(x) = \frac{1}{F^m(-x)} \sim \frac{1}{(F^m)'(\alpha_m)(-x-\alpha_m)}$$
$$= -\frac{1}{\alpha_m F^m(\frac{\alpha_m}{2^m})} \sum_{n=0}^{\infty} \left(-\frac{x}{\alpha_m}\right)^n,$$

and hence

$$R_n^m \sim -\frac{1}{\alpha_m F^m(\frac{\alpha_m}{2^m})} \left(-\frac{1}{\alpha_m}\right)^n 2^{\binom{n}{2}} n!. \tag{3}$$

Combining the estimations in (2) and (3) we obtain the following result.

Corollary 4.4. *When m is odd,*

$$\frac{O_n^m}{R_n^m} \sim \frac{K}{2^n},$$

where $K = 1 - F(2\alpha_m)$.

Here are some values for the constant K:

Table IV. Constant $K = 1 - F(2\alpha_m)$

m	1	3	5	7	9
$K \approx$	1.262	1.858	1.967	1.992	1.998

References

[1] Y. Chen and Y. Wang, Small covers over a product of simplices, Filomat **27** (2013), no. 5, 777–787.

[2] S. Choi, The number of small covers over cubes, Algebr. Geom. Topol. **8** (2008), no. 4, 2391–2399.

[3] S. Choi, The number of orientable small covers over cubes, Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), no. 6, 97–100.

[4] S. Choi, M. Masuda and D. Y. Suh, Quasitoric manifolds over a product of simplices, Osaka J. Math. **47** (2010), no. 1, 109–129.

[5] M. W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J. **62** (1991), no. 2, 417–451.

[6] A. Garrison and R. Scott, Small covers of the dodecahedron and the 120-cell, Proc. Amer. Math. Soc. **131** (2003), no. 3, 963–971.

[7] H. Nakayama and Y. Nishimura, The orientability of small covers and coloring simple polytopes, Osaka J. Math. **42** (2005), no. 1, 243–256.

[8] V. I. Rodionov, On the number of labeled acyclic digraphs, Discrete Math. **105** (1992), no. 1–3, 319–321.