The signs of the Stieltjes constants associated with the Dedekind zeta function

By Sumaia Saad Eddin

Institute of Financial Mathematics and Applied Number Theory, Johannes Kepler University,
Altenbergerstrasse 69, 4040 Linz, Austria

(Communicated by Heisuke Hironaka, m.j.a., Nov. 12, 2018)

Abstract: The Stieltjes constants $\gamma_n(K)$ of a number field K are the coefficients of the Laurent expansion of the Dedekind zeta function $\zeta_K(s)$ at its pole $s=1$. In this paper, we establish a similar expression of $\gamma_n(K)$ as Stieltjes obtained in 1885 for $\gamma_n(\mathbf{Q})$. We also study the signs of $\gamma_n(K)$.

Key words: Stieltjes constants; Riemann zeta function; Dedekind zeta function.

1. Introduction. Let K be a number field and $\mathcal{O}_K$ be its ring of integers. Define for $\Re s>1$ the *Dedekind zeta function*

$$\zeta_K(s)=\sum_{\mathfrak{a}}\frac{1}{N\mathfrak{a}^s}=\prod_{\mathfrak{p}}\frac{1}{1-N\mathfrak{p}^{-s}},$$

where $\mathfrak{a}$ runs over non-zero ideals in $\mathcal{O}_K$, $\mathfrak{p}$ runs over the prime ideals in $\mathcal{O}_K$ and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. It is known that $\zeta_K(s)$ can be analytically continued to $\mathbf{C}-\{1\}$, and that at $s=1$ it has a simple pole, with residue $\gamma_{-1}(K)$, given by the analytic class number formula:

$$\gamma_{-1}(K)=\frac{2^{r_1}(2\pi)^{r_2}h(K)R(K)}{\omega(K)\sqrt{|D(K)|}},$$

where r_1 denotes the number of real embeddings of K, r_2 is the number of complex embeddings of K, $h(K)$ is the class number of K, $R(K)$ is the regulator of K, $\omega(K)$ is the number of roots of unity contained in K and $D(K)$ is the discriminant of the extension $K/\mathbf{Q}$. The Laurent expansion of $\zeta_K(s)$ at $s=1$ is

$$\zeta_K(s)=\frac{\gamma_{-1}(K)}{s-1}+\gamma_0(K)+\gamma_1(K)(s-1)+\gamma_2(K)(s-1)^2+\cdots. \tag{1}$$

The constants $\gamma_n(K)$ are sometimes called the Stieltjes constants associated with the Dedekind zeta function. In [6] they are called by higher Euler's constants of K. While the constant $\gamma_K=\gamma_0(K)/\gamma_{-1}(K)$ is called the *Euler-Kronecker constant* in [7] and [16].

In case $K=\mathbf{Q}$, the Laurent expansion of the Riemann zeta function $\zeta(s)$ at its pole $s=1$ is given by

$$\zeta(s)=\frac{1}{s-1}+\sum_{n\geq 0}\gamma_n(s-1)^n,$$

where

$$\gamma_n=\frac{(-1)^n}{n!}\lim_{x\to\infty}\left(\sum_{m=1}^{x}\frac{(\log m)^n}{m}-\frac{(\log x)^{n+1}}{(n+1)}\right). \tag{2}$$

Stieltjes in 1885 was the first to propose this definition of γ_n for this reason these constants are today called by his name. The asymptotic behaviour of γ_n, as $n\to\infty$, has been widely studied by many authors (for instance: Briggs [3], Mitrović [12], Israilov [8], Matsuoka [11] and more recently Coffey [4] and [5], Knessl and Coffey [9], Adell [2], Adell and Lekuona [1] and Saad Eddin [14]). Their main interest is focused on the growth, the sign changes of the sequence (γ_n) and on giving explicit upper estimates for $|\gamma_n|$. Moreover, they obtained relations between this sequence and the zeros of $\zeta(s)$ (see [11], [15]). In this paper we are interested in the Stieltjes coefficients $\gamma_n(K)$ for the Dedekind zeta function. We first give the following formula of $\gamma_n(K)$ which is similar to Stieltjes's formula given by Eq. (2).

Theorem 1. *For any $n\geq 1$, we have*

$$\gamma_n(K)=\frac{(-1)^n}{n!}\lim_{x\to\infty}\left(\sum_{N\mathfrak{a}\leq x}\frac{(\log N\mathfrak{a})^n}{N\mathfrak{a}}-\gamma_{-1}(K)\frac{(\log x)^{n+1}}{n+1}\right),$$

and

$$\gamma_0(K)=\lim_{x\to\infty}\left(\sum_{N\mathfrak{a}\leq x}\frac{1}{N\mathfrak{a}}-\gamma_{-1}(K)\log x\right)+\gamma_{-1}(K).$$

2010 Mathematics Subject Classification. Primary 11R42; Secondary 11M06.

doi: 10.3792/pjaa.94.93

This result seems similar to another one obtained by Hashimoto et al. [6] for the higher Euler-Selberg constants. Despite a considerable effort the author has not been able to find Theorem 1 in the literature.

In 1962, Mitrović [12] studied the sign changes of the constants γ_n and prove that; each of the inequalities

$$\gamma_{2n} > 0,\ \gamma_{2n} < 0,\ \gamma_{2n-1} > 0,\ \gamma_{2n-1} < 0,$$

holds for infinitely many n. In [11], Matsuoka gave precise conditions for the sign of γ_n. By the same techniques used in [12], we prove that

Theorem 2. *For the coefficients in the expansion* (1), *each of the inequalities*

$$\gamma_{2n}(K) > 0,\ \gamma_{2n}(K) < 0,$$
$$\gamma_{2n-1}(K) > 0,\ \gamma_{2n-1}(K) < 0,$$

holds for infinitely many n.

It immediately follows that

Corollary 1. *Infinitely many* $\gamma_n(K)$ *are positive and infinitely many are negative.*

2. Proofs.

Proof of Theorem 1. By Eq. (1), we note that

$$\zeta_K(s) - \frac{\gamma_{-1}(K)s}{s-1} = \zeta_K(s) - \frac{\gamma_{-1}(K)}{s-1} - \gamma_{-1}(K) = \sum_{n\geq 0} \alpha_n(K)(s-1)^n, \tag{3}$$

where $\alpha_0(K) = \gamma_0(K) - \gamma_{-1}(K)$ and $\alpha_n(K) = \gamma_n(K)$ for $n \geq 1$. By the definition of $\zeta_K(s)$, we write

$$\zeta_K(s) = \int_{1^-}^{+\infty} \frac{dN_K(t)}{t^s} = s\int_{1^-}^{+\infty} \frac{N_K(t)}{t^{s+1}}\, dt,$$

where

$$N_K(t) = \sum_{N\mathfrak{a}\leq t} 1.$$

Then, we get

$$\zeta_K(s) - \frac{\gamma_{-1}(K)s}{s-1} = s\int_{1^-}^{+\infty} \frac{N_K(t) - \gamma_{-1}(K)t}{t^{s+1}}\, dt. \tag{4}$$

Put $\sum_{n\geq 0} \alpha_n(K)(s-1)^n = h(s)$. From Eqs. (3) and (4), we have

$$h(s) = s\int_{1^-}^{+\infty} \frac{N_K(t) - \gamma_{-1}(K)t}{t^{s+1}}\, dt.$$

From [10, Satz 210] we have $N_K(t) = \gamma_{-1}(K)t + \mathcal{O}(t^{1-1/m})$, where m is the degree of K and $\mathbf{Q}$. For $\Re s > 1 - 1/m$, it is easily seen that the n-th derivative of $h(s)$ at $s = 1$ is

$$h^{(n)}(1) = n!\alpha_n(K) = (-1)^n(I_1 - I_2), \tag{5}$$

where

$$I_1 = \int_{1^-}^{+\infty} N_K(t)\left(\frac{\log^n t - n(\log t)^{n-1}}{t^2}\right) dt,$$

and

$$I_2 = \gamma_{-1}(K)\int_{1^-}^{+\infty} \frac{\log^n t - n(\log t)^{n-1}}{t}\, dt.$$

On the other hand, we have

$$\sum_{N\mathfrak{a}\leq x} \frac{(\log N\mathfrak{a})^n}{N\mathfrak{a}} = \int_{1^-}^{x} \frac{\log^n t}{t}\, dN_K(t)$$
$$= N_K(x)\frac{\log^n x}{x} + \int_{1^-}^{x} N_K(t)\left(\frac{\log^n t - n(\log t)^{n-1}}{t^2}\right) dt.$$

Thus, we get

$$\int_{1^-}^{x} N_K(t)\left(\frac{\log^n t - n(\log t)^{n-1}}{t^2}\right) dt = \sum_{N\mathfrak{a}\leq x} \frac{(\log N\mathfrak{a})^n}{N\mathfrak{a}} - N_K(x)\frac{\log^n x}{x}.$$

Again using the fact that $N_K(t) = \gamma_{-1}(K)t + \mathcal{O}(t^{1-1/m})$, we find that

$$\int_{1^-}^{x} N_K(t)\left(\frac{\log^n t - n(\log t)^{n-1}}{t^2}\right) dt = \sum_{N\mathfrak{a}\leq x} \frac{(\log N\mathfrak{a})^n}{N\mathfrak{a}} - \gamma_{-1}(K)\log^n x + \mathcal{O}\left(\frac{\log^n x}{x^{1/m}}\right).$$

Taking $x \to +\infty$, the above becomes

$$I_1 = \lim_{x\to+\infty}\left[\sum_{N\mathfrak{a}\leq x} \frac{(\log N\mathfrak{a})^n}{N\mathfrak{a}} - \gamma_{-1}(K)\log^n x\right]. \tag{6}$$

Now, notice that

$$I_2 = \lim_{x\to+\infty}\left[\gamma_{-1}(K)\frac{(\log x)^{n+1}}{n+1} - \gamma_{-1}(K)\log^n x\right]. \tag{7}$$

From Eqs. (5), (6) and (7), we conclude that, for $n \geq 1$,

$$\gamma_n(K) = \alpha_n(K)$$
$$= \frac{(-1)^n}{n!}\lim_{x\to\infty}\left(\sum_{N\mathfrak{a}\leq x} \frac{(\log N\mathfrak{a})^n}{N\mathfrak{a}} - \gamma_{-1}(K)\frac{(\log x)^{n+1}}{n+1}\right)$$

and $\gamma_0(K) = \alpha_0(K) + \gamma_{-1}(K)$. This completes the proof. □

Proof of Theorem 2. To prove Theorem 2, we apply the same technique used in [12]. Let C be the set of all positive integers n such that $\gamma_n(K) \neq 0$. Define

$$C_1 = \{n : \quad \gamma_n(K) \neq 0 \text{ and } (-1)^n = 1\}$$
$$C_1^- = \{n : \quad \gamma_n(K) < 0 \text{ and } (-1)^n = 1\},$$
$$C_1^+ = \{n : \quad \gamma_n(K) > 0 \text{ and } (-1)^n = 1\},$$

and

$$C_2 = \{n : \quad \gamma_n(K) \neq 0 \text{ and } (-1)^n = -1\},$$
$$C_2^- = \{n : \quad \gamma_n(K) < 0 \text{ and } (-1)^n = -1\},$$
$$C_2^+ = \{n : \quad \gamma_n(K) > 0 \text{ and } (-1)^n = -1\}.$$

From [13], we have

$$\zeta_K(s) - \frac{\gamma_{-1}(K)}{s-1}$$

is an entire transcendental function. So the cardinal number of the set C is equal to the cardinal number of the set of all positive integers $\aleph_0$. Then, we can write

$$\zeta_K(s) - \frac{\gamma_{-1}(K)}{s-1} = \left(\sum_{n\in C_1^-} + \sum_{n\in C_1^+} + \sum_{n\in C_2^-} + \sum_{n\in C_2^+}\right)\gamma_n(K)(s-1)^n.$$

Replacing s by $t+1$ and then by $-t+1$ in the above. Adding and then subtracting the results, we find that

$$\zeta_K(t+1) + \zeta_K(-t+1) = 2\left(\sum_{n\in C_1^-} + \sum_{n\in C_1^+}\right)\gamma_n(K)t^n, \tag{8}$$

and

$$\zeta_K(t+1) - \zeta_K(-t+1) - \frac{2\gamma_{-1}(K)}{t} = 2\left(\sum_{n\in C_2^-} + \sum_{n\in C_2^+}\right)\gamma_n(K)t^n. \tag{9}$$

Taking $t = 2m+1$ with $m > 0$ and using the fact that the $\zeta_K(s)$ vanishes at all negative even integers. We find the left-hand side of Eq. (8) approaches to 1 when $m \to +\infty$. It follows that the right-hand side of this equation can't be polynomial. That means the cardinal of the set C_1 is $\aleph_0$. On the other hand, if we assume that the cardinal of the set C_1^- is less than $\aleph_0$. Then the right-hand side of Eq. (8) approaches $+\infty$. Similarly, if the cardinal of the set C_1^+ is less than $\aleph_0$. Then the right-hand side of Eq. (8) approaches $-\infty$, this leads to a contradiction. We thus conclude that the cardinal of the sets C_1^- and C_1^+ are $\aleph_0$. By a similar argument, we show that the cardinal of the sets C_2^- and C_2^+ in Eq. (9) are $\aleph_0$. That completes the proof. □

Acknowledgements. The author would like to thank Prof. Kohji Matsumoto for his valuable comments on an earlier version of this paper. The author is supported by the Austrian Science Fund (FWF): Project F5507-N26, which is part of the special Research Program "Quasi Monte Carlo Methods: Theory and Application". Part of this work was done while the author was supported by the Japan Society for the Promotion of Science (JSPS) "Overseas researcher under Postdoctoral Fellowship of JSPS".

References

[1] J. A. Adell and A. Lekuona, Fast computation of the Stieltjes constants, Math. Comp. **86** (2017), no. 307, 2479–2492.

[2] J. A. Adell, Asymptotic estimates for Stieltjes constants: a probabilistic approach, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **467** (2011), no. 2128, 954–963.

[3] W. E. Briggs, Some constants associated with the Riemann zeta-function, Michigan Math. J. **3** (1955–56), 117–121.

[4] M. W. Coffey, Hypergeometric summation representations of the Stieltjes constants, Analysis (Munich) **33** (2013), no. 2, 121–142.

[5] M. W. Coffey, Series representations for the Stieltjes constants, Rocky Mountain J. Math. **44** (2014), no. 2, 443–477.

[6] Y. Hashimoto, Y. Iijima, N. Kurokawa and M. Wakayama, Euler's constants for the Selberg and the Dedekind zeta functions, Bull. Belg. Math. Soc. Simon Stevin **11** (2004), no. 4, 493–516.

[7] Y. Ihara, On the Euler-Kronecker constants of global fields and primes with small norms, in *Algebraic geometry and number theory*, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006, pp. 407–451.

[8] M. I. Israilov, The Laurent expansion of the Riemann zeta function, Trudy Mat. Inst. Steklov. **158** (1981), 98–104.

[9] C. Knessl and M. W. Coffey, An effective asymptotic formula for the Stieltjes constants, Math. Comp. **80** (2011), no. 273, 379–386.

[10] E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Chelsea Publishing Company, New York, NY, 1949.

[11] Y. Matsuoka, Generalized Euler constants associated with the Riemann zeta function, in *Number theory and combinatorics. Japan 1984* (*Tokyo, Okayama and Kyoto, 1984*), 279–295, World Sci. Publishing, Singapore, 1985.

[12] D. Mitrović, The signs of some constants associated with the Riemann zeta-function, Michigan Math. J. **9** (1962), 395–397.

[13] A. Reich, Zur Universalität und Hypertranszendenz der Dedekindschen Zetafunktion, Abh. Braunschweig. Wiss. Ges. **33** (1982), 197–203.

[14] S. Saad Eddin, Explicit upper bounds for the Stieltjes constants, J. Number Theory **133** (2013), no. 3, 1027–1044.

[15] S. Saad Eddin, Applications of the Laurent-Stieltjes constants for Dirichlet L-series, Proc. Japan Acad. Ser. A Math. Sci. **93** (2017), no. 10, 120–123.

[16] M. A. Tsfasman, Asymptotic behaviour of the Euler-Kronecker constant, in *Algebraic geometry and number theory*, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006, pp. 453–458.