On the order of holomorphic curves with maximal deficiency sum, II

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Abstract: In this paper we consider the order of holomorphic curves with maximal deficiency sum in the complex plane. The purpose of this paper is to weaken the condition treated in the paper [9]. As a special case we obtain the result in [9].

Key words: Holomorphic curve; order; meromorphic function.

1. Introduction. Let $f=[f_1,\cdots,f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1,\cdots,f_{n+1}):\boldsymbol{C}\to\boldsymbol{C}^{n+1}\setminus\{\boldsymbol{0}\},$$

where n is a positive integer. We use the following notations:

$$\|f(z)\|=(|f_1(z)|^2+\cdots+|f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a}=(a_1,\cdots,a_{n+1})\in\boldsymbol{C}^{n+1}\setminus\{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\|=(|a_1|^2+\cdots+|a_{n+1}|^2)^{1/2},$$
$$(\boldsymbol{a},f)=a_1f_1+\cdots+a_{n+1}f_{n+1},$$
$$(\boldsymbol{a},f(z))=a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z).$$

We denote by $\boldsymbol{e}_j, 1\le j\le n+1$, the standard basis of $\boldsymbol{C}^{n+1}$.

The characteristic function of f is defined as follows (see [14]):

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

In addition, put

$$U(z)=\max_{1\le j\le n+1}|f_j(z)|,$$

then $U(z)\le\|f(z)\|\le(n+1)^{1/2}U(z)$ and we have

$$(1.1)\qquad T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log U(re^{i\theta})d\theta+O(1)$$

(see [1]).

We suppose throughout the paper that f is transcendental, that is to say,

$$\lim_{r\to\infty}\frac{T(r,f)}{\log r}=\infty$$

and that f is non-degenerate over $\boldsymbol{C}$; namely, $f_1,\cdots,f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

It is well-known that f is non-degenerate over $\boldsymbol{C}$ if and only if the Wronskian $W=W(f_1,\cdots,f_{n+1})$ of $f_1,\cdots,f_{n+1}$ is not identically equal to zero.

We denote the order of f by $\rho(f)$ and the lower order of f by $\mu(f)$ respectively:

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

$$\mu(f)=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

It is said that f is of regular growth if $\rho(f)=\mu(f)$.

We use the standard notation of the Nevanlinna theory of meromorphic functions in the complex plane ([3,4]).

For $\boldsymbol{a}\in\boldsymbol{C}^{n+1}\setminus\{\boldsymbol{0}\}$, we write

$$m(r,\boldsymbol{a},f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\cdot\|f(re^{i\theta})\|}{|(\boldsymbol{a},f(re^{i\theta}))|}d\theta,$$

$$N(r,\boldsymbol{a},f)=N\left(r,\frac{1}{(\boldsymbol{a},f)}\right).$$

We then have the First Fundamental Theorem ([14, p. 76]):

$$(1.2)\qquad T(r,f)=m(r,\boldsymbol{a},f)+N(r,\boldsymbol{a},f)+O(1).$$

We call the quantity

$$\delta(\boldsymbol{a},f)=1-\limsup_{r\to\infty}\frac{N(r,\boldsymbol{a},f)}{T(r,f)}=\liminf_{r\to\infty}\frac{m(r,\boldsymbol{a},f)}{T(r,f)}$$

the defect of $\boldsymbol{a}$ with respect to f. It is easy to see by (1.2) that

$$0\le\delta(\boldsymbol{a},f)\le1,$$

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since $m(r, \boldsymbol{a}, f) \geq 0$ and $N(r, \boldsymbol{a}, f) \geq 0$ $(r \geq 1)$.

We shall call an error term denoted by $S(r, f)$ a quantity such that if $\rho(f) = \infty$, $S(r, f) = o(T(r, f))$ as $r \to +\infty$, possibly outside a set of r of finite linear measure, and if $\rho(f) < \infty$, $S(r, f) = O(\log r)$ $(r \to +\infty)$.

Let X be a subset of $\boldsymbol{C}^{n+1} \setminus \{\boldsymbol{0}\}$ in N-subgeneral position satisfying $\#X \geq 2N - n + 1$, where N is an integer such that $N \geq n$. It is known that ([1], $N = n$; [5], $N > n$)

$$\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f) \leq 2N - n + 1. \tag{1.3}$$

The purpose of this paper is to prove

Theorem 1.1. *Let f be as cited above. Suppose that $\rho(f) < \infty$;*

(i) $\delta(\boldsymbol{e}_j, f) = 1$ $(j = 1, \cdots, n)$ *and that*

(ii) $\sum_{\boldsymbol{a} \in X} \delta(\boldsymbol{a}, f) = 2N - n + 1$.

Then f is of regular growth and $\rho(f)$ is a positive integer.

The case of $N = n$ was proved in [9].

2. Generalization of Nochka weight function. Let N, n and X be as in Section 1 such that $2N - n + 1 \leq \#X \leq \infty$. We note that X is in N-subgeneral position and that $\#X$ is not always finite. For a non-empty *finite* subset S of X, we denote by $V(S)$ the vector space spanned by elements of S and by $d(S)$ the dimension of $V(S)$. We put

$$\mathcal{O} = \{S \subset X \mid 0 < \#S \leq N + 1\}.$$

Lemma 2.1 ([2], p. 68). *For $R \subset S$ $(R, S \in \mathcal{O})$,*

$$\#R - d(R) \leq \#S - d(S) \leq N - n.$$

For $R \subsetneqq S$ $(R, S \in \mathcal{O})$, we put

$$\Lambda(R; S) = \frac{d(S) - d(R)}{\#S - \#R}.$$

Then, by Lemma 2.1 we have the following

Proposition 2.1 ([2], p. 67). $0 \leq \Lambda(R; S) \leq 1$.

Lemma 2.2 ([12], Lemma 2.3).

$\#\{d(S)/\#S \mid S \in \mathcal{O}\}$ *is finite.*

Definition 2.1 ([12], Definition 2.1).

$\lambda = \min_{S \in \mathcal{O}} d(S)/\#S$.

Proposition 2.2 ([12], Proposition 2.2).

$1/(N - n + 1) \leq \lambda \leq (n + 1)/(N + 1)$.

Proposition 2.3 ([12], Proposition 2.3). (I) *When $\lambda \geq (n + 1)/(2N - n + 1)$, for any $S \in \mathcal{O}$ it holds that*

$$\frac{n + 1}{2N - n + 1} \leq \frac{d(S)}{\#S}.$$

(II) *When $\lambda < (n + 1)/(2N - n + 1)$, there exist an integer p $(1 \leq p < (n + 1)/2)$ and a subfamily $\{T_i \mid 1 \leq i \leq p\}$ of $\mathcal{O}$ satisfying the following conditions*:

(i) $\phi = T_0 \subsetneqq T_1 \subsetneqq \cdots \subsetneqq T_p$, $d(T_p) < (n + 1)/2$;

(ii) $\Lambda(T_0; T_1) < \Lambda(T_1; T_2) < \cdots < \Lambda(T_{p-1}; T_p) < \dfrac{n + 1 - d(T_p)}{2N - n + 1 - \#T_p}$;

(iii) *Let $1 \leq i \leq p$. For any $U \in \mathcal{O}$ such that $T_{i-1} \subsetneqq U$ and $d(T_{i-1}) < d(U)$,*

(a) $\Lambda(T_{i-1}; T_i) \leq \Lambda(T_{i-1}; U)$,

and moreover

(b) $\Lambda(T_{i-1}; T_i) = \Lambda(T_{i-1}; U)$ *only if* $U \subseteqq T_i$;

(iv) *For any $U \in \mathcal{O}$ such that $T_p \subsetneqq U$ and $d(T_p) < d(U)$,*

$$\frac{n + 1 - d(T_p)}{2N - n + 1 - \#T_p} \leq \Lambda(T_p; U).$$

According to Proposition 2.3, we define a weight function w and a constant h for X as follows:

Definition 2.2 ([12], Definition 3.1). (I) When $\lambda \geq (n + 1)/(2N - n + 1)$, for any $\boldsymbol{a} \in X$ we set

$$w(\boldsymbol{a}) = \frac{n + 1}{2N - n + 1}, \quad h = \frac{2N - n + 1}{n + 1}.$$

(II) When $\lambda < (n + 1)/(2N - n + 1)$, we set

$$w(\boldsymbol{a}) = \begin{cases} \Lambda(T_{i-1}; T_i) & \text{for} \quad \boldsymbol{a} \in T_i \setminus T_{i-1} \\ & \qquad (i = 1, \cdots, p), \\ \dfrac{n + 1 - d(T_p)}{2N - n + 1 - \#T_p} & \text{for} \quad \boldsymbol{a} \in X \setminus T_p, \end{cases}$$

and

$$h = \frac{2N - n + 1 - \#T_p}{n + 1 - d(T_p)},$$

where $T_0 = \phi$, T_i and $\Lambda(T_{i-1}; T_i)$ $(i = 1, \cdots, p)$ are those given in Proposition 2.3 (II).

Proposition 2.4 ([12], Theorem 3.1). (a) *For any $\boldsymbol{a} \in X$, $0 < hw(\boldsymbol{a}) \leq 1$;*

(b-1) *For any $Q \subset X$ satisfying* (i) $Q \supset \{\boldsymbol{a} \in X \mid hw(\boldsymbol{a}) < 1\}$ *and* (ii) $2N - n + 1 \leq \#Q < \infty$,

$$\#Q - (2N - n + 1) = h\left(\sum_{\boldsymbol{a} \in Q} w(\boldsymbol{a}) - n - 1\right);$$

(b-2) $\sum_{\boldsymbol{a} \in X}(1 - hw(\boldsymbol{a}))$
$$= 2N - n + 1 - h(n + 1);$$

(c) $N/n \le h \le (2N-n+1)/(n+1)$;

(d) *For any* $S \in \mathcal{O}$, $\sum_{\boldsymbol{a}\in S} w(\boldsymbol{a}) \le d(S)$.

We consider the following set of weight functions on X:

Definition 2.3 ([13], Definition 4.1).

$$\mathcal{W} = \left\{ \tau : X \to (0,1] \mid \forall S \in \mathcal{O},\ \sum_{\boldsymbol{a}\in S} \tau(\boldsymbol{a}) \le d(S) \right\}.$$

Example 2.1 ([13], Example 4.1). (a) The weight function w in Definition 2.2 is in $\mathcal{W}$ by Definition 2.2 and Proposition 2.4 (d).

(b) Set $\tau_\lambda : X \to (0,1]$ such that $\tau_\lambda(\boldsymbol{a}) = \lambda$ for all $\boldsymbol{a} \in X$. Then $\tau_\lambda \in \mathcal{W}$. In fact, for any $S \in \mathcal{O}$,

$$\sum_{\boldsymbol{a}\in S} \tau_\lambda(\boldsymbol{a}) = \lambda \# S \le (d(S)/\#S)\#S = d(S).$$

(c) Let $\alpha_1, \cdots, \alpha_k$ $(k \ge 2)$ be positive numbers satisfying $\alpha_1 + \cdots + \alpha_k = 1$. If $w_1, \cdots, w_k \in \mathcal{W}$, then $\alpha_1 w_1 + \cdots + \alpha_k w_k \in \mathcal{W}$.

3. Lemmas and theorems. Let f and X etc. be as in Section 1 or 2.

Lemma 3.1 ([6], Théorème 2; [7], Théorème 3). *If there are $n+1$ linearly independent vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{n+1}$ in X such that*

$$\delta(\boldsymbol{a}_j, f) = 1 \quad (j = 1, \cdots, n+1),$$

then f is of regular growth and $\rho(f)$ is equal to a positive integer or ∞.

We note that f is not assumed to be non-degenerate in this lemma.

Now, suppose that f is non-degenerate. Let $d(z)$ be an entire function such that

$$f_j^{n+1}/d \ (j = 1, \cdots, n) \quad \text{and} \quad W(f_1, \cdots, f_{n+1})/d$$

are entire functions without common zeros.

Definition 3.1 ([8]). We call the holomorphic curve induced by the mapping

$$(f_1^{n+1}, \cdots, f_n^{n+1}, W(f_1, \cdots, f_{n+1})) : \boldsymbol{C} \to \boldsymbol{C}^{n+1}$$

the derived holomorphic curve of f and we write it by f^*:

$$f^* = [f_1^{n+1}/d, \cdots, f_n^{n+1}/d, W(f_1, \cdots, f_{n+1})/d].$$

Remark 3.1. When $n = 1$, f^* corresponds exactly to the derivative of the meromorphic function f_2/f_1.

Remark 3.2 ([8], Proposition 1). The definition of f^* does not depend on the choice of a reduced representation of f.

Lemma 3.2 ([8], Lemma 3). *We have*

$$T(r, f^*) \le (n+1)T(r,f) - N(r, 1/d) + S(r,f).$$

Proposition 3.1 ([8], Theorems 1, 2 and 3). (a) *f^* is transcendental.* (b) $\rho(f^*) = \rho(f)$. (c) *Even if f is non-degenerate, f^* can be degenerate.*

Lemma 3.3 ([11], Proposition 10). *Let $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ be in X $(N+1 \le q < \infty)$ and τ be in $\mathcal{W}$. Then the following inequalities hold*:

(I) $\sum_{j=1}^{q} \tau(\boldsymbol{a}_j) m(r, \boldsymbol{a}_j, f) \le (n+1)T(r,f) - N(r, 1/W) + S(r,f)$.

(II) $\sum_{\boldsymbol{a}\in X} \tau(\boldsymbol{a}) \delta(\boldsymbol{a}, f) \le n+1$.

Corollary 3.1. *If the equality holds in Lemma* 3.3 (II) *and $\rho(f) < \infty$, then*

$$\lim_{r\to\infty} N(r, 1/W(f_1, \cdots, f_{n+1}))/T(r,f) = 0.$$

Proof. From Lemma 3.3 (I) we obtain the inequality

$$\sum_{j=1}^{q} \tau(\boldsymbol{a}_j)\delta(\boldsymbol{a}_j, f) + \limsup_{r\to\infty} \frac{N(r, 1/W)}{T(r,f)} \le n+1,$$

since f is transcendental and $S(r,f) = O(\log r)$ as $\rho(f) < \infty$. So we easily obtain this corollary. □

For $X \in \mathcal{O}$ we put

$$X(0) = \{\boldsymbol{a} = (a_1, \cdots, a_{n+1}) \in X \mid a_{n+1} = 0\}.$$

Then, $\#X(0) \le N$ since X is in N-subgeneral position. We note that $d(X(0)) \le n$ from the definition of $X(0)$.

Proposition 3.2 (see [10], Theorem 1 (a)). *Let f be as in Section* 1 *and τ be in $\mathcal{W}$. For any $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_q$ in $X \setminus X(0)$ $(1 \le q < \infty)$ we have the following inequality*:

$$\sum_{j=1}^{q} \tau(\boldsymbol{a}_j) m(r, \boldsymbol{a}_j, f) \le m(r, \boldsymbol{e}_{n+1}, f^*) + S(r,f).$$

We used [10, Lemma 2] to prove [10, Theorem 1 (a)]. The property of ω used to prove [10, Lemma 2] is only that ω is in $\mathcal{W}$, and τ is also in $\mathcal{W}$. This implies that we can use τ instead of ω in [10, Lemma 2]. Due to this fact we can proceed the proof of Proposition 3.2 as in [10, Theorem 1 (a)] with τ and we obtain this inequality.

Corollary 3.2. *Let f and τ be as in Proposition* 3.2 *and $\rho(f) < \infty$. Then we have the following inequalities*:

$$\text{(3.1)} \qquad \frac{1}{n+1} \sum_{\boldsymbol{a}\in X\setminus X(0)} \tau(\boldsymbol{a})\delta(\boldsymbol{a}, f) \le \delta(\boldsymbol{e}_{n+1}, f^*),$$

$$(3.2)\quad \sum_{\boldsymbol{a}\in X\setminus X(0)} \tau(\boldsymbol{a})\delta(\boldsymbol{a},f) \le \liminf_{r\to\infty}\frac{T(r,f^*)}{T(r,f)} \le \limsup_{r\to\infty}\frac{T(r,f^*)}{T(r,f)} \le n+1.$$

Proof. First we prove the inequality (3.1). For any $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$ in $X\setminus X(0)$ $(1\le q<\infty)$, from Proposition 3.2 we obtain the inequality

$$\sum_{j=1}^{q}\tau(\boldsymbol{a}_j)\liminf_{r\to\infty}\frac{m(r,\boldsymbol{a}_j,f)}{T(r,f)} \le \liminf_{r\to\infty}\frac{m(r,\boldsymbol{e}_{n+1},f^*)}{T(r,f^*)}\cdot\frac{T(r,f^*)}{T(r,f)}.$$

Since f is transcendental and $\rho(f)<\infty$, $S(r,f)=O(\log r)$ and so

$$\lim_{r\to\infty}\frac{S(r,f)}{T(r,f)}=0.$$

We have $\limsup_{r\to\infty}T(r,f^*)/T(r,f)\le n+1$ by Lemma 3.2. Therefore we obtain that

$$(3.3)\quad \sum_{j=1}^{q}\tau(\boldsymbol{a}_j)\delta(\boldsymbol{a}_j,f)\le(n+1)\delta(\boldsymbol{e}_{n+1},f^*).$$

As $\boldsymbol{a}_j$ $(j=1,\cdots,q)$ are any elements of $X\setminus X(0)$, we easily obtain our inequality (3.1) from (3.3).

Next we prove the inequality (3.2). For any $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$ in $X\setminus X(0)$ $(1\le q<\infty)$, from Proposition 3.2 we obtain the inequality

$$\sum_{j=1}^{q}\tau(\boldsymbol{a}_j)m(r,\boldsymbol{a}_j,f)\le m(r,\boldsymbol{e}_{n+1},f^*)+S(r,f) \le T(r,f^*)+S(r,f),$$

so that from Lemma 3.2

$$\sum_{j=1}^{q}\tau(\boldsymbol{a}_j)\liminf_{r\to\infty}\frac{m(r,\boldsymbol{a}_j,f)}{T(r,f)}\le\liminf_{r\to\infty}\frac{T(r,f^*)}{T(r,f)} \le\limsup_{r\to\infty}\frac{T(r,f^*)}{T(r,f)} \le n+1.$$

Therefore we see that

$$\sum_{\boldsymbol{a}\in X\setminus X(0)}\tau(\boldsymbol{a})\delta(\boldsymbol{a},f)\le\liminf_{r\to\infty}\frac{T(r,f^*)}{T(r,f)} \le\limsup_{r\to\infty}\frac{T(r,f^*)}{T(r,f)}\le n+1,$$

which is the inequality (3.2). □

Lemma 3.4. *Let f and τ be as in Proposition 3.2. Then we have the inequality*

$$\sum_{\boldsymbol{a}\in X(0)}\tau(\boldsymbol{a})\delta(\boldsymbol{a},f)\le n.$$

Proof. We note that $X(0)\in\mathcal{O}$ since $\#X(0)\le N$. As $\tau\in\mathcal{W}$ and $\delta(\boldsymbol{a},f)\le 1$, we have the inequality

$$\sum_{\boldsymbol{a}\in X(0)}\tau(\boldsymbol{a})\delta(\boldsymbol{a},f)\le\sum_{\boldsymbol{a}\in X(0)}\tau(\boldsymbol{a})\le d(X(0))\le n.$$

□

Theorem 3.1. *Let f be as in Section 1 and τ be in $\mathcal{W}$. Suppose that $\rho(f)<\infty$ and that*

(i) $\delta(\boldsymbol{e}_j,f)=1$ $(j=1,\cdots,n)$;

(ii) *There exist $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_q$ in X $(2N-n+1\le q\le\infty)$ satisfying*

$$\sum_{j=1}^{q}\tau(\boldsymbol{a}_j)\delta(\boldsymbol{a}_j,f)=n+1.$$

Then f is of regular growth and $\rho(f)$ is a positive integer.

Proof. From Theorem 3.1 (ii), Corollary 3.2 and Lemma 3.4 we obtain the inequality

$$(3.4)\quad 1=n+1-n\le\sum_{\boldsymbol{a}\in X\setminus X(0)}\tau(\boldsymbol{a})\delta(\boldsymbol{a},f) \le\liminf_{r\to\infty}\frac{T(r,f^*)}{T(r,f)} \le\limsup_{r\to\infty}\frac{T(r,f^*)}{T(r,f)}\le n+1.$$

The inequality (3.4) implies that f^* is transcendental, and

$$(3.5)\quad \rho(f^*)=\rho(f),\quad \mu(f^*)=\mu(f).$$

From Definition 3.1, we obtain for $r\ge 1$

$$N(r,0,W/d)=N(r,1/W)-N(r,1/d) \le N(r,1/W).$$

From Corollary 3.1 and (3.4) it follows that

$$0\le\limsup_{r\to\infty}\frac{N(r,0,W/d)}{T(r,f^*)}\le\limsup_{r\to\infty}\frac{N(r,1/W)}{T(r,f^*)} \le\limsup_{r\to\infty}\frac{T(r,1/W)}{T(r,f)}\cdot\frac{T(r,f)}{T(r,f^*)}=0.$$

Thus we obtain the relation

$$(3.6)\quad \delta(\boldsymbol{e}_{n+1},f^*)=1-\limsup_{r\to\infty}\frac{N(r,0,W/d)}{T(r,f^*)}=1.$$

For $j=1,\cdots,n$, we obtain

$$N(r,0,f_j^{n+1}/d)=N(r,1/f_j^{n+1})-N(r,1/d) \le N(r,1/f_j^{n+1})\quad(r\ge 1)$$

and

$$0 \le \limsup_{r\to\infty} \frac{N(r,0,f_j^{n+1}/d)}{T(r,f^*)} \le \limsup_{r\to\infty} \frac{N(r,1/f_j^{n+1})}{T(r,f^*)}$$
$$\le (n+1) \limsup_{r\to\infty} \frac{N(r,1/f_j)}{T(r,f)} \cdot \frac{T(r,f)}{T(r,f^*)} = 0$$

from the condition (i) and (3.4), so that

$$(3.7) \quad \delta(\boldsymbol{e}_j, f^*) = 1 - \limsup_{r\to\infty} \frac{N(r,0,f_j^{n+1}/d)}{T(r,f^*)} = 1.$$

By using (3.6) and (3.7) we apply Lemma 3.1 to f^*. Then we see that f^* is of regular growth and $\rho(f^*)$ is a positive integer since $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}$ are linearly independent. From (3.5) we obtain that f is of regular growth and $\rho(f)$ is a positive integer.

□

Lemma 3.5. *We have the equality*

$$2N - n + 1 - \sum_{\boldsymbol{a}\in X} \delta(\boldsymbol{a}, f) = \sum_{\boldsymbol{a}\in X} (1 - hw(\boldsymbol{a}))(1 - \delta(\boldsymbol{a}, f)) + h\left(n + 1 - \sum_{\boldsymbol{a}\in X} w(\boldsymbol{a})\delta(\boldsymbol{a}, f)\right).$$

We obtain this equality as in [13], p. 371, Lemma 5.2.

Proof of Theorem 1.1. For any $\boldsymbol{a} \in X$ it follows from Proposition 2.4 (a) and the fact, $0 \le \delta(\boldsymbol{a}, f) \le 1$, that

$$(3.8) \quad (1 - hw(\boldsymbol{a}))(1 - \delta(\boldsymbol{a}, f)) \ge 0 \ (\boldsymbol{a} \in X).$$

Further from Lemma 3.3 (II) we obtain the inequality

$$(3.9) \qquad n + 1 - \sum_{\boldsymbol{a}\in X} w(\boldsymbol{a})\delta(\boldsymbol{a}, f) \ge 0.$$

From the condition (ii) of this theorem, Lemma 3.5, (3.8) and (3.9), we obtain that

$$n + 1 - \sum_{\boldsymbol{a}\in X} w(\boldsymbol{a})\delta(\boldsymbol{a}, f) = 0,$$

which is the condition (ii) of Theorem 3.1 with $\tau = w$. Considering the condition (i) of this theorem we obtain our conclusion from Theorem 3.1. □

Remark 3.3. We note that the condition (ii) of this theorem, Lemma 3.5, (3.8) and (3.9) imply

$$(1 - hw(\boldsymbol{a}))(1 - \delta(\boldsymbol{a}, f)) = 0 \ (\boldsymbol{a} \in X).$$

References

[1] E. Cartan, Sur les zéros des combinaisons linéaires de p fonctions holomorphes donées, Mathematica (Cluj) **7** (1933), 5–31.

[2] H. Fujimoto, *Value distribution theory of the Gauss map of minimal surfaces in* $\mathbf{R}^m$, Aspects of Mathematics, E21, Friedr. Vieweg & Sohn, Braunschweig, 1993.

[3] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[4] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Chelsea Publishing Co., New York, 1974.

[5] E. I. Nochka, On the theory of meromorphic curves, Dokl. Akad. Nauk SSSR **269** (1983), no. 3, 547–552.

[6] N. Toda, Sur une relation entre la croissance et le nombre de valeurs déficientes de fonctions algébroïdes ou de systèmes, Kōdai Math. Sem. Rep. **22** (1970), 114–121.

[7] N. Toda, Sur la croissance de fonctions algébroïdes à valeurs déficientes, Kōdai Math. Sem. Rep. **22** (1970), 324–337.

[8] N. Toda, An extension of the derivative of meromorphic functions to holomorphic curves, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), no. 6, 159–163.

[9] N. Toda, On the order of holomorphic curves with maximal deficiency sum, Kodai Math. J. **18** (1995), no. 3, 451–474.

[10] N. Toda, On the second fundamental inequality for holomorphic curves, Bull. Nagoya Inst. Tech. **50** (1998), 123–135.

[11] N. Toda, On the deficiency of holomorphic curves with maximal deficiency sum. II, in *Progress in Analysis, Vol. I, II* (*Berlin, 2001*), 287–300, World Sci. Publ., River Edge, NJ, 2003.

[12] N. Toda, A generalization of Nochka weight function, Proc. Japan Acad. Ser. A Math. Sci. **83** (2007), no. 9–10, 170–175.

[13] N. Toda, On the truncated defect relation for holomorphic curves, Kodai Math. J. **32** (2009), no. 2, 352–389.

[14] H. Weyl, *Meromorphic Functions and Analytic Curves*, Annals of Mathematics Studies, 12, Princeton University Press, Princeton, NJ, 1943.