SVV algebras

By Ruari Donald WALKER

Bureau 627, Université Paris Diderot-Paris VII, Bâtiment Sophie Germain,
75205 Paris Cedex 13, France

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Abstract: In 2010 Shan, Varagnolo and Vasserot introduced a family of graded algebras in order to prove a conjecture of Kashiwara and Miemietz which stated that the finite-dimensional representations of affine Hecke algebras of type D categorify a module over a certain quantum group. We study these algebras, and in various cases, show how they relate to Varagnolo-Vasserot algebras and to quiver Hecke algebras which in turn allows us to deduce various homological properties.

Key words: SVV algebras; Morita equivalence; affine cellular; affine quasi-hereditary.

Introduction. Following a conjecture of Enomoto and Kashiwara in [EK06] concerning categories of modules over affine Hecke algebras of type B, proved in general by Varagnolo and Vasserot [VV11], Kashiwara and Miemietz conjectured analogous results for type D affine Hecke algebras, see [KM07]. They predicted that categories of modules over affine Hecke algebras of type D categorify a highest weight module over a certain quantum group. This conjecture was confirmed by results of Shan, Varagnolo and Vasserot in [SVV11], in which they introduce and use a family of graded algebras, by showing that categories of modules over these algebras are equivalent to categories of modules over affine Hecke algebras of type D. These algebras are similar to KLR algebras and to VV algebras, the latter of which were studied in [Wal]. This note is a study of these algebras, which we call SVV algebras. We obtain a Morita equivalence between SVV algebras and a direct product of VV algebras. We use KLR algebras to show that, in certain settings, SVV algebras are graded affine quasi-hereditary and graded affine cellular.

1. Preliminaries and notation. Throughout this paper we will denote by k a field with $\mathrm{char}(k) \neq 2$ and by a grading we will always mean a $\mathbf{Z}$-grading. We write q to denote both a formal variable and a degree shift functor which shifts the degree by 1. So qM is a graded A-module with k^{th} graded component $(qM)_k = M_{k-1}$, where $M = \bigoplus_{n\in\mathbf{Z}} M_n$ is a graded A-module.

The Weyl group of type D, W_m^D has generators $s_0, s_1, \ldots, s_{m-1}$ which are subject to the relations s_i^2 $\forall i$, $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $1 \leq i \leq m-2$, $s_0s_i = s_is_0$ $\forall i \neq 2$, $s_is_j = s_js_i$ for $|i-j| > 1$ and $1 \leq i,j \leq m-1$, $s_0s_2s_0 = s_2s_0s_2$. Now let $\tau_0, \tau_1, \ldots, \tau_{m-1}$ be the generators of the type B Weyl group, W_m^B. We can consider W_m^D a subgroup of W_m^B via the injection $W_m^D \hookrightarrow W_m^B$ given by $s_0 \mapsto \tau_0\tau_1\tau_0$, $s_k \mapsto \tau_k$ for $1 \leq k \leq m-1$.

There are exactly two parabolic subgroups of W_m^D which are both isomorphic to the symmetric group on m letters. We label these subgroups $\mathfrak{S}_m$ and S_m; they are the subgroups generated by $s_1, s_2, \ldots, s_{m-1}$ and $s_0, s_2, \ldots, s_{m-1}$, respectively.

1.1. SVV algebras. In this section we recall a family of graded algebras which were introduced by Shan, Varagnolo and Vasserot [SVV11]. We will call them SVV algebras.

We start by fixing an element $p \in k^\times$. Consider the action of $\mathbf{Z} \rtimes \mathbf{Z}_2$ on $k^\times$ given by $(n, \pm 1)\cdot\lambda = p^{2n}\lambda^{\pm 1}$. Fix a $\mathbf{Z} \rtimes \mathbf{Z}_2$-orbit I_λ. So $I = I_\lambda = \{p^{2n}\lambda^{\pm 1} \mid n \in \mathbf{Z}\}$ is the $\mathbf{Z} \rtimes \mathbf{Z}_2$-orbit of λ. To I we associate a quiver $\Gamma = \Gamma_I$. The vertices of Γ are the elements $i \in I$ and we have arrows $p^2 i \longrightarrow i$ for every $i \in I$. We always assume that $\pm 1 \notin I$ and that $p \neq \pm 1$. If $p \notin I_\lambda$ then we can write $I_\lambda = I_\lambda^+ \sqcup I_\lambda^-$, where $I_\lambda^\pm = \{p^{2n}\lambda^{\pm 1} \mid n \in \mathbf{Z}\}$. Similarly, when $p \in I_p$ and p is not a root of unity then we can write $I_p = I_p^+ \sqcup I_p^-$, where $I_p^\pm = \{p^{\pm(2n+1)} \mid n \in \mathbf{Z}_{\geq 0}\}$. Let ${}^\theta\mathbf{N}I = \{\nu = \sum_{i\in I}\nu_i i \mid \nu_i = \nu_{i^{-1}}, \nu$ has finite support, $\nu_i \in \mathbf{Z}_{\geq 0}$ $\forall i\}$. Recall the height of $\nu \in {}^\theta\mathbf{N}I$ is denoted by $|\nu|$

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and is defined as $|\nu| = \sum_{i\in I} \nu_i$. We also recall that $i \in I$ has multiplicity one in ν if $\nu_i = 1$.

*Throughout this work we assume that if $p \in I$ then p is **not** a root of unity, so that we always have $I_\lambda = I_\lambda^+ \sqcup I_\lambda^-$.*

For any $\nu \in {}^\theta\mathbf{N}I$ we have $\nu_i = \nu_{i^{-1}}$ for all $i \in I$ which means that we can always write $\nu = \nu^+ + \nu^-$, where $\nu^\pm = \sum_{i\in I^\pm} \nu_i i \in \mathbf{N}I$. It now makes sense to talk about the KLR algebras associated to ν^+ and to ν^-. Denote these algebras by $\mathbf{R}_{\nu^+}$ and $\mathbf{R}_{\nu^-}$, respectively. Recall that the KLR algebras are a family of graded algebras that have been introduced in [KL09] and [Rou] in order to categorify quantum groups.

For $\nu \in {}^\theta\mathbf{N}I$ with $|\nu| = 2m$, define

$$ {}^\theta I^\nu := \left\{ \mathbf{i} = (i_1, \ldots, i_m) \in I^m \;\middle|\; \sum_{k=1}^m i_k + \sum_{k=1}^m i_k^{-1} = \nu \right\}. $$

For $\nu \in {}^\theta\mathbf{N}I$ with $|\nu| = 2m$, $m > 1$, the SVV algebra, denoted by ${}^\circ\mathbf{R}(\Gamma)_\nu$, is the graded k-algebra generated by elements

$$ \{e(\mathbf{i}) \mid \mathbf{i} \in {}^\theta I^\nu\} \cup \{x_1, \ldots, x_m\} \cup \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\} $$

which are subject to the following relations.

(a) $e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i})$, $\sigma_k e(\mathbf{i}) = e(s_k\mathbf{i})\sigma_k$, $x_l e(\mathbf{i}) = e(\mathbf{i})x_l$, $\sum_{\mathbf{i}\in{}^\theta I^\nu} e(\mathbf{i}) = 1$.

(b) The x_l's commute.

(c) For $1 \le k \le m-1$,

$$ \sigma_k^2 e(\mathbf{i}) = \begin{cases} (x_{k+1} - x_k)e(\mathbf{i}) & i_k \leftarrow i_{k+1} \\ (x_k - x_{k+1})e(\mathbf{i}) & i_k \rightarrow i_{k+1} \\ -(x_{k+1} - x_k)^2 e(\mathbf{i}) & i_k \leftrightarrow i_{k+1} \\ e(\mathbf{i}) & i_k \not\leftrightarrow i_{k+1} \\ 0 & i_k = i_{k+1}, \end{cases} $$

$$ \sigma_0^2 e(\mathbf{i}) = \begin{cases} -(x_1 + x_2)e(\mathbf{i}) & i_1^{-1} \rightarrow i_2 \\ (x_1 + x_2)e(\mathbf{i}) & i_1^{-1} \leftarrow i_2 \\ -(x_1 + x_2)^2 e(\mathbf{i}) & i_1^{-1} \leftrightarrow i_2 \\ e(\mathbf{i}) & i_1^{-1} \not\leftrightarrow i_2 \\ 0 & i_1^{-1} = i_2 \end{cases} $$

$\sigma_k\sigma_l = \sigma_l\sigma_k$ for $1 \le k < l-1 < m-1$ or $0 = k < l \ne 2$.

For $1 \le k < m-1$;

$$ (\sigma_{k+1}\sigma_k\sigma_{k+1} - \sigma_k\sigma_{k+1}\sigma_k)e(\mathbf{i}) = \begin{cases} e(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \rightarrow i_{k+1} \\ -e(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \leftarrow i_{k+1} \\ (2x_{k+1} - x_{k+2} - x_k)e(\mathbf{i}) & i_k = i_{k+2} \text{ and } i_k \leftrightarrow i_{k+1} \\ 0 & i_k \ne i_{k+2} \text{ or } i_k \not\leftrightarrow i_{k+1}, \end{cases} $$

$$ (\sigma_2\sigma_0\sigma_2 - \sigma_0\sigma_2\sigma_0)e(\mathbf{i}) = \begin{cases} e(\mathbf{i}) & i_1^{-1} = i_3 \text{ and } i_1^{-1} \rightarrow i_2 \\ -e(\mathbf{i}) & i_1^{-1} = i_3 \text{ and } i_1^{-1} \leftarrow i_2 \\ (x_1 + 2x_2 - x_3)e(\mathbf{i}) & i_1^{-1} = i_3 \text{ and } i_1^{-1} \leftrightarrow i_2 \\ 0 & i_1^{-1} \ne i_3 \text{ or } i_1^{-1} \not\leftrightarrow i_2. \end{cases} $$

(d) For $1 \le k < m$, $(\sigma_k x_l - x_{s_k(l)}\sigma_k)e(\mathbf{i}) = \begin{cases} -e(\mathbf{i}) & \text{if } l = k, i_k = i_{k+1} \\ e(\mathbf{i}) & \text{if } l = k+1, i_k = i_{k+1} \\ 0 & \text{else.} \end{cases}$

(e) For $\ell = 1, 2$, $(\sigma_0 x_\ell + x_{3-\ell}\sigma_0)e(\mathbf{i}) = \begin{cases} e(\mathbf{i}) & \text{if } i_1^{-1} = i_2 \\ 0 & \text{if } i_1^{-1} \ne i_2. \end{cases}$

The grading on ${}^\circ\mathbf{R}(\Gamma)_\nu$ is given as follows:

$$ \deg(e(\mathbf{i})) = 0, \quad \deg(x_l e(\mathbf{i})) = 2, $$

$$ \deg(\sigma_0 e(\mathbf{i})) = \begin{cases} |i_1^{-1} \rightarrow i_2| + |i_1^{-1} \leftarrow i_2| & \text{if } i_1^{-1} \ne i_2 \\ -2 & \text{if } i_1^{-1} = i_2, \end{cases} $$

$$ \deg(\sigma_k e(\mathbf{i})) = \begin{cases} |i_k \rightarrow i_{k+1}| + |i_k \leftarrow i_{k+1}| & \text{if } i_k \ne i_{k+1} \\ -2 & \text{if } i_k = i_{k+1} \end{cases} $$

where $|i \rightarrow j|$ denotes the number of arrows from i to j in the quiver Γ.

If $\nu = 0$ we set ${}^\circ\mathbf{R}(\Gamma)_\nu = k \oplus k$ and if $\nu = i + i^{-1}$, for some $i \in I$ then,

$$ {}^\circ\mathbf{R}(\Gamma)_\nu = k[x]e(i) \oplus k[x]e(i^{-1}). $$

The action of W_m^D on $\mathbf{i} = (i_1, \ldots, i_m) \in {}^\theta I^\nu$ is given via

$$ s_0 \cdot (i_1, i_2, \ldots, i_m) = (i_2^{-1}, i_1^{-1}, \ldots, i_m) $$
$$ s_k \cdot (\ldots, i_k, i_{k+1}, \ldots) = (\ldots, i_{k+1}, i_k, \ldots) $$

for $1 \le k < m$.

Take $w \in W_m^D$ and fix a reduced expression $w = s_{i_1} \cdots s_{i_k}$. We then set $\sigma_w e(\mathbf{i}) = \sigma_{i_1} \cdots \sigma_{i_k} e(\mathbf{i})$ and, for the identity element $1 \in W_m^D$, we have $\sigma_1 e(\mathbf{i}) = e(\mathbf{i})$. From the relations we can see that $\sigma_w e(\mathbf{i})$ is dependent upon the choice of reduced expression of w. Therefore whenever we write $\sigma_w e(\mathbf{i})$ it should be understood that, although not always specified, we are fixing a choice of reduced expression of w.

We can visualise the algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ as a quiver with the vertices given by the idempotents $e(\mathbf{i})$ and the arrows labelled by generators $x_1, \ldots, x_m$, $\sigma_0, \ldots, \sigma_{m-1}$ and determined by the relationship between idempotents. It is always the case that this quiver has two connected components so that we always have ${}^\circ\mathbf{R}(\Gamma)_\nu \cong \mathbf{e}_1 {}^\circ\mathbf{R}(\Gamma)_\nu \mathbf{e}_1 \times \mathbf{e}_2 {}^\circ\mathbf{R}(\Gamma)_\nu \mathbf{e}_2$, where $\mathbf{e}_1, \mathbf{e}_2$ are certain idempotents in ${}^\circ\mathbf{R}(\Gamma)_\nu$. In addition, we will see in Lemma 2.4 that, as algebras, these two components are isomorphic. Therefore it suffices to study one of these components in order to understand the algebra. For example, to show that ${}^\circ\mathbf{R}(\Gamma)_\nu$ is graded affine cellular and affine quasi-hereditary it is enough to show that one of the components, say $\mathbf{e}_2{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e}_2$, has these properties (using Remark 1.4 and Proposition 1.6).

Given the data in the definition, together with a fixed $\nu \in {}^\theta\mathbf{N}I$, we have a VV algebra $\mathfrak{W}_\nu$ (see [Wal] for the definition). In [SVV11], the authors note that there is a canonical inclusion of algebras ${}^\circ\mathbf{R}(\Gamma)_\nu \hookrightarrow \mathfrak{W}_\nu$ given by $e(\mathbf{i}), \sigma_k, x_i \mapsto e(\mathbf{i}), \sigma_k, x_i$ for $k = 1, \cdots m-1, i = 1, \cdots, m$, $\sigma_0 \mapsto \pi\sigma_1\pi$ so that the SVV algebras are unital subalgebras of the VV algebras.

Lemma 1.1 (Basis theorem for SVV algebras)**.** *Take $\nu \in {}^\theta\mathbf{N}I$ with $|\nu| = 2m$. For each element $w \in W_m^D$ fix a reduced expression. The set of elements*

$$\{\sigma_w x_1^{n_1} \cdots x_m^{n_m} e(\boldsymbol{i}) \mid w \in W_m^D, \boldsymbol{i} \in {}^\theta I^\nu, n_k \in \mathbf{Z}_{\geq 0}\ \forall k\}$$

forms a k-basis for ${}^\circ\mathbf{R}(\Gamma)_\nu$.

Proof. The proof is similar to that of [KL09, Theorem 2.5]. In particular we show this set is a spanning set in exactly the same way as for KLR algebras. For linear independence we use the polynomial representation of ${}^\circ\mathbf{R}(\Gamma)_\nu$ (see [SVV11]) and show that the elements in the set act by linearly independent operators. □

Remark 1.2. Suppose we take $\nu \in {}^\theta\mathbf{N}I$ such that $\mathrm{supp}(\nu^+)$ consists of two connected components. In other words, $\nu = \nu_1 + \nu_2$ where $\mathrm{supp}(\nu_1^+)$ and $\mathrm{supp}(\nu_2^+)$ are both connected and there are no arrows between any $i \in \mathrm{supp}(\nu_1^+)$ and $j \in \mathrm{supp}(\nu_2^+)$. It can be shown, using a similar proof as in [Wal, Proposition 2.6], that there is a Morita equivalence ${}^\circ\mathbf{R}(\Gamma)_\nu \sim {}^\circ\mathbf{R}(\Gamma)_{\nu_1} \otimes {}^\circ\mathbf{R}(\Gamma)_{\nu_2}$. So we may assume that ν is chosen in such a way that $\mathrm{supp}(\nu^+)$ is connected.

1.2. Affine cellularity. We now recall the definition of an affine cellular algebra [KX12]. Here we use the basis definition which is analogous to the way that Graham and Lehrer defined cellular algebras in [GL96]. The definition we recall is taken from [Cui], where it is shown to be equivalent to the basis-free definition in [KX12]. Let k be a noetherian domain and let A a unitary k-algebra. By an affine algebra, we mean a commutative k-algebra of the form $B = k[x_1, \ldots, x_t]/I$, for some ideal I and some positive integer t.

Definition 1.3. We say that $(\Lambda, M, B, C, *)$ is an affine cell datum for A, where $(\Lambda, \leq)$ is a finite poset, $M(\lambda)$ is a finite set for each $\lambda \in \Lambda$, B_λ is an affine k-algebra with an anti-involution σ_λ, $C = \{C_{s,t}^\lambda \mid \lambda \in \Lambda \text{ and } s, t \in M(\lambda)\}$ is a subset of A, and $*$ is a k-linear anti-involution on A, if the following are satisfied.

(a) For each $\lambda \in \Lambda$, let $\widehat{A}^\lambda$ be the right B_λ-span of $\{C_{s,t}^\lambda\}_{s,t\in M(\lambda)}$. Then $\{C_{s,t}^\lambda\}_{s,t\in M(\lambda)}$ is a B_λ-basis of the right B_λ-module $\widehat{A}^\lambda$, and $A = \bigoplus_{\lambda\in\Lambda} \widehat{A}^\lambda$ as k-modules.

(b) For each $\lambda \in \Lambda$, let $\widehat{A}^{>\lambda} = \sum_{\mu>\lambda} \widehat{A}^\mu$. For $\lambda \in \Lambda$, $s \in M(\lambda)$ and $a \in A$, $b \in B_\lambda$, there exist coefficients $r_v^s(a) \in B_\lambda$ such that for all $t \in M(\lambda)$,

$$a \cdot (C_{s,t}^\lambda \cdot b) \equiv \sum_{v\in M(\lambda)} C_{v,t}^\lambda \cdot r_v^s(a) b \ (\mathrm{mod}(\widehat{A}^{>\lambda})),$$

and the coefficients $r_v^s(a) \in B_\lambda$ are independent of t.

(c) For all $\lambda \in \Lambda$, $s, t \in M(\lambda)$, and for any $b \in B_\lambda$, $(C_{s,t}^\lambda \cdot b)^* = C_{t,s}^\lambda \cdot \sigma_\lambda(b)$.

The algebra A is said to be affine cellular if such an affine cell datum exists.

Remark 1.4. Let A_1 and A_2 be affine cellular algebras with affine cell data $(\Lambda_1, M_1, B_1, C_1, *_1)$ and $(\Lambda_2, M_2, B_2, C_2, *_2)$, respectively. Then $A_3 := A_1 \times A_2$ has an affine cell datum $(\Lambda_3, M_3, B_3, C_3, *_3)$ where $\Lambda_3 = \Lambda_1 \cup \Lambda_2$ with partial ordering given as follows: $\lambda \leq \mu$ if and only λ and μ both lie in Λ_i, for $i \in \{1, 2\}$, and moreover $\lambda \leq \mu$ in Λ_i. We also have, for any $\lambda \in \Lambda_3$,

$$M_3(\lambda) = \begin{cases} M_1(\lambda) & \text{if } \lambda \in \Lambda_1 \\ M_2(\lambda) & \text{if } \lambda \in \Lambda_2, \end{cases}$$

$$B_3(\lambda) = \begin{cases} B_1(\lambda) & \text{if } \lambda \in \Lambda_1 \\ B_2(\lambda) & \text{if } \lambda \in \Lambda_2, \end{cases}$$

with the obvious anti-involutions. Furthermore, $C_3 = C_1 \cup C_2$ and $*_3$ is the anti-involution on A_3 defined by, $*_3(a_1, a_2) = (*_1(a_1), *_2(a_2))$. One can quickly check that this really does define an affine cell datum for A_3 so that $A_3 = A_1 \times A_2$ is affine cellular.

Since ${}^\circ\mathbf{R}(\Gamma)_\nu \cong \mathbf{e}_1{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e}_1 \times \mathbf{e}_2{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e}_2$, in order to show that ${}^\circ\mathbf{R}(\Gamma)_\nu$ is affine cellular it suffices to show that each component is affine cellular.

1.3. Affine quasi-heredity. We also recall the notions of affine quasi-heredity, for left Noetherian Laurentian algebras, and affine highest weight categories, introduced by Kleshchev [Kle15] as a graded analogue of the theory of Cline, Parshall and Scott. KLR algebras of finite Lie type are graded affine quasi-hereditary, as are certain classes of VV algebras. Let $\mathcal{B}$ be the class of all positively graded polynomial algebras. Recall from [Kle15] that a two-sided ideal $J \subseteq A$ is an affine heredity ideal if; (SI1): $\mathrm{Hom}_A(J, A/J) = 0$, (SI2): as a left module $J \cong m(q)P(\pi)$ for some graded multiplicity $m(q) \in \mathbf{Z}[q, q^{-1}]$ and some $\pi \in \Pi$ such that $B_\pi := \mathrm{End}_A(P(\pi))^{\mathrm{op}} \in \mathcal{B}$, and (PSI): as a right B_π-module $P(\pi)$ is free finite rank.

By Lemma 6.5 in [Kle15], if J is an ideal in A which is projective as a left A-module, then (SI1) is equivalent J being an idempotent ideal, i.e. $J = AeA$, for an idempotent $e \in A$.

Definition 1.5. An algebra A is affine quasi-hereditary if there exists a finite chain of ideals

$$(0) = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n = A$$

with J_{i+1}/J_i an affine heredity ideal in A/J_i, for all $0 \leq i < n$. Such a chain of ideals is called an affine heredity chain.

Proposition 1.6. *If A, B are affine quasi-hereditary algebras then the direct product $A \times B$ is an affine quasi-hereditary algebra.*

Proof. Suppose A and B have affine heredity chains

$$(0) \subseteq A_1 \subseteq \cdots \subseteq A_n = A$$
$$(0) \subseteq B_1 \subseteq \cdots \subseteq B_m = B$$

respectively. Then one shows that

$$(0) \subseteq A_1 \times (0) \subseteq A_1 \times B_1 \subseteq A_1 \times B_2 \subseteq \cdots$$
$$\cdots \subseteq A_1 \times B \subseteq A_2 \times B \subseteq \cdots \subseteq A \times B$$

is an affine heredity chain for $A \times B$. □

2. Results. We remind the reader that if $p \in I$ then we assume that p is not a root of unity.

Lemma 2.1. *For $\nu \in {}^\theta\mathbf{N}I$ let $\mathbf{e} = \sum_{\mathbf{i} \in I^{\nu^+}} e(\boldsymbol{i})$. Then $\mathbf{e}{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e} \cong \mathbf{R}_{\nu^+}$, i.e. every SVV algebra has a distinguished idempotent subalgebra isomorphic to a KLR algebra.*

Proof. The proof is similar to [Wal, Proposition 1.17]. □

Remark 2.2. More generally, we can always express any $\nu \in {}^\theta\mathbf{N}I$ in the form $\nu = \tilde{\nu} + \tilde{\nu}^{-1}$, for some $\tilde{\nu} = \sum_{i \in I} \tilde{\nu}_i i \in \mathbf{N}I$, where $\tilde{\nu}^{-1} = \sum_{i \in I} \tilde{\nu}_i i^{-1} \in \mathbf{N}I$. Let $\mathbf{e} = \sum_{\mathbf{i} \in I^{\tilde{\nu}}} e(\mathbf{i})$. Provided $i + i^{-1}$ is not a summand of $\tilde{\nu}$, for any $i \in I$, one can use the same argument in Lemma 2.1 show that $\mathbf{e}{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e} \cong \mathbf{R}_{\tilde{\nu}}$, the KLR algebra associated to $\tilde{\nu}$.

Let $S_m := \langle s_0, s_2, s_3, \cdots, s_{m-1}\rangle$ be the parabolic subgroup of W_m^D generated by s_i, $i \neq 1$. We have already noted that S_m is isomorphic to the symmetric group $\mathfrak{S}_m$.

Take $\nu \in {}^\theta\mathbf{N}I$, $|\nu| = 2m$, and let $i_{\min}$ be the summand $p^{2k}\lambda$ of ν^+ such that k is minimal. Let $\mu = \nu^+ - i_{\min} + i_{\min}^{-1}$ and let $\mathbf{j} \in I^\mu = \{(j_1, \ldots, j_m) \in I^m \mid \sum_{k=1}^m j_k = \mu\}$ be such that $j_1 = i_{\min}^{-1}$ and $j_2, \ldots, j_m$ are ordered by power of p. For example, when $\nu = m\lambda + m\lambda^{-1}$ for some $m > 1$ and some $\lambda \in I$, we have $i_{\min} = \lambda$ and $\mathbf{j} = (\lambda^{-1}, \lambda, \lambda, \cdots, \lambda)$. Let J be the following subset of ${}^\theta I^\nu$.

$$J := \{w \cdot \mathbf{j} \mid w \in S_m \subset W_m^D\}$$

where the action of elements $w \in S_m$, considered elements of W_m^D, should be the obvious one.

Example 2.3. Take $\nu = \lambda + p^2\lambda + p^4\lambda + \lambda^{-1} + p^{-2}\lambda^{-1} + p^{-4}\lambda^{-1}$. Then $i_{\min} = \lambda$, $\mathbf{j} = (\lambda^{-1}, p^2\lambda, p^4\lambda)$ and

$$J = \left\{ \begin{array}{ll} (\lambda^{-1}, p^2\lambda, p^4\lambda), & (p^{-2}\lambda^{-1}, \lambda, p^4\lambda), \\ (p^{-2}\lambda^{-1}, p^4\lambda, \lambda), & (p^{-4}\lambda^{-1}, p^2\lambda, \lambda), \\ (p^{-4}\lambda^{-1}, \lambda, p^2\lambda), & (\lambda^{-1}, p^4\lambda, p^2\lambda) \end{array} \right\}.$$

Let $\mathcal{D} = \mathcal{D}(W_m^D/\mathfrak{S}_m)$ and $\mathcal{D}' = \mathcal{D}(W_m^D/S_m)$ denote the minimal length left coset representatives of $\mathfrak{S}_m$ and S_m in W_m^D, respectively. Note that $\mathcal{D}'$ consists of elements w' which are obtained from $w \in \mathcal{D}$ by replacing every occurrence of s_0 and every occurrence of s_1 in a reduced expression of w with s_1 and with s_0, respectively. Then put $\mathbf{e}_1 := \sum_{\substack{\mathbf{i} \in J \\ w' \in \mathcal{D}'}} e(w' \cdot \mathbf{i})$ and $\mathbf{e}_2 := \sum_{\substack{\mathbf{i} \in I^{\nu^+} \\ w \in \mathcal{D}}} e(w \cdot \mathbf{i})$. By considering the way in which the elements $w \in \mathcal{D}$, $w' \in \mathcal{D}'$ act on these tuples $\mathbf{i}$ we see that we have $\mathbf{e}_1 + \mathbf{e}_2 = 1$ in ${}^\circ\mathbf{R}(\Gamma)_\nu$ and therefore a decomposition ${}^\circ\mathbf{R}(\Gamma)_\nu = \mathbf{e}_1{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e}_1 \times \mathbf{e}_2{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e}_2$.

Lemma 2.4. *There is a k-algebra isomorphism $\boldsymbol{e}_2{}^\circ\mathbf{R}(\Gamma)_\nu\boldsymbol{e}_2 \cong \boldsymbol{e}_1{}^\circ\mathbf{R}(\Gamma)_\nu\boldsymbol{e}_1$.*

Proof. Define the map

$$\varphi : \mathbf{e}_2{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e}_2 \longrightarrow \mathbf{e}_1{}^\circ\mathbf{R}(\Gamma)_\nu\mathbf{e}_1,$$
$$e(i_1, i_2, \cdots, i_m) \mapsto e(i_1^{-1}, i_2, \cdots, i_m),$$
$$\sigma_\ell \mapsto \begin{cases} \sigma_{1-\ell} & \text{if } \ell = 0, 1 \\ \sigma_\ell & \text{if } \ell \neq 0, 1, \end{cases}$$
$$x_k \mapsto \begin{cases} -x_1 & \text{if } k = 1 \\ x_k & \text{if } k \neq 1 \end{cases}$$

and extend k-linearly and multiplicatively. Examining the defining relations of ${}^\circ\mathbf{R}(\Gamma)_\nu$ we see that this is well-defined and is therefore an algebra homomorphism. In fact, it is an isomorphism of k-vector spaces and hence an isomorphism of algebras as there is an obvious inverse map. □

As previously mentioned, the quiver which represents the algebra ${}^{\circ}\mathbf{R}(\Gamma)_{\nu}$ has two connected components. Lemma 2.4 tells us that the algebras which are associated to these connected components are isomorphic.

Corollary 2.5. *There is an algebra isomorphism* $\boldsymbol{e}_{-}{}^{\circ}\mathbf{R}(\Gamma)_{\nu}\boldsymbol{e}_{-} \cong \mathbf{R}_{\nu^+}$, *where* $\boldsymbol{e}_{-} := \sum_{\mathbf{i}\in J} e(\boldsymbol{i})$.

Proof. Under the isomorphism in Lemma 2.4 $\mathbf{e} = \sum_{\mathbf{i}\in I^{\nu^+}} e(\mathbf{i})$ is mapped to $\mathbf{e}_{-} = \sum_{\mathbf{i}\in J} e(\mathbf{i})$. It follows that $\mathbf{e}{}^{\circ}\mathbf{R}(\Gamma)_{\nu}\mathbf{e}$ is isomorphic to $\mathbf{e}_{-}{}^{\circ}\mathbf{R}(\Gamma)_{\nu}\mathbf{e}_{-}$. Now we use Lemma 2.1. □

Fix any element $q \in k^{\times}$ in such a way that $q \notin I$. We can then define a VV algebra $\mathfrak{W}_{\nu}$ for any given $\nu \in {}^{\theta}\mathbf{N}I$, see [Wal] for details.

Theorem 2.6. *There is a Morita equivalence* ${}^{\circ}\mathbf{R}(\Gamma)_{\nu} \sim \mathfrak{W}_{\nu} \times \mathfrak{W}_{\nu}$. *In particular, this demonstrates that any irreducible type D module arises from an irreducible type B module.*

Proof. Using Lemma 2.4, it suffices to show there is a Morita equivalence $\mathbf{e}_2{}^{\circ}\mathbf{R}(\Gamma)_{\nu}\mathbf{e}_2 \sim \mathfrak{W}_{\nu}$. Recall that we set $\mathbf{e}_2 := \sum_{\substack{\mathbf{i}\in I^{\nu^+}\\ w\in\mathcal{D}}} e(w\cdot\mathbf{i}) \in {}^{\circ}\mathbf{R}(\Gamma)_{\nu}$. We can also consider $\mathbf{e}_2$ an element of $\mathfrak{W}_{\nu}$. Proving that $\mathbf{e}_2\mathfrak{W}_{\nu}\mathbf{e}_2 \cong \mathbf{e}_2{}^{\circ}\mathbf{R}(\Gamma)_{\nu}\mathbf{e}_2$ as k-algebras together with the fact that $\mathbf{e}_2$ is full in $\mathfrak{W}_{\nu}$ implies that $\mathfrak{W}_{\nu}\mathbf{e}_2$ is a progenerator in $\mathfrak{W}_{\nu}$-Mod such that $\mathrm{End}_{\mathfrak{W}_{\nu}}(\mathfrak{W}_{\nu}\mathbf{e}_2) \cong \mathbf{e}_2{}^{\circ}\mathbf{R}(\Gamma)_{\nu}\mathbf{e}_2$. Then, by standard Morita theory, Morita equivalence between $\mathbf{e}_2{}^{\circ}\mathbf{R}(\Gamma)_{\nu}\mathbf{e}_2$ and $\mathfrak{W}_{\nu}$ follows. □

Remark 2.7. In order to define VV algebras one must fix $p, q \in k^{\times}$. We remark here that when $q \in I$ we do not have Morita equivalence between VV algebras and SVV algebras.

2.1. Setting: $\mathbf{p} \notin \mathbf{I}$. For the following two corollaries we will assume $p \notin I$.

Corollary 2.8. *Suppose $p \notin I$. For $\nu \in {}^{\theta}\mathbf{N}I$, there is a Morita equivalence* ${}^{\circ}\mathbf{R}(\Gamma)_{\nu} \sim \mathbf{R}_{\nu^+} \times \mathbf{R}_{\nu^+}$. *In particular, this demonstrates that any irreducible type D module arises from an irreducible type A module when $p \notin I$.*

Proof. Fix an element $q \in k^{\times}$ in such a way that $q \notin I$. Then we have, by Theorem 2.10 in [Wal], Morita equivalence $\mathfrak{W}_{\nu} \sim \mathbf{R}_{\nu^+}$. Now use Theorem 2.6. □

Corollary 2.9. *When $p \notin I$, with the additional constraint of p not a root of unity, the algebras ${}^{\circ}\mathbf{R}(\Gamma)_{\nu}$ are affine quasi-hereditary and affine cellular.*

Proof. We first note that affine quasi-heredity is a Morita invariant between unital algebras. Then by Proposition 1.6 and Corollary 2.8 it suffices to show that the algebras $\mathbf{R}_{\nu^+}$ are affine quasi-hereditary. This is true when p is not a root of unity (see [Kle15, Section 10.1] and results throughout [BKM14]). This proves affine quasi-heredity. It was shown in [KLM13] that KLR algebras of type A_{∞} are affine cellular. This, together with Remark 1.4, shows that the algebras $\mathbf{R}_{\nu^+} \times \mathbf{R}_{\nu^+}$ are affine cellular in this case. The proof is completed by applying Lemma 3.4 in [Yan14]. □

2.2. Setting: $\mathbf{p} \in \mathbf{I}$. Now suppose $p \in I$ and assume p is not a root of unity. Then we can write $I_p = I_p^+ \sqcup I_p^-$, where $I_p^{\pm} = \{p^{\pm(2n+1)} \mid n \in \mathbf{Z}_{\geq 0}\}$ and Γ_I is of the form

$$\begin{array}{ccccc} \cdots & \longrightarrow & p^3 & \longrightarrow & p \\ & & & & \downarrow \\ \cdots & \longleftarrow & p^{-3} & \longleftarrow & p^{-1}. \end{array}$$

Remark 2.10. If we take $\nu \in {}^{\theta}\mathbf{N}I$ such that the number of summands of ν equal to p is less than 2 (i.e. $\nu_p < 2$), then all the results of Subsection 2.1 apply to ${}^{\circ}\mathbf{R}(\Gamma)_{\nu}$. Namely, we are again reduced to type A via the Morita equivalence ${}^{\circ}\mathbf{R}(\Gamma)_{\nu} \sim \mathbf{R}_{\nu^+} \times \mathbf{R}_{\nu^+}$. Therefore, if $p \in I$ we may assume that p has multiplicity at least two in ν, i.e. $\nu_p \geq 2$.

Let A be the path algebra $A = k(a_1 \rightleftarrows a_2)$ where the arrow from a_1 to a_2 is labelled u_1 and the arrow from a_2 to a_1 is labelled u_2. We consider A a left $k[z]$-module with the action defined by; $z \cdot a_1 = u_2u_1a_1$ and $z \cdot a_2 = u_1u_2a_2$. Suppose $p \in I$, p is not a root of unity and we take $\nu \in {}^{\theta}\mathbf{N}I$. Recall that in [Wal, Section 2.6] we define the structure of a right $k[z]$-module on $\mathbf{R}_{\nu^+}$.

Corollary 2.11. *When p is not a root of unity and $\nu_p = 2$ there is a Morita equivalence* ${}^{\circ}\mathbf{R}(\Gamma)_{\nu} \sim (\mathbf{R}_{\nu^+} \otimes_{k[z]} A) \times (\mathbf{R}_{\nu^+} \otimes_{k[z]} A)$.

Proof. Use Theorem 2.6 and [Wal, Theorem 2.46]. □

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References

[BKM14] J. Brundan, A. Kleshchev and P. J. McNamara, Homological properties of finite-type Khovanov-Lauda-Rouquier algebras, Duke Math. J. **163** (2014), no. 7, 1353–1404.

[Cui] W. Cui, Affine cellularity of affine Birman-Murakami-Wenzl algebras, arXiv: 1406.3516.

[EK06] N. Enomoto and M. Kashiwara, Symmetric crystals and affine Hecke algebras of type B, Proc. Japan Acad. Ser. A Math. Sci. **82** (2006), no. 8, 131–136.

[GL96] J. J. Graham and G. I. Lehrer, Cellular algebras, Invent. Math. **123** (1996), no. 1, 1–34.

[KL09] M. Khovanov and A. D. Lauda, A diagrammatic approach to categorification of quantum groups. I, Represent. Theory **13** (2009), 309–347.

[Kle15] A. S. Kleshchev, Affine highest weight categories and affine quasihereditary algebras, Proc. Lond. Math. Soc. (3) **110** (2015), no. 4, 841–882.

[KLM13] A. S. Kleshchev, J. W. Loubert and V. Miemietz, Affine cellularity of Khovanov-Lauda-Rouquier algebras in type *A*, J. Lond. Math. Soc. (2) **88** (2013), no. 2, 338–358.

[KM07] M. Kashiwara and V. Miemietz, Crystals and affine Hecke algebras of type D, Proc. Japan Acad. Ser. A Math. Sci. **83** (2007), no. 7, 135–139.

[KX12] S. Koenig and C. Xi, Affine cellular algebras, Adv. Math. **229** (2012), no. 1, 139–182.

[Rou] R. Rouquier, 2-Kac-Moody algebras, arXiv: 0812.5023v1.

[SVV11] P. Shan, M. Varagnolo and E. Vasserot, Canonical bases and affine Hecke algebras of type *D*, Adv. Math. **227** (2011), no. 1, 267–291.

[VV11] M. Varagnolo and E. Vasserot, Canonical bases and affine Hecke algebras of type B, Invent. Math. **185** (2011), no. 3, 593–693.

[Wal] R. D. Walker, On Morita equivalences between KLR algebras and VV algebras, arXiv:1603.00796.

[Yan14] G. Yang, Affine cellular algebras and Morita equivalences, Arch. Math. (Basel) **102** (2014), no. 4, 319–327.