Analytic continuation of the multiple Fibonacci zeta functions

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Abstract: In this article, we prove the meromorphic continuation of the multiple Fibonacci zeta functions of depth 2:

$$\sum_{0 < n_1 < n_2} \frac{1}{F_{n_1}^{s_1} F_{n_2}^{s_2}},$$

where F_n is the n-th Fibonacci number, $\mathrm{Re}(s_1) > 0$ and $\mathrm{Re}(s_2) > 0$. We compute a complete list of its poles and their residues. We also prove that multiple Fibonacci zeta values at negative integer arguments are rational.

Key words: Analytic continuation; multiple Fibonacci zeta function; poles and residues.

1. Introduction. The Riemann zeta function $\zeta(s)$ is one of the most important objects in the study of number theory. It is a classical and well-known result that $\zeta(s)$, originally defined on the half plane $\mathrm{Re}(s) > 1$, can be analytically continued to a meromorphic function on the entire complex plane with the only pole at $s = 1$, which is a simple pole with residue 1 [3,6]. One way to generalize the Riemann zeta function is to define the "multiple (Euler-Riemann-Zagier) zeta function" of depth d as follows:

$$(1.1) \quad \zeta(s_1, s_2, \ldots, s_d) := \sum_{0 < n_1 < n_2 \cdots < n_d} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}},$$

$\mathrm{Re}(s_d) > 1$, $\sum_{j=1}^{d} \mathrm{Re}(s_j) > d$. Several authors have studied the analytic continuation of the multiple zeta function and proved that the multiple zeta function $\zeta(s_1, \ldots, s_d)$ of depth d can be analytically continued to a meromorphic function on all of $\mathbf{C}^d$. For example, Atkinson [5] first proved the analytic continuation of $\zeta(s_1, s_2)$, with applications to the study of the asymptotic behavior of the "mean values" of zeta-functions, using Poisson summation formula. In [4], Arakawa and Kaneko used analytic continuation of $\zeta(s_1, \ldots, s_d)$ as a function of one variable s_d when $s_1, \ldots, s_{d-1}$ are positive integers and discussed the relation among generalized

Bernoulli numbers. For a general d, Zhao [12] proved the analytic continuation of $\zeta(s_1, \ldots, s_d)$ as a function of d variables using the theory of generalized function and Akiyama, Egami and Tanigawa [1] proved the same result by applying the classical Euler-Maclaurin formula to the index of the summation n_d. Recently, Mehta et al., in [9] obtained the meromorphic continuation of multiple zeta functions by means of an elementary and simple translation formula for this multiple zeta function.

The sequence of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with initial values $F_0 = 0$, $F_1 = 1$. We denote the n-th term of the Fibonacci sequence by F_n and the Binet form of F_n is $\frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The Fibonacci zeta function is the series

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s}$$

and this series is absolutely convergent for $\mathrm{Re}(s) > 0$. Also it can be considered as an analogue of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}(s) > 1$. André-Jeannin [2] proved that $\zeta_F(1) = \sum_{n=1}^{\infty} \frac{1}{F_n}$ is an irrational number. Duverney et al., in [7] proved that $\zeta_F(2m)$ for $m = 1, 2, \cdots$ are all transcendental numbers and Elsner et al. [8] proved that $\zeta_F(2), \zeta_F(4)$ and $\zeta_F(6)$ are algebraically independent. In 2001, Navas [10] obtained analytic continuation of the Fibonacci Dirichlet series $\zeta_F(s)$.

Also Ram Murty [11] obtained that $\zeta_F(2m)$ are transcendental for $m \geq 1$ using the theory of modular form and a result of Nesterenko, which is a slight modification of Duverney's proof [7].

There is a connection between the special values of zeta function at positive integers with theoretical physics. In eighteenth century, Euler investigated the double zeta values. Though the multiple zeta values are extremely important, we do not want to discuss in details here. Similarly, the arithmetic nature of the multiple Fibonacci zeta values are important.

The special values of the Fibonacci zeta function stimulate us to study the analytic continuation of the multiple Fibonacci zeta function. In particular, we investigate the multiple Fibonacci zeta function which is defined as

$$(1.2) \qquad \zeta_F(s_1, \ldots, s_d) := \sum_{0 < n_1 < n_2 \cdots < n_d} \frac{1}{F_{n_1}^{s_1} \cdots F_{n_d}^{s_d}},$$

where F_n is the n-th Fibonacci number. In this situation, the sum $s_1 + \cdots + s_d$ is called the *weight* of $\zeta_F(s_1, \ldots, s_d)$ and d is called its *depth*. In this paper, we study the analytic continuation of the defined series in (1.2) for $d = 2$ on all of $\mathbf{C}^2$ with a complete list of poles and their corresponding residues. Moreover, we investigate the arithmetic nature of multiple Fibonacci zeta functions at negative integer arguments. To the best of our knowledge, this is the first work in this area.

2. Convergence of the multiple Fibonacci zeta functions. For the sake of completeness, we include here a small section on the convergence of the multiple Fibonacci zeta functions.

Proposition 1. *The infinite sum*

$$\sum_{0 < n_1 < n_2} \frac{1}{F_{n_1}^{s_1} F_{n_2}^{s_2}}$$

converges absolutely in the domain

$$D_2 := \{(s_1, s_2) \in \mathbf{C}^2 \mid \mathrm{Re}(s_1) > 0,\ \mathrm{Re}(s_2) > 0\}.$$

Proof. One can observe that

$$(2.1) \qquad \sum_{0 < n_1 < n_2} \frac{1}{F_{n_1}^{s_1} F_{n_2}^{s_2}} = \sum_{n_1=1}^{\infty} \frac{1}{F_{n_1}^{s_1}} \sum_{n_2=1}^{\infty} \frac{1}{F_{n_1+n_2}^{s_2}}.$$

We know that Fibonacci numbers grow exponentially and also $F_n \geq \alpha^n$. Thus, for $\sigma_1 := \mathrm{Re}(s_1) > 0$ and $\sigma_2 := \mathrm{Re}(s_2) > 0$, we have

$$(2.2) \qquad \left| \frac{1}{F_{n_1}^{s_1}} \right| = \frac{1}{F_{n_1}^{\sigma_1}} \leq \frac{1}{\alpha^{\sigma_1 n_1}},$$

and

$$(2.3) \qquad \left| \frac{1}{F_{n_1+n_2}^{s_2}} \right| = \frac{1}{F_{n_1+n_2}^{\sigma_2}} \leq \frac{1}{\alpha^{\sigma_2(n_1+n_2)}}.$$

From (2.1), (2.2) and (2.3), we get

$$\sum_{0 < n_1 < n_2} \left| \frac{1}{F_{n_1}^{s_1} F_{n_2}^{s_2}} \right| \leq \sum_{n_1=1}^{\infty} \left| \frac{1}{F_{n_1}^{s_1}} \right| \sum_{n_2=1}^{\infty} \left| \frac{1}{F_{n_1+n_2}^{s_2}} \right|$$

$$\leq \sum_{n_1=1}^{\infty} \frac{1}{\alpha^{\sigma_1 n_1}} \sum_{n_2=1}^{\infty} \frac{1}{\alpha^{\sigma_2(n_1+n_2)}}$$

$$= \sum_{n_1=1}^{\infty} \frac{1}{\alpha^{(\sigma_1+\sigma_2)n_1}} \sum_{n_2=1}^{\infty} \frac{1}{\alpha^{\sigma_2 n_2}}$$

$$= \frac{1}{(\alpha^{\sigma_1+\sigma_2} - 1)(\alpha^{\sigma_2} - 1)} < \infty.$$

This finishes the proof of the proposition. $\square$

It is natural to ask whether the domain of convergence of $\zeta_F(s_1, s_2)$ is extendable or not. In the following section, we give an affirmative answer to this question.

3. Analytic continuation of the multiple Fibonacci zeta functions.

Theorem 2. *The multiple Fibonacci zeta function $\zeta_F(s_1, s_2)$ of depth $d = 2$ can be analytically continued to a meromorphic function on $\mathbf{C}^2$. It has poles on the hyperplanes*

$$s_2 = -2\ell + \frac{i\pi(\ell + 2n)}{\log \alpha} \, (\ell, n \in \mathbf{Z}, \ell \geq 0),$$

and

$$s_1 + s_2 = -2(k + \ell) + \frac{i\pi(k + \ell + 2m)}{\log \alpha}$$

$$(k, \ell, m \in \mathbf{Z}, k, \ell \geq 0).$$

Moreover, all the poles are simple.

Proof. For any $z \in \mathbf{C}$,

$$F_n^z = \left(\frac{\alpha^n - \beta^n}{\sqrt{5}} \right)^z = 5^{-z/2} \alpha^{nz} \left(1 - \left(\frac{\beta}{\alpha} \right)^n \right)^z$$

$$= 5^{-z/2} \alpha^{nz} \left(1 + (-1)^{n+1} \frac{1}{\alpha^{2n}} \right)^z$$

$$= 5^{-z/2} \sum_{k=0}^{\infty} \binom{z}{k} (-1)^{(n+1)k} \alpha^{n(z-2k)}.$$

The above binomial series converges as $\alpha > 1$. Substituting this into the multiple Fibonacci zeta function in (1.2) for $d = 2$, we get

(3.1)

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{F_{n_1}^{s_1}}\frac{1}{F_{n_1+n_2}^{s_2}}$$

$$=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\left(5^{s_1/2}\sum_{k=0}^{\infty}\binom{-s_1}{k}(-1)^{(n_1+1)k}\alpha^{-n_1(s_1+2k)}\right)$$

$$\times\left(5^{s_2/2}\sum_{\ell=0}^{\infty}\binom{-s_2}{\ell}(-1)^{(n_1+n_2+1)\ell}\alpha^{-(n_1+n_2)(s_2+2\ell)}\right).$$

Since $\left|\binom{-s_1}{k}\right|\leq(-1)^k\binom{-|s_1|}{k}$ and $\left|\binom{-s_2}{\ell}\right|\leq(-1)^{\ell}\binom{-|s_2|}{\ell}$, we have

$$\sum_{n_1,n_2=1}^{\infty}\left|5^{s_1/2}\sum_{k=0}^{\infty}\binom{-s_1}{k}(-1)^{(n_1+1)k}\alpha^{-n_1(s_1+2k)}5^{s_2/2}\right.$$

$$\left.\times\sum_{\ell=0}^{\infty}\binom{-s_2}{\ell}(-1)^{(n_1+n_2+1)\ell}\alpha^{-(n_1+n_2)(s_2+2\ell)}\right|$$

$$\leq\sum_{n_1,n_2=1}^{\infty}5^{(\sigma_1+\sigma_2)/2}\alpha^{-n_1\sigma_1}\sum_{k=0}^{\infty}\binom{-|s_1|}{k}(-1)^k\alpha^{-2n_1k}$$

$$\times\alpha^{-(n_1+n_2)\sigma_2}\sum_{\ell=0}^{\infty}\binom{-|s_2|}{\ell}(-1)^{\ell}\alpha^{-2(n_1+n_2)\ell}$$

$$=\sum_{n_1,n_2=1}^{\infty}5^{(\sigma_1+\sigma_2)/2}\alpha^{-n_1\sigma_1}\left(1-\alpha^{-2n_1}\right)^{-|s_1|}$$

$$\times\alpha^{-(n_1+n_2)\sigma_2}\left(1-\alpha^{-2(n_1+n_2)}\right)^{-|s_2|}$$

$$\leq\sum_{n_1,n_2=1}^{\infty}5^{(\sigma_1+\sigma_2)/2}\alpha^{-n_1\sigma_1}(1-\alpha^{-2})^{-|s_1|}$$

$$\times\alpha^{-(n_1+n_2)\sigma_2}(1-\alpha^{-4})^{-|s_2|}$$

$$=(1-\alpha^{-2})^{-|s_1|}(1-\alpha^{-4})^{-|s_2|}5^{(\sigma_1+\sigma_2)/2}$$

$$\times\sum_{n_1=1}^{\infty}\alpha^{-(\sigma_1+\sigma_2)n_1}\sum_{n_2=1}^{\infty}\alpha^{-\sigma_2 n_2}<\infty.$$

Let us denote

$$M:=5^{(s_1+s_2)/2}\sum_{k=0}^{\infty}\binom{-s_1}{k}(-1)^k\sum_{\ell=0}^{\infty}\binom{-s_2}{\ell}(-1)^{\ell}.$$

By interchanging the order of summation in (3.1), we have

(3.2) $\quad\zeta_F(s_1,s_2)$

$$=M\sum_{n_1=1}^{\infty}\left((-1)^{k+\ell}\alpha^{-(s_1+s_2+2k+2\ell)}\right)^{n_1}$$

$$\times\sum_{n_2=1}^{\infty}\left((-1)^{\ell}\alpha^{-(s_2+2\ell)}\right)^{n_2}$$

$$=M\frac{(-1)^{k+\ell}\alpha^{-(s_1+s_2+2k+2\ell)}}{\left(1-(-1)^{k+\ell}\alpha^{-(s_1+s_2+2k+2\ell)}\right)}$$

$$\times\frac{(-1)^{\ell}\alpha^{-(s_2+2\ell)}}{\left(1-(-1)^{\ell}\alpha^{-(s_2+2\ell)}\right)}$$

$$=M\frac{(-1)^{k+\ell}}{\left(\alpha^{(s_1+s_2+2k+2\ell)}+(-1)^{k+\ell+1}\right)}$$

$$\times\frac{(-1)^{\ell}}{\left(\alpha^{(s_2+2\ell)}+(-1)^{\ell+1}\right)}$$

$$=5^{(s_1+s_2)/2}\sum_{k=0}^{\infty}\binom{-s_1}{k}\sum_{\ell=0}^{\infty}\binom{-s_2}{\ell}$$

$$\times\frac{(-1)^{\ell}}{\left(\alpha^{s_1+s_2+2k+2\ell}+(-1)^{k+\ell+1}\right)}$$

$$\times\frac{1}{\left(\alpha^{s_2+2\ell}+(-1)^{\ell+1}\right)}.$$

For any $s_1,s_2\in\mathbf{C}$, we have

(3.3) $\quad|\alpha^{s_1+s_2+2k+2\ell}+(-1)^{k+\ell+1}|\geq\alpha^{\sigma_1+\sigma_2+2k+2\ell}-1$
$$>\alpha^{\sigma_1+\sigma_2+k+\ell}$$

for $k\geq k_1,\ell\geq\ell_1$ and $|\alpha^{(s_2+2\ell)}+(-1)^{\ell+1}|>\alpha^{\sigma_2+\ell}$ for $\ell\geq\ell_2$, where k_1,ℓ_1 and ℓ_2 are constants given by $k_1=k_1(\sigma_1,\sigma_2,\alpha)\gg0$, $\ell_1=\ell_1(\sigma_1,\sigma_2,\alpha)\gg0$ and $\ell_2=\ell_2(\sigma_2,\alpha)\gg0$. Define $\ell_0=\max\{\ell_1,\ell_2\}$. Thus,

$$\sum_{k>k_1}^{\infty}\sum_{\ell>\ell_0}\left|\binom{-s_1}{k}(-1)^k\binom{-s_2}{\ell}(-1)^{\ell}\right.$$

$$\times\frac{(-1)^{k+\ell}}{\left(\alpha^{(s_1+s_2+2k+2\ell)}+(-1)^{k+\ell+1}\right)}$$

$$\left.\times\frac{(-1)^{\ell}}{\left(\alpha^{(s_2+2\ell)}+(-1)^{\ell+1}\right)}\right|$$

$$\leq\sum_{\substack{k>k_1\\\ell>\ell_0}}^{\infty}\binom{-|s_1|}{k}(-1)^k\binom{-|s_2|}{\ell}(-1)^{\ell}$$

$$\times\frac{1}{\alpha^{(\sigma_1+\sigma_2+k+\ell)}}\frac{1}{\alpha^{(\sigma_2+\ell)}}$$

$$\leq\alpha^{-(\sigma_1+2\sigma_2)}\sum_{\substack{k>k_1\\\ell>\ell_0}}^{\infty}\binom{-|s_1|}{k}(-1)^k$$

$$\times\alpha^{-k}\binom{-|s_2|}{\ell}(-1)^{\ell}\alpha^{-2\ell}$$

$$\leq\alpha^{-(\sigma_1+2\sigma_2)}(1-\alpha^{-1})^{-|s_1|}(1-\alpha^{-2})^{-|s_2|}.$$

This bound is uniform, when (s_1, s_2) varies over a compact subsets of $\mathbf{C}^2$. Thus, the series in (3.2) converges uniformly and absolutely on compact subsets of $\mathbf{C}^2$ without containing any of the poles of the functions

$$g_{k,\ell}(s_1, s_2) := \frac{1}{\left(\alpha^{(s_1+s_2+2k+2\ell)} + (-1)^{k+\ell+1}\right)} \quad \text{and}$$

$$g_\ell(s_2) := \frac{1}{\left(\alpha^{(s_2+2\ell)} + (-1)^{\ell+1}\right)}.$$

Hence, (3.2) defines the analytic continuation of $\zeta_F(s_1, s_2)$ to a meromorphic function on $\mathbf{C}^2$ with simple poles at $s_2 = -2\ell + \frac{i\pi(\ell+2n)}{\log\alpha}$ $(\ell, n \in \mathbf{Z}, \ell \geq 0)$ and $s_1 + s_2 = -2(k+\ell) + \frac{i\pi(k+\ell+2m)}{\log\alpha}$ $(k, \ell, m \in \mathbf{Z}, k, \ell \geq 0)$. $\qquad\square$

Remark. The poles $s_2 = -2\ell + \frac{i\pi(\ell+2n)}{\log\alpha}$ lie on the lines $\mathrm{Re}(s_2) = -2\ell$ spaced at the intervals of length $\frac{2\pi i}{\log\alpha}$; $s_2 = -2\ell$ is a pole, when ℓ is even, and $s_2 = -2\ell + \frac{\pi i}{\log\alpha}$ is a pole, when ℓ is odd. Similarly, $s_1 + s_2 = -2k - 2\ell$ is a pole, when k, ℓ are both even or both odd and $s_1 + s_2 = -2k - 2\ell + \frac{\pi i}{\log\alpha}$ is a pole when either k is odd and ℓ is even or k is even and ℓ is odd.

4. Residues of the multiple Fibonacci zeta functions at poles. From Theorem 2, we know that $s_2 = -2\ell + \frac{i\pi(\ell+2n)}{\log\alpha}$ for $\ell \geq 0, n \in \mathbf{Z}$ are the simple poles of $\zeta_F(s_1, s_2)$. We define the residue along the hyperplane given by the equation $s_2 = -2\ell + \frac{i\pi(\ell+2n)}{\log\alpha}$ to be the restriction to the hyperplane of the meromorphic function $\left(s_2 + 2\ell - \frac{i\pi(\ell+2n)}{\log\alpha}\right)\zeta_F(s_1, s_2)$. Here we obtain the corresponding residues.

Theorem 3. *Let* $\ell, n \in \mathbf{Z}$ *and* $\ell \geq 0$. *Then the residue of the multiple Fibonacci zeta function* $\zeta_F(s_1, s_2)$ *at* $a_{\ell,n} := -2\ell + \frac{i\pi(\ell+2n)}{\log\alpha}$ *is*

$$\zeta_F(s_1) \frac{(-1)^\ell}{\log\alpha} \frac{5^{a_{\ell,n}/2}}{} \binom{-a_{\ell,n}}{\ell}.$$

Proof. First note that $\alpha^{s_2+2\ell} + (-1)^{\ell+1}$ is an analytic function with simple zeros at $a_{\ell,n} = -2\ell + \frac{i\pi(\ell+2n)}{\log\alpha}$. Then, we have

$$(4.1) \quad \lim_{s_2 \to a_{\ell,n}} \frac{s_2 - a_{\ell,n}}{\left(\alpha^{s_2+2\ell} + (-1)^{\ell+1}\right)}$$

$$= \mathrm{Res}_{s_2=a_{\ell,n}} \frac{1}{\left(\alpha^{s_2+2\ell} + (-1)^{\ell+1}\right)}$$

$$= \frac{1}{\frac{d}{ds_2}\left(\alpha^{s_2+2\ell} + (-1)^{\ell+1}\right)}\bigg|_{s_2=a_{\ell,n}} = \frac{(-1)^\ell}{\log\alpha}.$$

The residue of the multiple Fibonacci zeta function $\zeta_F(s_1, s_2)$ at $a_{\ell,n} := -2\ell + \frac{i\pi(\ell+2n)}{\log\alpha}$ is equivalent to take the restriction to the hyperplane $s_2 = a_{\ell,n}$. Hence, the residue of $\zeta_F(s_1, s_2)$ along the hyperplane $s_2 = a_{\ell,n}$ is

$$\lim_{s_2 \to a_{\ell,n}} (s_2 - a_{\ell,n}) 5^{(s_1+s_2)/2} \sum_{k=0}^{\infty} \binom{-s_1}{k} \sum_{\ell=0}^{\infty} \binom{-s_2}{\ell}$$

$$\times \frac{(-1)^\ell}{\left(\alpha^{(s_1+s_2+2k+2\ell)} + (-1)^{k+\ell+1}\right)} \frac{1}{\left(\alpha^{s_2+2\ell} + (-1)^{\ell+1}\right)}$$

$$= 5^{s_1/2} \sum_{\substack{k=0 \\ \ell=0}}^{\infty} \binom{-s_1}{k} \frac{(-1)^\ell}{\left(\alpha^{(s_1+s_2+2k+2\ell)} + (-1)^{k+\ell+1}\right)}\bigg|_{s_2=a_{\ell,n}}$$

$$\times \lim_{s_2 \to a_{\ell,n}} 5^{s_2/2} \binom{-s_2}{\ell} \frac{(s_2 - a)}{\left(\alpha^{s_2+2\ell} + (-1)^{\ell+1}\right)}$$

$$= 5^{s_1/2} \sum_{k=0}^{\infty} \binom{-s_1}{k} \frac{1}{\left(\alpha^{(s_1+2k)} + (-1)^{k+1}\right)}$$

$$\times 5^{a_{\ell,n}/2} \binom{-a_{\ell,n}}{\ell} \frac{(-1)^\ell}{\log\alpha}$$

$$= \zeta_F(s_1) \frac{(-1)^\ell}{\log\alpha} 5^{a_{\ell,n}/2} \binom{-a_{\ell,n}}{\ell}.$$

$$\square$$

Further, we compute the residues at the poles which lie on the hyperplane $s_1 + s_2 = -2(k+\ell) + \frac{i\pi(k+\ell+2m)}{\log\alpha}$. Similarly, we define the residue along the hyperplane given by the equation

$$s_1 + s_2 = -2(k+\ell) + \frac{i\pi(k+\ell+2m)}{\log\alpha}$$

to be the restriction to the hyperplane of the meromorphic function

$$\left(s_1 + s_2 + 2(k+\ell) - \frac{i\pi(k+\ell+2m)}{\log\alpha}\right)\zeta_F(s_1, s_2).$$

Theorem 4. *Let* $k', \ell', m' \in \mathbf{Z}$ *with* $k', \ell' \geq 0$. *Then the residue of the multiple Fibonacci zeta function* $\zeta_F(s_1, s_2)$ *at* $b_{k',\ell',m'} := -2(k'+\ell') + \frac{i\pi(k'+\ell'+2m')}{\log\alpha}$ *is*

$$5^{\frac{b_{k',\ell',m'}}{2}} \frac{(-1)^{k'+\ell'}}{\log\alpha}$$

$$\times \sum_{\substack{\ell,k \geq 0 \\ \ell+k=\ell'+k'}} \binom{-s_2}{\ell}\binom{-s_1}{k} \frac{(-1)^\ell}{\left(\alpha^{(s_2+2\ell)} + (-1)^{\ell+1}\right)}.$$

Proof. By proceeding as in Theorem 3, we have

$$\lim_{s_1+s_2 \to b_{k',\ell',m'}} \frac{s_1+s_2-b_{k',\ell',m'}}{\left(\alpha^{s_1+s_2+2k'+2\ell'} + (-1)^{k'+\ell'+1}\right)}$$

$$= \operatorname{Res}_{s_1+s_2=b_{k',\ell',m'}} \frac{1}{\left(\alpha^{s_1+s_2+2k'+2\ell'} + (-1)^{k'+\ell'+1}\right)}$$

$$= \frac{(-1)^{k'+\ell'}}{\log \alpha}.$$

Hence the residue of $\zeta_F(s_1, s_2)$ along the hyperplane $s_1 + s_2 = b_{k',\ell',m'}$ is

$$\lim_{s_1+s_2 \to b_{\ell',m',n'}} (s_1+s_2-b_{\ell',m',n'}) 5^{(s_1+s_2)/2} \sum_{\ell=0}^{\infty} \binom{-s_2}{\ell}$$

$$\sum_{k=0}^{\infty} \binom{-s_1}{k} \frac{(-1)^{\ell}}{\left(\alpha^{(s_2+2\ell)} + (-1)^{\ell+1}\right)}$$

$$\times \frac{1}{\left(\alpha^{s_1+s_2+2k+2\ell} + (-1)^{k+\ell+1}\right)}$$

$$= 5^{(s_1+s_2)/2} \sum_{\ell=0}^{\infty} \binom{-s_2}{\ell} \sum_{k=0}^{\infty} \binom{-s_1}{k}$$

$$\times \frac{(-1)^{\ell}}{\left(\alpha^{(s_2+2\ell)} + (-1)^{\ell+1}\right)}$$

$$\times \lim_{s_1+s_2 \to b_{\ell,m,n}} \frac{s_1+s_2-b_{\ell,m,n}}{\left(\alpha^{s_1+s_2+2k+2\ell} + (-1)^{k+\ell+1}\right)}$$

$$= 5^{\frac{b_{k',\ell',m'}}{2}} \frac{(-1)^{k'+\ell'}}{\log \alpha}$$

$$\times \sum_{\substack{\ell,k \geq 0 \\ \ell+k=\ell'+k'}} \binom{-s_2}{\ell} \binom{-s_1}{k} \frac{(-1)^{\ell}}{\left(\alpha^{(s_2+2\ell)} + (-1)^{\ell+1}\right)}.$$

$\square$

5. Values at negative integers. Now we discuss the values of $\zeta_F(s_1, s_2)$ at the negative integers. We already know that $s_2 = 0, -4, -8, \cdots$ are simple poles and there are also other poles which lie on the hyperplane $s_1 + s_2 = -2k - 2\ell$ when both k, ℓ are even or odd simultaneously.

Theorem 5. *Let* $m, n \in \mathbf{Z}_{\geq 0}$ *with* $n \equiv 2$ (mod 4), $m \equiv 0$ (mod 4) *or* n *is odd,* $m \equiv 0$ (mod 4). *Then*

$$\zeta_F(-m, -n) \in \mathbf{Q}.$$

Proof. From (3.2), we have

$$(5.1) \quad \zeta_F(-m, -n) = 5^{-(m+n)/2} \sum_{k=0}^{\infty} \binom{m}{k} \sum_{\ell=0}^{\infty} \binom{n}{\ell}$$

$$\times \frac{(-1)^{\ell}}{\left(\alpha^{-m-n+2k+2\ell} + (-1)^{k+\ell+1}\right)} \frac{1}{\left(\alpha^{-n+2\ell} + (-1)^{\ell+1}\right)}.$$

Note that $\binom{m}{k}$ and $\binom{n}{\ell}$ are 0 for $k > m$ and $\ell > n$ respectively. Therefore this is a finite sum belonging to $\mathbf{Q}(\sqrt{5})$. Let

$$\sigma_\ell := \binom{n}{\ell} \frac{1}{\left(\alpha^{-n+2\ell} + (-1)^{\ell+1}\right)}$$

and

$$\theta_{k,\ell} := \binom{m}{k} \frac{(-1)^{\ell}}{\left(\alpha^{-m-n+2k+2\ell} + (-1)^{k+\ell+1}\right)}.$$

Thus (5.1) can be rewritten as

$$(5.2) \quad \zeta_F(-m, -n) = 5^{-(m+n)/2} \sum_{\ell=0}^{n} \sum_{k=0}^{m} \sigma_\ell \theta_{k,\ell}$$

$$= \frac{5^{-(m+n)/2}}{2} \sum_{\ell=0}^{n} \left[\sum_{k=0}^{m} \sigma_\ell \theta_{k,\ell} + \sum_{k=0}^{m} \sigma_{n-\ell} \theta_{k,(n-\ell)} \right]$$

$$= \frac{5^{-(m+n)/2}}{4} \sum_{\ell=0}^{n} \left[\left(\sum_{k=0}^{m} \sigma_\ell \theta_{k,\ell} + \sum_{k=0}^{m} \sigma_\ell \theta_{m-k,\ell} \right) \right.$$

$$\left. + \left(\sum_{k=0}^{m} \sigma_{n-\ell} \theta_{k,(n-\ell)} + \sum_{k=0}^{m} \sigma_{n-\ell} \theta_{m-k,(n-\ell)} \right) \right].$$

Let us denote by

$$I := \sum_{\ell=0}^{n} \sum_{k=0}^{m} \sigma_\ell \theta_{k,\ell}, \quad II := \sum_{\ell=0}^{n} \sum_{k=0}^{m} \sigma_\ell \theta_{m-k,\ell},$$

$$III := \sum_{\ell=0}^{n} \sum_{k=0}^{m} \sigma_{n-\ell} \theta_{k,(n-\ell)}$$

and

$$IV := \sum_{\ell=0}^{n} \sum_{k=0}^{m} \sigma_{n-\ell} \theta_{m-k,(n-\ell)}.$$

Put $\alpha_{\ell,k} = I + II + III + IV$. Let $\psi \neq \mathrm{Id}$ be an automorphism of $\mathrm{Gal}(\mathbf{Q}(\sqrt{5})/\mathbf{Q})$ and hence $\psi(\alpha) = \beta$. Consider,

(5.3)

$$\sigma_{n-\ell} \theta_{m-k,(n-\ell)}$$

$$= \binom{n}{n-\ell} \frac{1}{\alpha^{-n+2(n-\ell)} + (-1)^{n-\ell+1}}$$

$$\times \binom{m}{m-k} \frac{(-1)^{n-\ell}}{\alpha^{-(n+m)+2(m-k+n-\ell)} + (-1)^{(m-k+n-\ell+1)}}$$

$$= \binom{n}{\ell} \frac{1}{\alpha^{n-2\ell} + (-1)^{n+\ell+1}}$$

$$\times \binom{m}{k} \frac{(-1)^{n+\ell}}{\alpha^{(n+m)-2(k+\ell)} + (-1)^{(m+k+n+\ell+1)}} \, .$$

Using $\alpha\beta = -1$ and $(\alpha\beta)^{n-2\ell} = (-1)^{n-2\ell} = (-1)^n$, we have $\alpha^{n-2\ell} = (-1)^n \beta^{-n+2\ell}$. Similarly, we get $\alpha^{m+n-2(k+\ell)} = (-1)^{m+n} \beta^{-(m+n)+2(k+\ell)}$. Substituting the above expression in (5.3), we obtain

$$(5.4) \quad \sigma_{n-\ell} \theta_{m-k,(n-\ell)}$$

$$= \binom{n}{\ell} \frac{1}{(-1)^n \beta^{-n+2\ell} + (-1)^{n+\ell+1}} \binom{m}{k}$$

$$\times \frac{(-1)^{n+\ell}}{(-1)^{m+n} \beta^{-(n+m)+2(k+\ell)} + (-1)^{m+n+k+\ell+1}}$$

$$= \binom{n}{\ell} \frac{(-1)^n}{\beta^{-n+2\ell} + (-1)^{\ell+1}}$$

$$\times \binom{m}{k} \frac{(-1)^{m+\ell}}{\beta^{-(n+m)+2(k+\ell)} + (-1)^{k+\ell+1}} \, .$$

Similarly,

$$(5.5) \quad \sigma_{n-\ell} \theta_{k,(n-\ell)} = \binom{n}{\ell} \frac{(-1)^n}{\beta^{-n+2\ell} + (-1)^{\ell+1}} \binom{m}{k}$$

$$\times \frac{(-1)^{m+\ell}}{\beta^{m-n+2(\ell-k)} + (-1)^{k+\ell+m+1}} \, .$$

Case I: (m and n are both even or both odd).

Note that $(m + n)/2$ is an integer. Therefore from (5.4) and (5.5), we conclude that $\psi(I) = IV$ and $\psi(II) = III$. Thus, from (5.2), $\alpha_{\ell,k} \in \mathbf{Q}$. Since $5^{(m+n)/2} \in \mathbf{Q}$, we have $\zeta_F(-m, -n) \in \mathbf{Q}$.

Case II: (Either m is even and n is odd or m is odd and n is even).

In this case, $5^{(m+n)/2} \in \sqrt{5}\mathbf{Q}$. From (5.4) and (5.5), we have $\psi(I) = -IV$ and $\psi(II) = -III$. Also we know that, if $\psi(x) = -x$, then $x - \psi(x) \in \sqrt{5}\mathbf{Q}$. Thus, from (5.2), $\alpha_{\ell,k}$ is of the form $\beta_{\ell,k}\sqrt{5}$ for some $\beta_{\ell,k} \in \mathbf{Q}$ and hence $\zeta_F(-m, -n) \in \mathbf{Q}$. □

6. Concluding remark. We know that Fibonacci zeta function has trivial zero at $-2, -6, -10, \cdots$. The determination of the zeros of the multiple Fibonacci zeta functions depth $d = 2$ is still unknown. It seems to be a delicate problem to find out the zeros of $\zeta_F(s_1, s_2)$.

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