A note on the dimension of global sections of adjoint bundles for polarized 4-folds

By Yoshiaki Fukuma

Department of Mathematics and Physics, Faculty of Science and Technology, Kochi University, 2-5-1 Akebono-cho, Kochi 780-8520, Japan

(Communicated by Shigefumi Mori, M.J.A., April 12, 2018)

Abstract: Let (X, L) be a polarized manifold defined over the field of complex numbers. In this paper, we consider the case where $\dim X = 4$ and we prove that the second Hilbert coefficient $A_2(X, L)$ of (X, L), which was defined in our previous paper, is non-negative. Furthermore we consider a question proposed by H. Tsuji for $\dim X = 4$.

Key words: Polarized manifold; adjoint bundle; sectional geometric genus.

1. Introduction. Let X be a projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then (X, L) is called a *polarized variety.* If X is smooth, then we say that (X, L) is a polarized *manifold.*

In [2, Conjecture 7.2.7], Beltrametti and Sommese proposed the following conjecture.

Conjecture 1.1. Let (X, L) be a polarized manifold of dimension n. Assume that $K_X + (n-1)L$ is nef. Then $h^0(K_X + (n-1)L) > 0$.

At present, there are some answers for Conjecture 1.1. For example, it is known that this conjecture is true if $\dim X \leq 4$ ([2, Theorem 7.2.6], [8, Theorem 2.4], [4] and [12, Theorem 3.1]) or $h^0(L) > 0$ ([14, 1.2 Theorem]). But it is unknown whether this conjecture is true or not in general. The following conjecture is a generalization of Conjecture 1.1.

Conjecture 1.2 (Ionescu [16, Open problems, p. 321], Ambro [1] and Kawamata [15])**.** Let (X, L) be a polarized manifold of dimension n. Assume that $K_X + L$ is nef. Then $h^0(K_X + L) > 0$.

At present, there are some partial answers for this conjecture (for example, [9, Theorem 3.2], [3, Théorème 1.8]). Höring [14, 1.5 Theorem] gave a proof of Conjecture 1.2 for the case of $n = 3$. But we do not know whether this conjecture is true or not for the case of $n \geq 4$.

These conjectures motivated the author to begin investigating $h^0(K_X + tL)$ for a positive integer t. Our aim is not only to know the positivity of $h^0(K_X + tL)$ but also to evaluate a lower bound for $h^0(K_X + tL)$. In [10], in order to investigate $h^0(K_X + tL)$ systematically, we introduced an invariant $A_i(X, L)$ for every integer i with $0 \leq i \leq n$, which is called the *i-th Hilbert coefficient of* (X, L) (see Definition 2.2 (ii) below). From the following theorem which shows a relationship between $h^0(K_X + tL)$ and $A_i(X, L)$, we see that it is important to study the value of $A_i(X, L)$ in order to know the value of $h^0(K_X + tL)$.

Theorem 1.1 ([10, Corollary 3.1])**.** *Let (X, L) be a polarized manifold of dimension n, and let t be a positive integer. Then we have*

$$h^0(K_X + tL) = \sum_{j=0}^{n} \binom{t-1}{n-j} A_j(X, L).$$

So it is interesting and important to study the non-negativity of $A_i(X, L)$. In general, there is the following conjecture.

Conjecture 1.3 (see [10, Conjecture 5.1])**.** Let (X, L) be a polarized manifold of dimension n. Then $A_i(X, L) \geq 0$ holds for every integer i with $0 \leq i \leq n$.

In [10] we studied the invariant $A_i(X, L)$ in the case where L is ample and spanned by global sections. In particular we proved that $A_i(X, L) \geq 0$ for every integer i with $0 \leq i \leq n$ for the case where L is ample and spanned.

And we obtained a lower bound of $h^0(K_X + tL)$ by using some properties of $A_i(X, L)$ (see [10]). In [11, Theorem 3.1.1], we proved that this conjecture for $i = 2$ is true if either (i) $n \leq 3$ or (ii) $n \geq 4$ and $\kappa(X) \geq 0$. Finally we studied the following question

2010 Mathematics Subject Classification. Primary 14C20; Secondary 14J35.

doi: 10.3792/pjaa.94.53

of H. Tsuji ([17, Problem 1]).

Problem 1.1. Let (X, L) be a polarized manifold of dimension n. Then is it true that

$$h^0(K_X + mL) \geq h^0(K_X + (m-1)L) \tag{1}$$

for every integer m with $m \geq 2$?

In [11, Theorem 4.3.1] we proved that this inequality (1) holds for the following cases; (i) $n \leq 3$, (ii) $n = 4$ and $\kappa(X) \geq 0$.

Main purposes of this paper are (i) to prove $A_2(X, L) \geq 0$ for $n = 4$ and (ii) to prove that (1) in Problem 1.1 is true for $n = 4$ and every integer $m \geq 3$.

In this paper, varieties are always assumed to be defined over the field of complex numbers. We use the standard notation from algebraic geometry.

2. Preliminaries.

Notation 2.1. Let X be a projective variety of dimension n and let L be a line bundle on X. Then $\chi(tL)$ is a polynomial in t of degree at most n, and we can write $\chi(tL)$ as $\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L)\binom{t}{j}$.

Definition 2.1 ([7, Definition 2.1])**.** Let X be a projective variety of dimension n and let L be a line bundle on X. For every integer i with $0 \leq i \leq n$, the *ith sectional geometric genus* $g_i(X, L)$ of (X, L) is defined by the following

$$g_i(X, L) = (-1)^i(\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

Remark 2.1. (i) Since $\chi_{n-i}(X, L) \in \mathbf{Z}$, we see that $g_i(X, L)$ is an integer.
(ii) If $i = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$.
(iii) If $i = 0$, then $g_0(X, L) = L^n$.
(iv) If $i = 1$, then $g_1(X, L) = g(X, L)$, where $g(X, L)$ is the sectional genus of (X, L). If X is smooth, then the sectional genus $g(X, L)$ is written as $g(X, L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1}$.

Theorem 2.1. *Let X be a smooth projective variety with* $\dim X = n$ *and let L be a nef and big line bundle on X. Then for any integer i with $0 \leq i \leq n-1$, we have*

$$g_i(X, L) = \sum_{j=0}^{n-i-1}(-1)^j\binom{n-i}{j}h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i}(-1)^{n-i-k}h^{n-k}(\mathcal{O}_X).$$

Proof. See [7, Theorem 2.3]. □

Definition 2.2 ([10, Definitions 3.1 and 3.2])**.** Let (X, L) be a polarized manifold of dimension n.
(i) Let t be a positive integer. Then we set

$$F_0(t) := h^0(K_X + tL),$$
$$F_i(t) := F_{i-1}(t+1) - F_{i-1}(t) \quad \text{for every integer } i \text{ with } 1 \leq i \leq n.$$

(ii) For every integer i with $0 \leq i \leq n$, the *ith Hilbert coefficient* $A_i(X, L)$ *of* (X, L) is defined by $A_i(X, L) = F_{n-i}(1)$.

Remark 2.2. (i) If $1 \leq i \leq n$, then $A_i(X, L)$ can be written as follows (see [10, Proposition 3.2]):

$$A_i(X, L) = g_i(X, L) + g_{i-1}(X, L) - h^{i-1}(\mathcal{O}_X).$$

(ii) By Definition 2.2 and [10, Proposition 3.1 (2)], we have the following
(ii.1) $A_i(X, L) \in \mathbf{Z}$ for every integer i with $0 \leq i \leq n$,
(ii.2) $A_0(X, L) = L^n$,
(ii.3) $A_1(X, L) = g_1(X, L) + g_0(X, L) - h^0(\mathcal{O}_X) = \frac{1}{2}K_XL^{n-1} + \frac{n+1}{2}L^n$,
(ii.4) $A_n(X, L) = h^0(K_X + L)$.

Theorem 2.2. *Let (X, L) be a polarized manifold of dimension n and let t be a positive integer. Then for every integer i with $0 \leq i \leq n$ we have*

$$F_{n-i}(t) = \sum_{j=0}^{i}\binom{t-1}{i-j}A_j(X, L).$$

Proof. See [10, Theorem 3.1]. Here we note that if $i = n$, then this result is Theorem 1.1 in Introduction. □

Definition 2.3. (i) Let X (resp. Y) be an n-dimensional smooth projective variety, and L (resp. H) an ample line bundle on X (resp. Y). Then (X, L) is called a *simple blowing up of* (Y, H) if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $L = \pi^*(H) - E$, where E is the π-exceptional effective reduced divisor.
(ii) Let X (resp. M) be an n-dimensional smooth projective variety, and L (resp. A) an ample line bundle on X (resp. M). Then we say that (M, A) is a *reduction of* (X, L) if there exists a birational morphism $\mu : X \to M$ such that μ is a composition of simple blowing ups and (M, A) is not obtained by a simple blowing up of any other polarized manifolds.

Remark 2.3. Let (X,L) be a polarized manifold and let (M,A) be a reduction of (X,L). Let $\mu : X \to M$ be the reduction map, and let γ be the number of simple blowing ups of its reduction. Then by [7, Proposition 2.6]

$$g_i(X,L) = \begin{cases} g_i(M,A) & \text{if } 1 \le i \le n, \\ A^n - \gamma & \text{if } i = 0. \end{cases}$$

Hence

$$A_i(X,L) = \begin{cases} A_i(M,A) & \text{if } 2 \le i \le n, \\ A_i(M,A) - \gamma & \text{if } i = 0, 1. \end{cases}$$

3. Main results. First we prove the following

Theorem 3.1. *Let (X,L) be a polarized manifold of dimension 4. Then $A_2(X,L) \ge 0$.*

Proof. (A) Assume that $h^0(K_X+L) > 0$. Then by [13, Claim 2.1] we obtain that $\Omega_X\langle L\rangle$ is generically nef. So by [14, 2.11 Corollary] we have

$$c_2(X)L^2 \ge -3K_XL^3 - 6L^4. \tag{2}$$

Hence by [12, Remark 2.3 (iii)]

$$\begin{aligned} &A_2(X,L) \\ &= \frac{25}{12}L^4 + K_XL^3 + \frac{1}{12}(K_X^2 + c_2(X))L^2 \\ &\ge \frac{25}{12}L^4 + K_XL^3 + \frac{1}{12}K_X^2L^2 \\ &\quad - \frac{1}{12}(3K_XL^3 + 6L^4) \\ &= \frac{1}{12}(K_X+L)(K_X+8L)L^2 + \frac{11}{12}L^4 > 0. \end{aligned}$$

(B) Assume that $h^0(K_X+L) = 0$. By [13, Remark 2.4] we may assume that $\kappa(K_X+2L) \ge 0$. Moreover, by Remark 2.3, we may assume that (X,L) is the reduction of itself. Then we note that K_X+2L is nef by the adjunction theory (see [2, Proposition 7.2.2 and Theorem 7.2.4]). In particular, K_X+3L is ample. In this case, we take the MRC-fibration of X. (For the definition of the MRC-fibration, see, e.g., [12, Theorem 2.3 and Definition 2.4].) Then there exist smooth projective varieties Y and B, a birational morphism $\pi : Y \to X$ and a surjective morphism with connected fibers $f : Y \to B$ such that B is not uniruled and the fiber of f is rationally connected. Let b be the dimension of the base space B of the MRC-fibration.

(B.i) Assume that $b \ge 3$. Then by [12, Remark 2.4 (2)] and the argument of [14, Step 2, p. 741]

$$\begin{aligned} A_2(X,L) &= \frac{1}{24}L^2(2(K_X^2 + c_2(X)) + 24K_XL + 50L^2) \\ &> 0. \end{aligned}$$

(B.ii) Assume that $b \le 2$. Then we note that $h^i(\mathcal{O}_X) = 0$ for $i \ge 3$. Hence by Theorem 2.1 and the assumption that $h^0(K_X+L) = 0$ we have

$$g_2(X,L) = h^0(K_X+2L) + h^2(\mathcal{O}_X) \ge 0. \tag{3}$$

(B.ii.1) Assume that $b = 2$. Then since $h^2(\mathcal{O}_X) \ge h^2(\mathcal{O}_B)$ and $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_B)$ we have

$$\begin{aligned} &g_2(X,L) - h^1(\mathcal{O}_X) \\ &= h^0(K_X+2L) + h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) \\ &\ge h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X) \\ &\ge \chi(\mathcal{O}_B) - 1. \end{aligned} \tag{4}$$

Here we note that we may assume that $g_1(X,L) \ge 2$ because we see from [5, (12.1) Theorem and (12.3) Theorem] and [13, Remark 2.4] that $A_2(X,L) \ge 0$ holds for any (X,L) with $g_1(X,L) \le 1$. Since $A_2(X,L) = g_2(X,L) + g_1(X,L) - h^1(\mathcal{O}_X)$ and $\kappa(B) \ge 0$, by (4) we have

$$\begin{aligned} A_2(X,L) &\ge \chi(\mathcal{O}_B) + g_1(X,L) - 1 \\ &> \chi(\mathcal{O}_B) \ge 0. \end{aligned}$$

(B.ii.2) Assume that $b = 1$. In this case, $h^1(\mathcal{O}_X) = g(B)$ and $g_1(X,L) - h^1(\mathcal{O}_X) = g_1(X,L) - g(B) \ge 0$ by [6, Theorem 1.2.1], where $g(B)$ is the genus of B. Hence by (3) we have $A_2(X,L) = g_2(X,L) + g_1(X,L) - h^1(\mathcal{O}_X) \ge g_2(X,L) \ge 0$.

(B.ii.3) Assume that $b = 0$. Then $h^1(\mathcal{O}_X) = 0$. Hence by (3) and [5, (12.1) Theorem] we get $A_2(X,L) = g_2(X,L) + g_1(X,L) - h^1(\mathcal{O}_X) = g_2(X,L) + g_1(X,L) \ge 0$.

These complete the proof of Theorem 3.1. □

Next we consider Problem 1.1 for $\dim X = 4$ and $m \ge 3$.

Theorem 3.2. *Let (X,L) be a polarized manifold of dimension 4. Then for every integer m with $m \ge 3$, we have*

$$h^0(K_X+mL) - h^0(K_X+(m-1)L) \ge 0.$$

Proof. In this case, by using Theorem 1.1, we have

$$\begin{aligned} &h^0(K_X+mL) - h^0(K_X+(m-1)L) \\ &= \binom{m-2}{3}A_0(X,L) + \binom{m-2}{2}A_1(X,L) \end{aligned} \tag{5}$$

$$+\binom{m-2}{1}A_2(X,L)+\binom{m-2}{0}A_3(X,L).$$

(I) Assume that $h^0(K_X+L)>0$. Then we see from [13, Claim 2.1] that $\Omega_X\langle L\rangle$ is generically nef. We note that $\kappa(K_X+3L)\geq 0$. Therefore K_X+3L is nef by the adjunction theory ([2, Proposition 7.2.2, Theorems 7.2.3 and 7.2.4]). Hence $K_X+(2m-1)L$ is nef for every integer $m\geq 2$. So by [14, 2.11 Corollary] we have

$$\begin{aligned}(6)\quad & c_2(X)(K_X+(2m-1)L)L\\ &\geq -\bigl(3K_XL+6L^2\bigr)(K_X+(2m-1)L)L\\ &= -3K_X^2L^2-(6m+3)K_XL^3\\ &\quad -6(2m-1)L^4.\end{aligned}$$

We note that by Remark 2.2 (ii.2), (ii.3) and [12, Remark 2.3 (iii)]

$$(7)\quad A_0(X,L)=L^4,$$

$$(8)\quad A_1(X,L)=\frac{1}{2}K_XL^3+\frac{5}{2}L^4,$$

$$\begin{aligned}(9)\quad A_2(X,L)&=\frac{25}{12}L^4+K_XL^3\\ &\quad+\frac{1}{12}(K_X^2+c_2(X))L^2,\end{aligned}$$

$$\begin{aligned}(10)\quad A_3(X,L)&=\frac{5}{8}L^4+\frac{7}{12}K_XL^3+\frac{1}{8}K_X^2L^2\\ &\quad+\frac{1}{24}c_2(X)(K_X+3L)L.\end{aligned}$$

By (5), (6), (7), (8), (9) and (10), we have

$$\begin{aligned}(11)\quad & h^0(K_X+mL)-h^0(K_X+(m-1)L)\\ &=\left(\frac{1}{6}m^3-\frac{1}{4}m^2+\frac{1}{6}m-\frac{1}{24}\right)L^4\\ &\quad+\left(\frac{1}{4}m^2-\frac{1}{4}m+\frac{1}{12}\right)K_XL^3\\ &\quad+\left(\frac{1}{12}m-\frac{1}{24}\right)K_X^2L^2\\ &\quad+\frac{1}{24}c_2(X)(K_X+(2m-1)L)L\\ &\geq\left(\frac{1}{6}m^3-\frac{1}{4}m^2-\frac{1}{3}m+\frac{5}{24}\right)L^4\\ &\quad+\left(\frac{1}{4}m^2-\frac{1}{2}m-\frac{1}{24}\right)K_XL^3\\ &\quad+\left(\frac{1}{12}m-\frac{1}{6}\right)K_X^2L^2\\ &=\frac{m-2}{12}(K_X+L)(K_X+3L)L^2\\ &\quad+\left\{\frac{1}{12}(3m-1)(m-3)\right.\\ &\quad\left.+\frac{9}{24}\right\}(K_X+2L)L^3\\ &\quad+\left\{\frac{1}{12}m(2m-6)\left(m-\frac{3}{2}\right)\right.\\ &\quad\left.+\frac{1}{3}m-\frac{13}{24}\right\}L^4.\end{aligned}$$

If $m\geq 3$, we have

$$\frac{m-2}{12}>0,$$

$$\frac{1}{12}(3m-1)(m-3)+\frac{9}{24}\geq\frac{9}{24},$$

$$\frac{1}{12}m(2m-6)\left(m-\frac{3}{2}\right)+\frac{1}{3}m-\frac{13}{24}\geq\frac{11}{24}.$$

Here we note that $(K_X+L)(K_X+3L)L^2\geq 0$ since $h^0(K_X+L)>0$ and K_X+3L is nef. Moreover $(K_X+2L)L^3>0$ since $h^0(K_X+L)>0$ and L is ample. Hence, for every integer m with $m\geq 3$, we have $h^0(K_X+mL)-h^0(K_X+(m-1)L)>0$.

(II) Assume that $h^0(K_X+L)=0$. First we note that in this case $A_3(X,L)\geq 0$ because $A_4(X,L)=h^0(K_X+L)=0$ and $0\leq h^0(K_X+2L)=A_4(X,L)+A_3(X,L)$. Since $A_2(X,L)\geq 0$ by Theorem 3.1, we get $h^0(K_X+mL)-h^0(K_X+(m-1)L)\geq 0$ for every integer $m\geq 3$ by [11, Remark 2.2 (2.2) and Theorem 3.1.1 (1)] and (5). □

Next we consider the case of $\dim X=4$ and $m=2$ in Problem 1.1.

Theorem 3.3. *Let (X,L) be a polarized manifold of dimension 4 such that (X,L) does not have the following structure $(*)$. Then $h^0(K_X+2L)\geq h^0(K_X+L)$.*

$(*)$ *There exist smooth projective varieties $\widetilde{X}$ and Y with $\dim\widetilde{X}=4$ and $\dim Y=3$, a birational morphism $\mu:\widetilde{X}\to X$, and a fiber space $f:\widetilde{X}\to Y$ such that $F\cong\mathbf{P}^1$ and $(\mu^*L)_F=\mathcal{O}_{\mathbf{P}^1}(2)$, where F is a general fiber of f.*

Proof. If $h^0(K_X+L)=0$, then $h^0(K_X+2L)-h^0(K_X+L)=h^0(K_X+2L)\geq 0$. So we may assume that $h^0(K_X+L)>0$. Then we can prove the following

Claim 3.1. $\Omega_X\langle\frac{3}{4}L\rangle$ is generically nef.

Proof. Assume that $\Omega_X\langle\frac{3}{4}L\rangle$ is not generically nef. By [14, 3.1 Theorem] there exist a smooth projective variety $\widetilde{X}$ of dimension 4, a smooth projective variety Y of dimension m with $m \leq 3$, a birational morphism $\mu : \widetilde{X} \to X$, and a surjective morphism $f : \widetilde{X} \to Y$ with connected fibers such that the following (#) holds:

(#) Any general fiber F of f is rationally connected and $h^0(D) = 0$ for any Cartier divisor D on F such that $D \sim_{\mathbf{Q}} K_F + j\mu^*(\frac{3}{4}L)_F$ for any $j \in [0, n-m] \cap \mathbf{Q}$, where $\sim_{\mathbf{Q}}$ denotes the linear equivalence of $\mathbf{Q}$-divisors.

(a) Assume that $\dim Y \leq 2$. Then we see from (#) that $h^0(K_F + \mu^*(L)_F) = h^0(K_F + \frac{4}{3}\mu^*(\frac{3}{4}L)_F) = 0$ for any general fiber F of f. But since $h^0(K_{\widetilde{X}} + \mu^*(L)) = h^0(K_X + L) > 0$, we have $h^0(K_F + \mu^*(L)_F) > 0$ holds for any general fiber F. Hence this is a contradiction.

(b) Assume that $\dim Y = 3$. In this case $F \cong \mathbf{P}^1$. If $\deg \mu^*(L)_F \geq 3$, then there exists $j \in [0,1] \cap \mathbf{Q}$ such that $K_F + j\mu^*(\frac{3}{4}L)_F$ is a Cartier divisor with $\deg(K_F + j\mu^*(\frac{3}{4}L)_F) \geq 0$. Hence $h^0(K_F + j\mu^*(\frac{3}{4}L)_F) > 0$ and this contradicts (#). So we have $\deg(\mu^*(L)_F) \leq 2$. On the other hand, we get $\deg(\mu^*(L)_F) \geq 2$ because $h^0(K_F + \mu^*(L)_F) > 0$. Therefore $\deg(\mu^*(L)_F) = 2$. But this case is excluded by the assumption that (X, L) does not have the structure $(*)$. Therefore we get the assertion of Claim 3.1. □

We note that $K_X + 3L$ is nef because $h^0(K_X + L) > 0$ (see (I) in the proof of Theorem 3.2). By the same argument as in the proof of Theorem 3.2, we see from Claim 3.1 and [14, 2.11 Corollary] that

$$(12) \qquad c_2(X)(K_X + 3L)L \geq -\frac{81}{8}L^4 - \frac{81}{8}K_X L^3 - \frac{9}{4}K_X^2 L^2.$$

On the other hand by (11) in the proof of Theorem 3.2, we have

$$(13) \qquad h^0(K_X + 2L) - h^0(K_X + L) = \frac{5}{8}L^4 + \frac{7}{12}K_X L^3 + \frac{1}{8}K_X^2 L^2 + \frac{1}{24}c_2(X)(K_X + 3L)L.$$

Hence, by noting that $h^0(K_X + L) > 0$, L is ample and $K_X + 3L$ is nef, we see from (12) and (13) that

$$\begin{aligned} h^0(K_X + 2L) - h^0(K_X + L) &\geq \frac{5}{8}L^4 + \frac{7}{12}K_X L^3 + \frac{1}{8}K_X^2 L^2 \\ &\quad - \frac{3}{64}(9L^4 + 9K_X L^3 + 2K_X^2 L^2) \\ &= \frac{13}{64}L^4 + \frac{31}{192}K_X L^3 + \frac{1}{32}K_X^2 L^2 \\ &= \frac{1}{32}(K_X + L)(K_X + 4L)L^2 \\ &\quad + \frac{1}{192}(K_X + 15L)L^3 > 0. \end{aligned}$$

This completes the proof of Theorem 3.3. □

Acknowledgements. This work is supported by JSPS KAKENHI Grant Number 16K05103. The author would like to thank the referee for giving a comment.

References

[1] F. Ambro, Ladders on Fano varieties, J. Math. Sci. (New York) **94** (1999), no. 1, 1126–1135.

[2] M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties*, De Gruyter Expositions in Mathematics, 16, Walter de Gruyter & Co., Berlin, 1995.

[3] A. Broustet, Non-annulation effective et positivité locale des fibrés en droites amples adjoints, Math. Ann. **343** (2009), no. 4, 727–755.

[4] A. Broustet and A. Höring, Effective non-vanishing conjectures for projective threefolds, Adv. Geom. **10** (2010), no. 4, 737–746.

[5] T. Fujita, *Classification theories of polarized varieties*, London Mathematical Society Lecture Note Series, 155, Cambridge University Press, Cambridge, 1990.

[6] Y. Fukuma, A lower bound for sectional genus of quasi-polarized manifolds, J. Math. Soc. Japan **49** (1997), no. 2, 339–362.

[7] Y. Fukuma, On the sectional geometric genus of quasi-polarized varieties. I, Comm. Algebra **32** (2004), no. 3, 1069–1100.

[8] Y. Fukuma, On a conjecture of Beltrametti-Sommese for polarized 3-folds, Internat. J. Math. **17** (2006), no. 7, 761–789.

[9] Y. Fukuma, On the dimension of global sections of adjoint bundles for polarized 3-folds and 4-folds, J. Pure Appl. Algebra **211** (2007), no. 3, 609–621.

[10] Y. Fukuma, A study on the dimension of global sections of adjoint bundles for polarized manifolds, J. Algebra **320** (2008), no. 9, 3543–3558.

[11] Y. Fukuma, A study on the dimension of global sections of adjoint bundles for polarized manifolds, II, Hokkaido Math. J. **40** (2011), no. 2, 251–277.

[12] Y. Fukuma, On a conjecture of Beltrametti-Sommese for polarized 4-folds, Kodai Math. J. **38** (2015), no. 2, 343–351.

[13] Y. Fukuma, On the dimension of $H^0(K_X + mL)$ of polarized n-folds (X, L) with $m \geq n$, Kyushu J. Math. **71** (2017), no. 1, 115–128.

[14] A. Höring, On a conjecture of Beltrametti and Sommese, J. Algebraic Geom. **21** (2012), no. 4, 721–751.
[15] Y. Kawamata, On effective non-vanishing and base-point-freeness, Asian J. Math. **4** (2000), no. 1, 173–181.
[16] A. Lanteri, M. Palleschi and D. C. Struppa (Eds.), *Geometry of complex projective varieties, Proceedings of the conference held in Cetraro, May 28–June 2, 1990*, Seminars and Conferences, 9, Mediterranean Press, Rende, 1993.
[17] R. Miyaoka and M. Kotani (Eds.), *Mathematics in the 21st century, unscaled peaks of geometry*, Nippon Hyoron sha, Tokyo, 2004. (in Japanese).