Self-similar measures for iterated function systems driven by weak contractions

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Abstract: We show the existence and uniqueness for self-similar measures for iterated function systems driven by weak contractions. Our main idea is using the duality theorem of Kantorovich-Rubinstein and equivalent conditions for weak contractions established by Jachymski. We also show collage theorems for such iterated function systems.

Key words: Self-similar measures; iterated function systems; weak contractions; Kantorovich-Rubinstein duality theorem.

1. Introduction and main result.

Hutchinson [Hu81] showed the following result: Let $N \geq 2$. Let X be a complete metric space. Let $p_1, \dots, p_N \in (0,1)$ such that $\sum_{i=1}^N p_i = 1$. Let $f_1, \dots, f_N$ be contractions on X. Then, there exist a unique compact set K and a unique probability measure μ on K such that $K = \cup_{i=1}^N f_i(K)$ and

$$\mu(A) = \sum_{i=1}^N p_i \mu(f_i^{-1}(A))$$

for any Borel subset A of K.

In this paper we consider the case that $f_1, \dots, f_N$ are *weak contractions*. Iterated function systems driven by weak contractions are considered in [AF04, Ha85-1, Ha85-2, L04], for example. There are several different definitions of weak contractions, here we adopt the following definition.

Definition 1.1 (Weak contractions in the sense of Browder [Br68], cf. [J97])**.** Let (X, d) be a metric space and $f: X \to X$ be a map. Then, we say that f is a weak contraction in the sense of Browder if there exists an increasing right-continuous function $\phi: [0, +\infty) \to [0, +\infty)$ such that

$$\phi(t) < t, \quad t > 0,$$

$$d(f(x), f(y)) \leq \phi(d(x,y)), \quad x, y \in X.$$

Hata [Ha85-1, Ha85-2] extended the result of [Hu81] and showed that if each f_i is a weak contradiction on X, then there exists a unique compact subset K of X such that $K = \cup_{i=1}^N f_i(K)$. Hata's definition is different from the Browder's one, but it follows that they are equivalent.

In this paper we show that

Theorem 1.2. *Let (X, d) be a complete metric space and $f_1, \dots, f_N$ be weak contractions. Let K be the unique compact subset of X such that $K = \cup_{i=1}^N f_i(K)$. Let $p_1, \dots, p_N \in (0,1)$ such that $\sum_{i=1}^N p_i = 1$. Then, there exists a unique probability measure μ on K such that*

$$\mu(A) = \sum_{i=1}^N p_i \mu(f_i^{-1}(A)) \tag{1}$$

for any Borel subset A of K.

Barnsley [Ba05, Ba06] considered an inhomogeneous version of this result, specifically, he showed that there exists a unique Borel probability measure μ on a topological space X such that

$$\mu(A) = p\mu_0(A) + \sum_{i=1}^N p_i \mu(f_i^{-1}(A)),$$
$$\forall A\text{: Borel subset of } X,$$

where each f_i is a continuous transformation on X, $p + \sum_{i=1}^N p_i = 1$, $p > 0$, $p_i \geq 0$ for each i, and, μ_0 is a probability measure on X. This framework is general, however, the assumption that $p > 0$ is essential.

Our second result is a collage theorem.

Theorem 1.3. *Let (X, d) be a complete metric space and $f_1, \dots, f_N$ be weak contractions. Let K be the unique compact subset of X such that $K = \cup_{i=1}^N f_i(K)$. Let d_{Haus} be the Hausdorff distance between compact subsets of X. Then, for any $M > \epsilon > 0$, there exists $\delta > 0$ such that if a compact subset L of X satisfies that*

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$$d_{\mathrm{Haus}}(L, \cup_{i=1}^N f_i(L)) \le \delta, \tag{2}$$

and

$$d_{\mathrm{Haus}}(K, L) \le M, \tag{3}$$

then,

$$d_{\mathrm{Haus}}(K, L) \le \epsilon.$$

If $f_1, \dots, f_N$ are contractions, then, the collage theorem is shown by [BEHL86]. Since we add (3), the above result is not an extension of [BEHL86]. However, we believe that (3) is not a large constraint. If (X, d) is compact, there exists M such that (3) is satisfied for any compact subset L of X.

Finally we state a collage theorem for probability measures. Let (X, d) be a complete metric space and $f_1, \dots, f_N$ be weak contractions. Let K be the unique compact subset of X such that $K = \cup_{i=1}^N f_i(K)$. Let $\mathcal{P}(K)$ be the set of probability measures on K. For $f : K \to \mathbf{R}$, let $\mathrm{Lip}(f)$ be the Lipschitz constant for f. For $\mu, \nu \in \mathcal{P}(K)$, let

$$D(\mu, \nu) := \sup\left\{ \int_K f d\mu - \int_K f d\nu : \mathrm{Lip}(f) \le 1 \right\}.$$

This is called the Monge-Kantorovich metric. $(\mathcal{P}(K), D)$ is a compact metric space. See [Ba06, Theorem 2.4.15 and Definition 2.4.16] for details.

Theorem 1.4. *Let (X, d) be a complete metric space and $f_1, \dots, f_N$ be weak contractions. Let K be the unique compact subset of X such that $K = \cup_{i=1}^N f_i(K)$ and μ be the solution for (1). Let $p_1, \dots, p_N \in (0, 1)$ such that $\sum_{i=1}^N p_i = 1$. Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that if a probability measure ν on K satisfies that*

$$D\left(\nu, \sum_{i=1}^N p_i \nu \circ f_i^{-1}\right) \le \delta, \tag{4}$$

then,

$$D(\nu, \mu) \le \epsilon.$$

Before we proceed to proof, we give an example.

Example 1.5. Let $X = [0, 1]$, $N = 2$, $p_1 = p_2 = 1/2$, $f_1(x) = x/(x + 1)$, and $f_2(x) = 1/(2 - x)$. Then, the distribution function of the solution μ of (1) is the Minkowski question-mark function [M1905]. In this particular case, it is shown in Kesseböhmer-Stratmann [KeSt08] that the Hausdorff dimension for μ is strictly smaller than one.

2. Proofs.

Definition 2.1 (Hata's definition of weak contractions [Ha85-2, Definition 2.1])**.** Let (X, d) be a metric space and $f : X \to X$ be a map. Then, we say that f is a weak contraction in the sense of Hata if for any $t > 0$

$$\lim_{s \to t, s > t} \sup_{x, y \in X, d(x,y) \le s} d(f(x), f(y)) < t.$$

Lemma 2.2 (Cf. [J97, Theorem 1])**.** *Let (X, d) be a metric space and $f : X \to X$ be a map. Then, f is a weak contraction in the sense of Hata if and only if f is a weak contraction in the sense of Browder.*

Proof. If f is a weak contraction in the sense of Browder, that is, [J97, Condition (a) of Theorem 1] holds, then it is obvious that f is a weak contraction in the sense of Hata. Conversely, assume that f is a weak contraction in the sense of Hata. Then, [J97, Condition (f) of Theorem 1] holds for

$$\phi(s) := \sup_{d(x,y) \le s} d(f(x), f(y)), \ s \ge 0.$$

Then, by [J97, Theorem 1], f is a weak contraction in the sense of Browder. □

[W91, Proposition A4.5] also discusses several conditions for Hata's definition of weak contractions.

Now we proceed to the proof of Theorem 1.2.

If $f : X \to X$ is a weak contraction and not a contraction on a metric space X, then, $\mathrm{Lip}(g \circ f) = \mathrm{Lip}(g)$ may occur for a function g on X, and it would be difficult to give an upper bound for

$$\sup\left\{ \int_X g \circ f d\mu - \int_X g \circ f d\nu : \mathrm{Lip}(g) \le 1 \right\}, \quad \mu, \nu \in \mathcal{P}(X).$$

Therefore, it seems that the proof of [Hu81] does not work well in a direct manner. Our idea is that we first show the metric D is identical with the first *Wasserstein metric* on $\mathcal{P}(K)$ thanks to the duality theorem of Kantorovich-Rubinstein [KR58] (see also Villani's book [V09, Particular Case 5.16]), and then use several definitions for weak contractions which are equivalent to Browder's definition. Their equivalences are established by [J97, Theorem 1].

Proof. By the fixed point theorem of Browder, it suffices to show that for any $t > 0$,

$$\lim_{s \to t, s > t} \sup_{D(\mu,\nu) \le s} D\left(\sum_{i=1}^N p_i \mu \circ f_i^{-1}, \sum_{i=1}^N p_i \nu \circ f_i^{-1} \right) < t.$$

Since

$$D\left(\sum_{i=1}^N p_i\mu\circ f_i^{-1}, \sum_{i=1}^N p_i\nu\circ f_i^{-1}\right) \le \sum_{i=1}^N p_i D(\mu\circ f_i^{-1}, \nu\circ f_i^{-1}),$$

it suffices to show that for each i,

$$\lim_{s\to t, s>t} \sup_{D(\mu,\nu)\le s} D\big(\mu\circ f_i^{-1}, \nu\circ f_i^{-1}\big) < t. \tag{5}$$

For $\mu,\nu \in \mathcal{P}(K)$, let $\Pi(\mu,\nu)$ be the set of probability measures on $X\times X$ whose marginal distributions to the first and second coordinates are μ and ν respectively. By the duality theorem of [KR58],

$$D(\mu,\nu) = \inf\left\{\int_K\int_K d(x,y)\gamma(dxdy) : \gamma\in\Pi(\mu,\nu)\right\}.$$

If $\gamma\in\Pi(\mu,\nu)$, then, $\gamma\circ(f_i,f_i)^{-1}\in\Pi(\mu\circ f_i^{-1}, \nu\circ f_i^{-1})$. Hence, for any $\gamma\in\Pi(\mu,\nu)$,

$$D(\mu\circ f_i^{-1}, \nu\circ f_i^{-1}) \le \int_K\int_K d(f_i(x), f_i(y))\gamma(dxdy).$$

Since f_i is a weak contraction, by the condition of Krasnoselskii-Stetsenko [KrSt69], whose equivalence with Browder's definition is established by Jachymski [J97, Theorem 1 (d)], there exists a continuous function $\psi_i : [0,+\infty)\to[0,+\infty)$ such that $\psi_i(t) > 0$ if $t > 0$, and,

$$d(f_i(x), f_i(y)) \le d(x,y) - \psi_i(d(x,y)),\ x,y\in K. \tag{6}$$

We show that a contradiction occurs if we take a sufficiently small $\delta > 0$.

Since K is compact, there exists M such that $\sup_{x,y\in K} d(x,y) \le M$. Take sufficiently small $\epsilon\in(0,1)$ so that $4\epsilon t\le M$. Then,

$$\begin{aligned} D(\mu,\nu) &\le \int_{d(x,y)\le\epsilon t} d(x,y)\gamma(dxdy) + \int_{d(x,y)>\epsilon t} d(x,y)\gamma(dxdy)\\ &\le (\epsilon t)\gamma(\{(x,y)\in K^2 : d(x,y)\le\epsilon t\})\\ &\quad + M\gamma(\{(x,y)\in K^2 : d(x,y)>\epsilon t\}).\end{aligned}$$

Hence,

$$\gamma(\{(x,y)\in K^2 : d(x,y)>\epsilon t\}) \ge \frac{D(\mu,\nu)-\epsilon t}{M}.$$

Since ψ_i is positive and continuous,

$$\inf_{M\ge u>\epsilon t}\psi_i(u) > 0.$$

Therefore,

$$\begin{aligned}\int_{K\times K}\psi_i(d(x,y))\gamma(dxdy) &\ge \frac{(D(\mu,\nu)-\epsilon t)_+}{M}\inf_{M\ge u>\epsilon t}\psi_i(u)\\ &\ge \frac{D(\mu,\nu)-\epsilon t}{M}\inf_{M\ge u>\epsilon t}\psi_i(u).\end{aligned}$$

Hence,

$$\begin{aligned} &D(\mu\circ f_i^{-1}, \nu\circ f_i^{-1})\\ &\le \int_{K\times K} d(f_i(x), f_i(y))\gamma(dxdy)\\ &\le \int d(x,y)\gamma(dxdy) - \frac{D(\mu,\nu)-\epsilon t}{M}\inf_{M\ge u>\epsilon t}\psi_i(u).\end{aligned}$$

By taking infimum with respect to γ,

$$\begin{aligned} &D(\mu\circ f_i^{-1}, \nu\circ f_i^{-1})\\ &\le \left(1 - \frac{\inf_{M\ge u>\epsilon t}\psi_i(u)}{M}\right)D(\mu,\nu)\\ &\quad + \frac{\epsilon t}{M}\inf_{M\ge u>\epsilon t}\psi_i(u).\end{aligned} \tag{7}$$

By (6), $\psi_i(u)\le u$ for any $u\le M$, and hence,

$$\inf_{M\ge u>\epsilon t}\psi_i(u) < M.$$

Hence,

$$\begin{aligned} &\sup_{D(\mu,\nu)\le s} D(\mu\circ f_i^{-1}, \nu\circ f_i^{-1})\\ &\le \left(1 - \frac{\inf_{M\ge u>\epsilon t}\psi_i(u)}{M}\right)s + \frac{\epsilon t}{M}\inf_{M\ge u>\epsilon t}\psi_i(u).\end{aligned}$$

Hence,

$$\begin{aligned} &\lim_{s\to t, s>t}\sup_{D(\mu,\nu)\le s} D(\mu\circ f_i^{-1}, \nu\circ f_i^{-1})\\ &\le \left(1 - \frac{\inf_{M\ge u>\epsilon t}\psi_i(u)}{M}\right)t + \frac{\epsilon t}{M}\inf_{M\ge u>\epsilon t}\psi_i(u)\\ &= t\left(1 - (1-\epsilon)\frac{\inf_{M\ge u>\epsilon t}\psi_i(u)}{M}\right) < t.\end{aligned}$$

Thus (5) follows. □

Now we show the collage theorem.

Proof of Theorem 1.3. Assume that there exist $M > \epsilon > 0$ such that for any $\delta > 0$ there exists a compact subset L of X satisfying (2), (3) and

$$d_{\mathrm{Haus}}(K,L) > \epsilon. \tag{8}$$

Since $K = \cup_{i=1}^N f_i(K)$,

$$\begin{aligned} d_{\mathrm{Haus}}(K,L) &\le d_{\mathrm{Haus}}(L, \cup_{i=1}^N f_i(L))\\ &\quad + d_{\mathrm{Haus}}(\cup_{i=1}^N f_i(K), \cup_{i=1}^N f_i(L))\end{aligned}$$

Hence,

$$(9)\quad d_{\mathrm{Haus}}(K,L)-\delta < d_{\mathrm{Haus}}(\cup_{i=1}^N f_i(K), \cup_{i=1}^N f_i(L)) \leq \max_{1\leq i\leq N} d_{\mathrm{Haus}}(f_i(K), f_i(L)).$$

Since f_i is a weak contraction, there exists a continuous function $\psi_i : [0,+\infty) \to [0,+\infty)$ such that $\psi_i(t) > 0$ if $t > 0$, and, (6) holds. It follows that

$$\begin{aligned} &d_{\mathrm{Haus}}(f_i(K), f_i(L)) \\ &\quad\leq \max\{d(f_i(x), f_i(y)) : x \in K, y \in L, d(x,y) \leq d_{\mathrm{Haus}}(K,L)\} \\ &\quad\leq \max\{d(x,y) - \psi_i(d(x,y)) : d(x,y) \leq d_{\mathrm{Haus}}(K,L)\}. \end{aligned}$$

Since $\psi_i(t) \geq 0$ for any $t \geq 0$,

$$\begin{aligned} &\max\{d(x,y) - \psi_i(d(x,y)) : d(x,y) \leq d_{\mathrm{Haus}}(K,L)\} \\ &\leq \max\left\{\frac{\epsilon}{2}, \max\left\{d(x,y) - \psi_i(d(x,y)) : \frac{\epsilon}{2} \leq d(x,y) \leq d_{\mathrm{Haus}}(K,L)\right\}\right\}. \end{aligned}$$

By (3),

$$\begin{aligned} &\max\{d(x,y) - \psi_i(d(x,y)) : \epsilon/2 \leq d(x,y) \leq d_{\mathrm{Haus}}(K,L)\} \\ &\quad\leq d_{\mathrm{Haus}}(K,L) - \inf_{s\in[\epsilon/2,M]} \psi_i(s). \end{aligned}$$

By this and (9),

$$d_{\mathrm{Haus}}(K,L) - \delta < \max\left\{\epsilon/2, d_{\mathrm{Haus}}(K,L) - \min_{1\leq i\leq N} \inf_{s\in[\epsilon/2,M]} \psi_i(s)\right\}.$$

We remark that by the continuity and positivity for ψ_i,

$$\inf_{s\in[\epsilon/2,M]} \psi_i(s) > 0.$$

Hence if we take

$$\delta < \min\left\{\epsilon/4, \min_{1\leq i\leq N} \inf_{s\in[\epsilon/2,M]} \psi_i(s)\right\}$$

and an associated L, then, by (8), a contradiction occurs. □

Remark 2.3. (i) We are not sure whether we can drop (3) or not. It is added because we do not know about the long-time behavior of $\psi_i(t)$ appearing in the above proof. If $\lim_{t\to\infty} \psi_i(t) > 0$, we can remove (3). If f_i is contractive, then, we can take $\psi_i(t) := (1 - \mathrm{Lip}(f_i))t$.
(ii) [AF04, Proposition 4.3] considers a weak contractivity for the Barnsley-Hutchinson operator. However, their definition of weak contractions [AF04, Definition 3.1], which is also adopted by [R01], is stronger than the one we adopt. If [AF04, Definition 3.1] is adopted, we can drop (3).

Finally we show Theorem 1.4.

Proof. The outline is the same as in the proof of Theorem 1.3, so we give a sketch only. Assume that there exists $\epsilon > 0$ such that for any $\delta \in (0,\epsilon)$ there exists a probability measure ν on K satisfying (4), and

$$(10)\qquad D(\mu,\nu) > \epsilon.$$

We have that for some i,

$$(11)\qquad D(\mu,\nu) - \delta \leq D(\mu\circ f_i^{-1}, \nu\circ f_i^{-1}).$$

Let M such that $\sup_{x,y\in K} d(x,y) \leq M$. Then, we can show that by replacing ϵt with $D(\mu,\nu)/4$ in the proof of Theorem 1.2, and by recalling (7),

$$D(\mu\circ f_i^{-1}, \nu\circ f_i^{-1}) \leq \left(1 - \frac{3\inf_{M\geq u > D(\mu,\nu)/4} \psi_i(u)}{4M}\right) D(\mu,\nu).$$

By this, (10) and (11),

$$(12)\qquad \delta \geq \frac{3\inf_{M\geq u > D(\mu,\nu)/4} \psi_i(u)}{4M} D(\mu,\nu)$$

$$(13)\qquad \geq \frac{3\epsilon \inf_{M\geq u > \epsilon/4} \psi_i(u)}{4M}.$$

Hence a contradiction occurs if

$$0 < \delta < \frac{3\epsilon \min_{1\leq i\leq N} \inf_{M\geq u>\epsilon/4} \psi_i(u)}{4M}.$$

□

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