Complete flat fronts as hypersurfaces in Euclidean space

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Abstract: By Hartman–Nirenberg's theorem, any complete flat hypersurface in Euclidean space must be a cylinder over a plane curve. However, if we admit some singularities, there are many non-trivial examples. *Flat fronts* are flat hypersurfaces with admissible singularities. Murata–Umehara gave a representation formula for complete flat fronts with non-empty singular set in Euclidean 3-space, and proved the four vertex type theorem. In this paper, we prove that, unlike the case of $n = 2$, there do not exist any complete flat fronts with non-empty singular set in Euclidean $(n+1)$-space $(n \geq 3)$.

Key words: Flat hypersurface; flat front; Hartman–Nirenberg's theorem; singular point; wave front; coherent tangent bundle.

1. Introduction. Let $\mathbf{R}^{n+1}$ be the Euclidean $(n+1)$-space. By Hartman–Nirenberg's theorem [2], *any complete flat hypersurface in* $\mathbf{R}^{n+1}$ *must be a cylinder over a plane curve.* Here, a cylinder is a regular hypersurface which is congruent to $f : \mathbf{R}^n \to \mathbf{R}^{n+1}$ defined by

$$f(t, w_2, \dots, w_n) := (x(t), y(t), w_2, \dots, w_n),$$

where $t \mapsto (x(t), y(t))$ is a regular curve in $\mathbf{R}^2$. We remark that Massey [8] gave an alternative proof for $n = 2$.

However, in $\mathbf{R}^3$, there are non-trivial flat surfaces with admissible singularities called *flat fronts*. Here, a *front* is a generalized notion of regular surfaces (more generally, regular hypersurfaces) with admissible singular points. See Section 2 for precise definitions. Murata–Umehara [9] gave a representation formula for complete flat fronts with non-empty singular set, and proved the four vertex type theorem: *Let* $\xi : S^1 \to S^2$ *be a regular curve without inflection points, and* $\alpha = a(t)dt$ *a* 1*-form on* $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ *such that* $\int_{S^1} \xi\,\alpha = 0$ *holds. Then,* $f_{\xi,\alpha} : S^1 \times \mathbf{R} \to \mathbf{R}^3$ *defined by*

$$f_{\xi,\alpha}(t, v) := \hat{\sigma}(t) + v\,\xi(t) \quad \left(\hat{\sigma}(t) := \int_0^t a(\tau)\,\xi(\tau)d\tau\right) \tag{1.1}$$

is a complete flat front with non-empty singular set. Conversely, let $f : M^2 \to \mathbf{R}^3$ *be a complete flat front defined on a connected smooth* 2*-manifold* M^2*. If the singular set* $S(f)$ *of* f *is not empty, then* f *is umbilic-free, co-orientable,* M^2 *is diffeomorphic to* $S^1 \times \mathbf{R}$*, and* f *is given by* (1.1)*. Moreover, if the ends of* f *are embedded,* f *has at least four singular points other than cuspidal edges.*

Therefore, it is natural to ask what occurs in the higher dimensional cases. In this paper, we prove that there do not exist any non-trivial flat fronts in higher dimensions:

Theorem 1. *If* $n \geq 3$*, there do not exist any complete flat fronts with non-empty singular set in* $\mathbf{R}^{n+1}$*.*

Table. Classification of complete flat fronts in $\mathbf{R}^{n+1}$

	Singular set $= \emptyset$	Singular set $\neq \emptyset$
$n = 2$	Cylinder	$\exists\infty$ ([9])
$n \geq 3$	([2])	$\nexists$ (Theorem 1)

Combining Hartman–Nirenberg's theorem [2], Murata–Umehara's theorem [9] and Theorem 1, we have the classification of complete flat fronts in $\mathbf{R}^{n+1}$.

We remark that, although there do not exist any complete flat fronts in $\mathbf{R}^{n+1}$ $(n \geq 3)$, there are many *weakly complete* ones. For example, we can construct a weakly complete flat front by a pair $(\gamma(t), a(t))$ of a complete regular curve $\gamma(t)$ in S^n and a smooth function $a(t)$ on $\mathbf{R}$ (cf. Proposi-

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tion 8). Here, we denote by S^n the n-sphere of constant sectional curvature 1. Moreover, by a regular curve in $\mathbf{R}^{n+1}$, one may construct a flat front called *tangent developable*. (See [7] for more details and properties of singularities of tangent developables.)

We also remark that there are several works related to Murata–Umehara's theorem. Naokawa [10] gave an estimation of singular points other than cuspidal edges on asymptotic completions of developable Möbius strips. On the other hand, flat fronts can be considered as fronts with one principal curvature zero. In a previous paper [5], the author gave a classification of weakly complete fronts with one principal curvature non-zero constant.

With respect to the case of non-flat ambient spaces, it is known that flat fronts in $\mathbf{R}^{n+1}$ are identified with fronts of constant sectional curvature 1 (CSC-1 fronts) in S^{n+1} via the central projection of a hemisphere to a tangent space. Therefore, the local nature of flat fronts in $\mathbf{R}^{n+1}$ is the same as that of CSC-1 fronts in S^{n+1}. However, they may display different global properties. In [6], the author gave a classification of complete CSC-1 fronts, which is a generalization of O'Neill–Stiel's theorem [11]. In particular, in the case of $n \geq 3$, there exist many non-trivial complete CSC-1 fronts in S^{n+1}, although there do not exist any complete flat fronts other than cylinders in $\mathbf{R}^{n+1}$. (See also [4] for the case of negative sectional curvature.)

This paper is organized as follows: In Section 2, we shall review the definition and fundamental properties of flat fronts. Using them, we shall prove Theorem 1 in Section 3.

2. Preliminaries. We denote by $\mathbf{R}^{n+1}$ the Euclidean $(n+1)$-space, and S^n the unit sphere

$$S^n := \{\boldsymbol{x} \in \mathbf{R}^{n+1};\ \boldsymbol{x} \cdot \boldsymbol{x} = 1\},$$

where the dot '$\cdot$' is the canonical inner product on $\mathbf{R}^{n+1}$. Let M^n be a connected smooth n-manifold and

$$f : M^n \to \mathbf{R}^{n+1}$$

a smooth map. A point $p \in M^n$ is called a *singular point* if f is not an immersion at p. Otherwise, we call p a *regular point*. Denote by $S(f)$ $(\subset M^n)$ the set of singular points. If $S(f)$ is empty, we call f a (regular) hypersurface.

A smooth map $f : M^n \to \mathbf{R}^{n+1}$ is called a *frontal*, if for each point $p \in M^n$, there exist a neighborhood U of p and a smooth map $\nu : U \to S^n$ such that

$$df_q(\boldsymbol{v}) \cdot \nu(q) = 0$$

holds for each $q \in U$ and $\boldsymbol{v} \in T_qM^n$. Such a ν is called the unit normal vector field or the Gauss map of f. If ν can be defined throughout M^n, f is called *co-orientable*. On the other hand, we say *orientable* if M^n is orientable. If

$$(L :=)\ (f, \nu) : U \to \mathbf{R}^{n+1} \times S^n$$

gives an immersion, f is called a *wave front* (or a *front*, for short). The map $L = (f, \nu)$ is called the Legendrian lift of f.

2.1. Completeness, weak completeness, umbilic points. The first fundamental form (i.e., the induced metric) is given by $ds^2 := df \cdot df$. For a front $f : M^n \to \mathbf{R}^{n+1}$ with a (possibly locally defined) unit normal vector field ν,

$$ds^2_{\#} := ds^2 + d\nu \cdot d\nu$$

gives a positive definite Riemannian metric called the lift metric. If the lift metric $ds^2_{\#}$ is complete, f is called *weakly complete*. On the other hand, f is called *complete*, if there exists a symmetric covariant $(0,2)$-tensor T on M^n with compact support such that $ds^2 + T$ gives a complete metric on M^n. In this case, the singular set $S(f)$ must be compact. As noted in [9], if $S(f)$ is empty, then $f : M^n \to \mathbf{R}^{n+1}$ is complete as a front if and only if f is complete as a regular hypersurface (i.e., (M^n, ds^2) is a complete Riemannian manifold).

Fact 2 ([9, Lemma 4.1])**.** A complete front is weakly complete.

A point $p \in M^n$ is called an *umbilic point*, if there exist real numbers δ_1, δ_2 such that

$$\delta_1 (df)_p = \delta_2 (d\nu)_p, \quad (\delta_1, \delta_2) \neq (0,0)$$

hold. For a positive number $\delta > 0$, set

$$f^\delta := f + \delta\nu, \quad \nu^\delta := \nu. \tag{2.1}$$

Then we can check that f^δ is a front and ν^δ gives a unit normal along f^δ. Such an f^δ is called the *parallel front* of f. Umbilic points are common in its parallel family.

Fact 3 ([6, Lemma 2.7])**.** Let $p \in M^n$ be a singular point of a front f. Then, p is an umbilic point if and only if $\operatorname{rank}(df)_p = 0$ holds. In this case, we have $\operatorname{rank}(d\nu)_p = n$ for any unit normal vector field of f.

2.2. Flat fronts. In [12,14], Saji–Umehara–Yamada introduced *coherent tangent bundles*, which is a generalized notion of Riemannian manifolds.

Let $\mathcal{E}$ be a vector bundle of rank n over a smooth n-manifold M^n. We equip a fiber metric $\langle\ ,\ \rangle$ on $\mathcal{E}$ and a metric connection D on $(\mathcal{E}, \langle\ ,\ \rangle)$. Let $\varphi : TM^n \to \mathcal{E}$ be a bundle homomorphism such that

$$D_X\varphi(Y) - D_Y\varphi(X) - \varphi([X, Y]) = 0 \tag{2.2}$$

holds for arbitrary smooth vector fields X, Y on M^n. Then

$$\mathcal{E} = (\mathcal{E}, \langle\ ,\ \rangle, D, \varphi)$$

is called a *coherent tangent bundle* over M^n.

It is known that, in general, coherent tangent bundles can be constructed only from positive semi-definite metrics called *Kossowski metrics* (cf. [3,15]).

We shall review the coherent tangent bundles induced from frontals (cf. [14, Example 2.4]). For a frontal $f : M^n \to \mathbf{R}^{n+1}$, set $\mathcal{E}_f$, $\langle\ ,\ \rangle_f$, D_f and φ_f, respectively, as follows:

- $\mathcal{E}_f$ is the subbundle of the pull-back bundle $f^*T\mathbf{R}^{n+1}$ perpendicular to ν,
- $\langle\ ,\ \rangle_f$ is the metric on $\mathcal{E}_f$ induced from the canonical metric on $\mathbf{R}^{n+1}$,
- D_f is the tangential part of the Levi–Civita connection on $\mathbf{R}^{n+1}$,
- $\varphi_f : TM^n \to \mathcal{E}_f$ defined as $\varphi_f(X) := df(X)$.

Then, $\mathcal{E}_f = (\mathcal{E}_f, \langle\ ,\ \rangle_f, D_f, \varphi_f)$ is a coherent tangent bundle, which we call the *induced coherent tangent bundle*.

Definition 4 ([6])**.** A coherent tangent bundle is said to be *flat* if

$$R^D(X, Y)\xi = 0$$

holds for all smooth vector fields X, Y on M^n and each smooth section ξ of $\mathcal{E}$, where R^D is the curvature tensor of the connection D given by

$$R^D(X, Y)\xi := D_XD_Y\xi - D_YD_X\xi - D_{[X,Y]}\xi.$$

A frontal f is called *flat*, if the induced coherent tangent bundle $\mathcal{E}_f$ is flat.

In [6], the following characterization of flatness was proved by using the Gauss equation for frontals given by Saji–Umehara–Yamada [13, Proposition 2.4].

Fact 5 ([6, Lemma 3.3])**.** Let $f : M^n \to \mathbf{R}^{n+1}$ be a frontal with a unit normal vector field ν. Then f is flat if and only if

$$\operatorname{rank}(d\nu) \leq 1 \tag{2.3}$$

holds on M^n.

We remark that Murata–Umehara [9] defined the flatness for frontals in $\mathbf{R}^3$ by the condition (2.3). Therefore, our definition of flatness is compatible to that given by Murata–Umehara. We also remark that, if $S(f)$ is empty, then $f : M^n \to \mathbf{R}^{n+1}$ is flat as a front if and only if f is flat as a regular hypersurface.

3. Proof of Theorem 1.

Denote by $\mathcal{U}_f$ the set of umbilic points. Since $\mathcal{U}_f$ is a closed subset in M^n, the non-umbilic point set $M^n \setminus \mathcal{U}_f$ is open.

Lemma 6. *Let $f : M^n \to \mathbf{R}^{n+1}$ be a non-totally-umbilic flat front. For each non-umbilic point $q \in M^n \setminus \mathcal{U}_f$, there exist a local coordinate neighborhood $(U; u_1, \ldots, u_n)$ of q and a smooth function $\rho = \rho(u_1, \ldots, u_n)$ on U such that*

$$-\rho\nu_{u_1} = f_{u_1}, \quad \nu_{u_j} = 0, \quad \nu_{u_1} \cdot f_{u_j} = 0 \tag{3.1}$$

hold for each $j = 2, \ldots, n$, and $\{\nu_{u_1}, f_{u_2}, \ldots, f_{u_n}\}$ is a frame on U. For each u_1, set the slice U_{u_1} of U as

$$U_{u_1} := \{\boldsymbol{u} \in \mathbf{R}^{n-1};\ (u_1, \boldsymbol{u}) \in U\}.$$

Then, the restriction $f|_{U_{u_1}} : U_{u_1} \to \mathbf{R}^{n+1}$ is a totally geodesic embedding for each u_1.

Proof. Since f is flat, Fact 5 implies that there exists a local coordinate system $(V; v_1, \ldots, v_n)$ around $q \in M^n \setminus \mathcal{U}_f$ such that

$$\nu_{v_j} = 0 \quad (j = 2, \ldots, n)$$

holds. Then, we have

$$L_{v_1} = (f_{v_1}, \nu_{v_1}), \quad L_{v_j} = (f_{v_j}, 0) \quad (j = 2, \ldots, n).$$

Since f is a front, $\{L_{v_1}, L_{v_2}, \ldots, L_{v_n}\}$ is linearly independent around q. And hence, we have that $\{f_{v_1}, f_{v_2}, \ldots, f_{v_n}\}$ or $\{\nu_{v_1}, f_{v_2}, \ldots, f_{v_n}\}$ is linearly independent around q. In each case, there exists a positive number $\delta > 0$ such that the parallel front $f^\delta := f + \delta\,\nu$ is a flat immersion around q (cf. (2.1)). Since f is umbilic-free around q, so is f^δ. Let $(U; u_1, \ldots, u_n)$ be a curvature line coordinate system of f^δ around $q \in M^n \setminus \mathcal{U}_f$. That is, for each $j = 2, \ldots, n$,

$$(\nu^\delta)_{u_j} = 0, \quad (f^\delta)_{u_1} \cdot (f^\delta)_{u_j} = 0 \tag{3.2}$$

and

$$-(\nu^\delta)_{u_1} = \alpha(f^\delta)_{u_1} \tag{3.3}$$

hold, where $\alpha = \alpha(u_1, \ldots, u_n)$ is a smooth function on U. In this case, the principal curvatures $\lambda_1^\delta, \ldots, \lambda_n^\delta$ of f^δ are given by $\lambda_1^\delta = \alpha$, $\lambda_j^\delta = 0$ ($j = 2, \ldots, n$). Since f^δ is umbilic-free, $\alpha \neq 0$ on U. Substituting (2.1) into Eqs. (3.2) and (3.3), we may conclude that (3.1) holds with

$$\rho := \frac{1 + \delta\alpha}{\alpha}.$$

With respect to the third assertion, $\boldsymbol{n} := \nu_{u_1}/|\nu_{u_1}|$ gives a unit normal vector field of $f|_{U_{u_1}}$. Set $\psi := 1/|\nu_{u_1}|$. Then, for each $j = 2, \ldots, n$,

$$\boldsymbol{n}_{u_j} = \psi_{u_j}\nu_{u_1} + \psi\nu_{u_1 u_j} = \psi^{-1}\psi_{u_j}\boldsymbol{n}$$

and $\boldsymbol{n} \cdot \boldsymbol{n}_{u_j} = 0$ yield $\boldsymbol{n}_{u_j} = 0$ on U_{u_1}. Together with $\nu_{u_j} = 0$ ($j = 2, \ldots, n$) on U_{u_1}, we have the conclusion. □

By Lemma 6, since the image of $f|_{U_{u_1}}$ is included in an $(n-1)$-dimensional affine subspace $A_{u_1}^{n-1}$ of $\mathbf{R}^{n+1}$ for each u_1, by a coordinate change of $(u_2, \ldots, u_n)$, we may take a new coordinate system $(U'; u_1, w_2, \ldots, w_n)$ such that $(w_2, \ldots, w_n)$ is the canonical Euclidean coordinate system of $A_{u_1}^{n-1}$ for each u_1. Namely, $f_{w_j} \cdot f_{w_k} = \delta_{jk}$ holds for $j, k = 2, \ldots, n$.

Set $\sigma(u_1)$ and $\boldsymbol{e}_j(u_1)$ ($j = 2, \ldots, n$) as

$$\sigma(u_1) := f(u_1, 0, \ldots, 0),$$
$$\boldsymbol{e}_j(u_1) := f_{w_j}(u_1, 0, \ldots, 0),$$

respectively. Then, we have

$$f(u_1, w_2, \ldots, w_n) = \sigma(u_1) + w_2\boldsymbol{e}_2(u_1) + \cdots + w_n\boldsymbol{e}_n(u_1).$$

Since f has no umbilic point on U, the Gauss map ν depends only on u_1 and $\nu_{u_1} \neq 0$ holds. Therefore,

$$\gamma(u_1) := \nu(u_1, 0, \ldots, 0)$$

is a regular curve in S^n. By a coordinate change of u_1, we may take a new coordinate system $(W; t, w_2, \ldots, w_n)$ such that the spherical regular curve $t \mapsto \gamma(t)$ is parametrized by arc-length. Thus, we have

$$f(t, w_2, \ldots, w_n) = \sigma(t) + w_2\boldsymbol{e}_2(t) + \cdots + w_n\boldsymbol{e}_n(t). \tag{3.4}$$

Denote by $\boldsymbol{e}(t) := \gamma'(t)$ the unit tangent vector of $\gamma(t)$. Since $f_{w_j} = \boldsymbol{e}_j$ for each $j = 2, \ldots, n$ and $\gamma(t)$ is the Gauss map of f, we have

$$\gamma(t) \cdot \boldsymbol{e}_j(t) = 0 \quad (j = 2, \ldots, n). \tag{3.5}$$

In addition, the third equation of (3.1) yields

$$\gamma'(t) \cdot \boldsymbol{e}_j(t) = 0 \quad (j = 2, \ldots, n). \tag{3.6}$$

Therefore, $\{\boldsymbol{e}_j(t)\}_{j=2,\ldots,n}$ is an orthonormal frame of the normal bundle $(\gamma'(t))^\perp$ along the spherical regular curve $\gamma(t)$. Moreover, Eqs. (3.5) and (3.6) yield

$$\gamma(t) \cdot \boldsymbol{e}_j'(t) = 0 \quad (j = 2, \ldots, n). \tag{3.7}$$

Hence, by (3.4), $f_t \cdot \gamma = 0$ implies $\sigma'(t) \cdot \gamma(t) = 0$. Therefore, there exist smooth functions $a_j = a_j(t)$ ($j = 1, \ldots, n$) such that

$$\sigma'(t) = a_1(t)\boldsymbol{e}(t) + a_2(t)\boldsymbol{e}_2(t) + \cdots + a_n(t)\boldsymbol{e}_n(t). \tag{3.8}$$

Thus, we have the following

Lemma 7. *Let $f : M^n \to \mathbf{R}^{n+1}$ be a non-totally-umbilic flat front. For each non-umbilic point $q \in M^n \setminus \mathcal{U}_f$, there exist a local coordinate neighborhood $(W; t, w_2, \ldots, w_n)$ of q, a regular curve $\gamma(t)$ in S^n, an orthonormal frame $\{\boldsymbol{e}_2(t), \ldots, \boldsymbol{e}_n(t)\}$ of the normal bundle $(\gamma')^\perp$ along $\gamma(t)$ and smooth functions $\{a_j(t)\}_{j=1,\ldots,n}$ such that f is given by* (3.4) *on W, where*

$$\sigma(t) := \int_0^t \eta(\tau)\, d\tau \quad \left(\eta(\tau) := a_1(\tau)\boldsymbol{e}(\tau) + \sum_{j=2}^n a_j(\tau)\boldsymbol{e}_j(\tau)\right) \tag{3.9}$$

and $\boldsymbol{e}(t) := \gamma'(t)$.

Finally, we shall reduce the numbers of functions. For a unit speed regular curve $\gamma = \gamma(t) : I \to S^n$ defined on an open interval I, set $\boldsymbol{e}(t) := \gamma'(t)$. Then, there exist an orthonormal frame $\{\boldsymbol{e}_j(t)\}_{j=2,\ldots,n}$ of the normal bundle along γ and smooth functions $\mu_j(t)$ ($j = 2, \ldots, n$) such that

$$\boldsymbol{e}_j'(t) = -\mu_j(t)\boldsymbol{e}(t)$$

for each $j = 2, \ldots, n$. Such a frame $\{\boldsymbol{e}_j(t)\}_{j=2,\ldots,n}$ is called the *Bishop frame* (cf. [1]).

Let $f = f(t, w_2, \ldots, w_n)$ be the flat front given by (3.4) with the Bishop frame $\{\boldsymbol{e}_j(t)\}_{j=2,\ldots,n}$. Set

$$\rho(t, w_2, \ldots, w_n) := a_1(t) - \sum_{j=2}^n w_j\mu_j(t).$$

Since $f_{w_j} = \boldsymbol{e}_j(t)$ for $j = 1, \ldots, n$,

$$f_t = \rho(t, w_2, \ldots, w_n)\, \boldsymbol{e}(t) + a_2(t)\boldsymbol{e}_2(t) + \cdots + a_n(t)\boldsymbol{e}_n(t),$$

and $d\nu \cdot d\nu = dt^2$, the lift metric $ds^2_\# = df \cdot df + d\nu \cdot d\nu$ is given by

$$ds^2_\# = \left(1 + \rho^2 + \sum_{j=2}^{n} (a_j(t))^2\right) dt^2 + \sum_{j=2}^{n} \left(2a_j(t) dw_j dt + (dw_j)^2\right).$$

By a straightforward calculation, it can be checked that each w_j-curve ($j = 2, \ldots, n$) gives a geodesic of the lift metric $ds^2_\#$. Thus, if f is weakly complete, every w_j-curve ($j = 2, \ldots, n$) can be defined on the whole real line $\mathbf{R}$. For each $j = 2, \ldots, n$, set

$$b_j(t) := -\int_0^t a_j(\tau) d\tau.$$

By a coordinate change

$$(t, w_2, \ldots, w_n) \mapsto (t, w_2 + b_2(t), \ldots, w_n + b_n(t)),$$

we have

$$\begin{aligned} f(t, w_2 + b_2(t), \ldots, w_n + b_n(t)) &= \sigma(t) + \sum_{j=2}^{n} (w_j + b_j(t)) \boldsymbol{e}_j(t) \\ &= \hat{\sigma}(t) + \sum_{j=2}^{n} w_j \boldsymbol{e}_j(t), \end{aligned}$$

where we set $\hat{\sigma}(t)$ as

$$\hat{\sigma}(t) := \sigma(t) + b_2(t)\boldsymbol{e}_2(t) + \cdots + b_n(t)\boldsymbol{e}_n(t).$$

By (3.9), we have $\hat{\sigma}'(t) = a(t)\boldsymbol{e}(t)$, where

$$a(t) := a_1(t) - b_2(t)\mu_2(t) - \cdots - b_n(t)\mu_n(t).$$

Therefore, we have the following

Proposition 8. *Let $f : M^n \to \mathbf{R}^{n+1}$ be a weakly complete flat front which is not totally-umbilic. Around each non-umbilic point, there exist an interval I, a local coordinate system $(I \times \mathbf{R}^{n-1}; t, w_2, \ldots, w_n)$, a regular curve $\gamma : I \to S^n$ parametrized by arc-length, an orthonormal frame $\{\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ of the normal bundle $(\gamma')^\perp$ along γ and a smooth function $a(t)$ on I such that f is given by*

$$f(t, w_2, \ldots, w_n) = \hat{\sigma}(t) + \sum_{j=2}^{n} w_j \boldsymbol{e}_j(t) \quad \left(\hat{\sigma}(t) := \int_0^t a(\tau)\gamma'(\tau) d\tau\right) \tag{3.10}$$

on $I \times \mathbf{R}^{n-1}$. Conversely, for a given unit speed regular curve $\gamma : I \to S^n$ defined on an interval I, an orthonormal frame $\{\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ of the normal bundle $(\gamma')^\perp$ along γ and a smooth function $a(t)$ on I, $f : I \times \mathbf{R}^{n-1} \to \mathbf{R}^{n+1}$ defined as (3.10) is an umbilic-free flat front.

Theorem 9. *Let $f : M^n \to \mathbf{R}^{n+1}$ be a weakly complete flat front. If $n \geq 3$ and the singular set $S(f)$ is not empty, then $S(f)$ cannot be compact.*

Proof. Take a singular point $q \in S(f)$. By Facts 3 and 5, q is not an umbilic point. By Proposition 8, we have that f is given by (3.10) on $U := I \times \mathbf{R}^{n-1}$. Without loss of generality, $\{\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is the Bishop frame such that $\boldsymbol{e}_j'(t) = -\mu_j(t)\boldsymbol{e}(t)$ holds for each $j = 2, \ldots, n$. We remark that the curvature function $\kappa_\gamma(t)$ of $\gamma(t)$ is given by $\kappa_\gamma(t) = \sqrt{(\mu_2(t))^2 + \cdots + (\mu_n(t))^2}$.

Differentiating (3.10), we have

$$f_t = \hat{\rho}(t, w_2, \ldots, w_n)\boldsymbol{e}(t), \quad f_{w_j} = \boldsymbol{e}_j(t)$$

for $j = 1, \ldots, n$, where

$$\hat{\rho}(t, w_2, \ldots, w_n) := a(t) - \sum_{j=2}^{n} w_j \mu_j(t).$$

Since

$$f_t \wedge f_{w_2} \wedge \cdots \wedge f_{w_n} = \hat{\rho}(t, w_2, \ldots, w_n)\boldsymbol{e}(t) \wedge \boldsymbol{e}_2(t) \wedge \cdots \wedge \boldsymbol{e}_n(t),$$

we have

$$S(f) \cap U = \{p \in U;\ \hat{\rho}(p) = 0\}.$$

Let S_1, S_2 be the subsets of $S(f) \cap U$ defined by

$$\begin{aligned} S_1 &:= \{(t, w_2, \ldots, w_n) \in U;\ a(t) = \kappa_\gamma(t) = 0\}, \\ S_2 &:= \{(t, w_2, \ldots, w_n) \in U; \\ &\qquad \hat{\rho}(t, w_2, \ldots, w_n) = 0, \kappa_\gamma(t) \neq 0\}, \end{aligned}$$

respectively. Then, we have $S(f) \cap U = S_1 \cup S_2$.

Since $\nu(t, w_2, \ldots, w_n) = \gamma(t)$ gives a unit normal vector field along f, the lift metric $ds^2_\#$ is given by

$$ds^2_\# = \left(1 + \hat{\rho}^2\right) dt^2 + \sum_{j=2}^{n} dw_j^2$$

on U.

If $q = (t^o, w_2^o, \ldots, w_n^o) \in S_1$, $a(t^o) = \kappa_\gamma(t^o) = 0$ holds. In this case, we have $(t^o, w_2, \ldots, w_n) \in S_1$ for any $w_j \in \mathbf{R}$ ($j = 2, \ldots, n$). In particular, $c_1 : \mathbf{R} \to S_1\ (\subset M^n)$ given by

$$c_1(x) := (t^o, x, 0, \ldots, 0)$$

is a geodesic with respect to the lift metric $ds^2_\#$ such that $\hat{c}_1 := f \circ c_1$ is a straight line in $\mathbf{R}^{n+1}$, and hence $S(f)\ (\supset S_1)$ cannot be compact.

If $q = (t^o, w_2^o, \dots, w_n^o) \in S_2$, we have $\kappa_\gamma(t^o) \neq 0$. Without loss of generality, we may assume that $\mu_n(t^o) \neq 0$. Then, there exists $\varepsilon > 0$ such that $\mu_n(t) \neq 0$ for each $t \in I(t^o, \varepsilon) := (t^o - \varepsilon, t^o + \varepsilon)$. Thus,

$$S_2[t^o] := \left\{ (t, w_2, \dots, w_n) \in I(t^o, \varepsilon) \times \mathbf{R}^{n-1}; \right.$$
$$\left. w_n = \frac{a(t)}{\mu_n(t)} - \sum_{j=2}^{n-1} \hat{\mu}_j(t) w_j \right\}$$

is a subset of S_2, where $\hat{\mu}_j(t) := \mu_j(t)/\mu_n(t)$ for $j = 2, \dots, (n-1)$. Set a positive number k^o as $k^o := \sqrt{1 + (\hat{\mu}_2(t^o))^2}$. Since $c_2 : \mathbf{R} \to S_2[t^o]$ given by

$$c_2(x) := \left(t^o, \frac{1}{k^o} x, 0, \dots, 0, \frac{a(t^o)}{\mu_n(t^o)} - \frac{\hat{\mu}_2(t^o)}{k^o} x \right)$$

is a geodesic with respect to the lift metric $ds^2_{\#}$ such that $\hat{c}_2 := f \circ c_2$ is a straight line in $\mathbf{R}^{n+1}$, and hence $S(f)\,(\supset S_2[t^o])$ cannot be compact. □

Proof of Theorem 1. We shall give a proof by contradiction. Let $f : M^n \to \mathbf{R}^{n+1}$ a complete flat front $(n \geq 3)$. By Fact 2, f is weakly complete. Assume that the singular set $S(f)$ is not empty. By the completeness of f, the singular set $S(f)$ must be compact, which contradicts Theorem 9. Hence, we have that $S(f)$ must be empty, and then f is a complete flat regular hypersurface. □

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