Examples of genus two fibrations with no sections on rational surfaces

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Abstract: We construct explicit examples of genus two fibrations with no sections on rational surfaces by the double covering method. For the proof of non-existence of sections, we use the theory of the virtual Mordell-Weil groups.

Key words: Sections of fibrations; Mordell-Weil groups; rational surfaces.

1. Introduction. We shall work over the complex number field $\mathbf{C}$. Let S be a smooth projective rational surface and $\varphi : S \to \mathbf{P}^1$ a fibration whose general fibre F is a projective curve of genus $g \geq 1$. We assume that φ is relatively minimal, i.e., there are no (-1)-curves contained in fibres. In the case where $g = 1$, it is well-known that the Picard number $\rho(S)$ equals ten. Furthermore, as a consequence of the canonical bundle formula of Kodaira [10, Theorem 12], φ admits a section if and only if φ does not have any multiple fibres (e.g., [4, Proposition (1.1.)]). When $g \geq 2$, it can be shown that $11 \leq \rho(S) \leq 4g + 6$ by a similar argument to [7, §2] (cf. [9], see also [14, Theorem 2.8] and [3]). Furthermore, if either $\rho(S) = 4g + 6$ with $g \neq 1$ or $\rho(S) = 13$ with $g = 2$ holds, then φ always admits a section (cf. [8, Theorem 2.2] and [6, Lemma 1.4], see also [14], [13] and [2, §10.5]).

We restrict ourselves to the case where $g = 2$ and $\rho(S) = 12$. Suppose that φ admits a section. We can regard it as a horizontal curve D on S such that $D.F = 1$. Therefore the fibres of φ contain at least one irreducible component with multiplicity one (cf. Remark 5). The purpose of the paper is to construct examples for the converse:

Theorem 1. *Let (t, x) be local coordinates of $(\mathbf{P}^1 \setminus \{\infty\}) \times (\mathbf{P}^1 \setminus \{\infty\})$ and $pr_1 : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ the projection map onto the first factor. Put $\Gamma_0 = pr_1^{-1}(0)$ and $\Gamma_\infty = pr_1^{-1}(\infty)$. Let σ and τ be complex numbers with $\sigma \neq 0, 1$ and $\tau \neq 0, 1/2, 2/3, 1$. Let $A(\gamma, \tau)$ denote the closure of the zero set of a polynomial $\gamma\tau x^3 - \gamma t x^2 - (3\tau - 2)tx + (2\tau - 1)t^2$ in t, x on $\mathbf{P}^1 \times \mathbf{P}^1$ for $\gamma \in \mathbf{C} \setminus \{0\}$. Put $B = A(1, \tau) + A(\sigma, \tau) + \Gamma_0 + \Gamma_\infty$. Let $\pi : \hat{X} \to \mathbf{P}^1 \times \mathbf{P}^1$ be the finite double cover branched along B and $\mu : \tilde{X} \to \hat{X}$ the canonical resolution of singularities of $\hat{X}$. Then a general fibre of $pr_1 \circ \pi \circ \mu : \tilde{X} \to \mathbf{P}^1$ is a smooth curve of genus two. Let $f : X \to \mathbf{P}^1$ be the relatively minimal model of $pr_1 \circ \pi \circ \mu$.*

Then X is a smooth rational surface with $\rho(X) = 12$. Furthermore, $f : X \to \mathbf{P}^1$ has no sections, the fibres $f^{-1}(0)$ and $f^{-1}(\infty)$ are as in [11, p. 172, 3–II^*_{3-0}], and the other fibres of f are irreducible and reduced. In particular, the fibres of f contain at least one irreducible component with multiplicity one.

Let us explain the organization of the paper. At first we construct a genus two fibration $f : X \to \mathbf{P}^1$ as in Theorem 1 from a $\mathbf{P}^1$-bundle Σ_0 with a given branch divisor by the double covering method. In Proposition 3, we show the assertions as in Theorem 1 except for non-existence of a section of f. If f has a section, then f has at least two sections since the branch divisor does not contain a section of Σ_0 (cf. Lemma 6).

Next we consider the ruling associated to $2K_X + F$, where K_X denotes the canonical divisor. The degenerate fibres consist of the (-2)-curves contained in fibres of f and the (-1)-bisections of f. Here a (-1)-bisection of f means a (-1)-curve meeting F at two points. In Proposition 7, through a birational morphism from X to a relatively minimal model of the ruling, we have an explicit description of the Néron-Severi group $\mathrm{NS}(X)$. Then we see that there is a (non-effective) divisor D with $D.F = 1$. In fact, Corollary 8 gives us that D and the irreducible components of the fibres of f generate $\mathrm{NS}(X)$.

In §3, we introduce the theory of the *virtual*

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Mordell-Weil groups. Further, Corollary 8 yields that the virtual Mordell-Weil group of f is trivial. As a result, a section of f is unique if it exists. This contradicts Lemma 6. Therefore f has no sections. This completes the proof of Theorem 1.

2. Construction. In this section we shall construct a smooth projective rational surface together with a relatively minimal fibration of genus two as in Theorem 1. Furthermore, we shall describe singular fibres of the fibration and the Néron-Severi group of the surface, which coincides with the Picard group in our situation.

Put $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$. Denote by (t, x) the inhomogeneous coordinates on Σ_0. Let $pr_n : \Sigma_0 \to \mathbf{P}^1$ be the projection map onto the n-th factor. Put $\Gamma_q = pr_1^{-1}(q)$ and $\Delta_q = pr_2^{-1}(q)$ for any point $q \in \mathbf{P}^1$. Let σ and τ be complex numbers with $\sigma \neq 0, 1$ and $\tau \neq 0, 1/2, 1$. Let $A(\gamma, \tau)$ denote the closure of the zero set of a polynomial $\gamma\tau x^3 - \gamma t x^2 - (3\tau - 2)tx + (2\tau - 1)t^2$ in t, x on $\mathbf{P}^1 \times \mathbf{P}^1$ for $\gamma \in \mathbf{C} \setminus \{0\}$. Put $B = A(1, \tau) + A(\sigma, \tau) + \Gamma_0 + \Gamma_\infty$.

Let $\phi_1 : W_1 \to \Sigma_0$ be the blow-up at two points $(0, 0)$ and (∞, ∞) with the exceptional curves $E_{0,1}$ and $E_{\infty,1}$, i.e., $\phi_1(E_{0,1}) = (0, 0)$ and $\phi_1(E_{\infty,1}) = (\infty, \infty)$. Let $P_{i,2}$ be the intersection point of $E_{i,1}$ and the strict transform to W_1 of Γ_i for $i = 0, \infty$. The strict transform $A_1(\gamma, \tau)$ to W_1 of $A(\gamma, \tau)$ passes through the two points $P_{0,2}$ and $P_{\infty,2}$. The local intersection number at $P_{0,2}$ of $A_1(1, \tau)$ and $A_1(\sigma, \tau)$ is two if and only if $\tau = 2/3$. Next let $\phi_2 : W_2 \to W_1$ be the blow-up at two points $P_{0,2}$ and $P_{\infty,2}$ with $E_{0,2} = \phi_2^{-1}(P_{0,2})$ and $E_{\infty,2} = \phi_2^{-1}(P_{\infty,2})$. The strict transform to W_2 of Δ_0 and that of Δ_∞ are (-1)-curves on W_2. Let $\phi_2' : W_2 \to W_1'$ be the contraction of the two (-1)-curves. The image by ϕ_2' of the strict transform to W_2 of Γ_0 and that of Γ_∞ are (-1)-curves on W_1'. By contracting them, we get another $\mathbf{P}^1 \times \mathbf{P}^1$ and denote it by Σ_0'. Then the images of $E_{0,2}$ and $E_{\infty,2}$ by the other contraction $\phi' : W_2 \to \Sigma_0'$ are two distinct fibres of the projection map $pr_2' : \Sigma_0' \to \mathbf{P}^1$ onto the second factor. Similarly, the images by ϕ' of the strict transforms of $E_{0,1}$ and $E_{\infty,1}$ are two distinct fibres of the projection map $pr_1' : \Sigma_0' \to \mathbf{P}^1$ onto the first factor. Therefore we may assume that the strict transform to W_2 of Γ_i contracts by ϕ' to the point (i, i) for $i = 0, \infty$.

Lemma 2. *Let $\phi' : W_2 \to \Sigma_0'$ be the above contraction. Denote by (v, z) the inhomogeneous coordinates on Σ_0'. Let $\psi : \Sigma_0 \dashrightarrow \Sigma_0'$ be the rational map given by $(v, z) = (t/x^2, t/x)$. Then ψ is the birational map satisfying $\phi' = \psi \circ \phi_1 \circ \phi_2$. Furthermore, the closure of the zero set of a polynomial $\gamma(\tau - z) - (3\tau - 2)v + (2\tau - 1)vz$ in v, z on Σ_0' is isomorphic to the strict transform $A_2(\gamma, \tau)$ to W_2 of $A(\gamma, \tau)$ for $\gamma \in \mathbf{C} \setminus \{0\}$. In particular, $A(\gamma, \tau)$ is irreducible and tangent to $\Gamma_{1/\gamma}$ at the point given by $(t, x) = (1/\gamma, 1/\gamma)$, and B is reduced.*

Proof. We obtain ψ^{-1} by setting $(t, x) = (z^2/v, z/v)$. The indeterminacy of ψ^{-1} consists of $(0, 0), (\infty, \infty)$ and their infinitely near points. Let $A'(\gamma, \tau)$ denote the closure of the zero set of a polynomial $\gamma(\tau - z) - (3\tau - 2)v + (2\tau - 1)vz$ in v, z on Σ_0'. From $\tau \neq 1$, we show that $A'(\gamma, \tau)$ is irreducible. In fact, $A'(\gamma, \tau)$ is tangent to the closure of the zero set of a polynomial $v - \gamma z^2$ in v, z on Σ_0' at $(\gamma, 1)$. Therefore $A(\gamma, \tau)$ is tangent to $\Gamma_{1/\gamma}$ at $(1/\gamma, 1/\gamma)$. It follows from $\tau \neq 0, 1/2$ that $A'(\gamma, \tau)$ does not pass through $(0, 0), (\infty, \infty)$. Thus $A'(\gamma, \tau)$ is isomorphic to $A_2(\gamma, \tau)$ through ϕ'. In particular, $A(\gamma, \tau)$ is also irreducible and B is reduced. □

Notice that B is divisible by two in the Picard group $\mathrm{Pic}(\Sigma_0)$ of Σ_0. Since $\mathrm{Pic}(\Sigma_0)$ is torsion free, there is a unique element $\delta \in \mathrm{Pic}(\Sigma_0)$ with $B \sim 2\delta$, where the symbol $\sim$ means the linear equivalence of divisors. Thus a finite double cover of Σ_0 branched along B is uniquely constructed from B up to isomorphism. We denote it by $\pi : \hat{X} \to \Sigma_0$. Let us resolve singularities of $\hat{X}$ according to Horikawa [5]. In fact, Γ_0 and Γ_∞ are singular fibres of type (V) as in [5, p. 84, Definition] for all σ, τ with $\sigma \neq 0, 1$ and $\tau \neq 0, 1/2, 1$. Furthermore, W_2 coincides with $W^\flat$ as in [5, p. 85, Lemma 7]. In our situation singularities of the induced branch divisor $B^\flat$ are two simple triple points, which correspond to the point $(\infty, (3\tau - 2)/(2\tau - 1))$ and the point $(0, \tau)$ on Σ_0' through ϕ' as in Lemma 2. On W_2, in the corresponding points, $A_2(1, \tau)$ and $A_2(\sigma, \tau)$ meet transversally. One of the two points, which is denoted by $P_{0,3}$, is on the strict transform of $E_{0,1}$. We denote by $P_{\infty,3}$ the other one, which is on the strict transform of $E_{\infty,1}$. Denote by $\phi_3 : W_3 \to W_2$ the blow-up at the two points $P_{0,3}$ and $P_{\infty,3}$ with $E_{0,3} = \phi_3^{-1}(P_{0,3})$ and $E_{\infty,3} = \phi_3^{-1}(P_{\infty,3})$. Let $A_3(\gamma, \tau)$ denote the strict transform to W_3 of $A(\gamma, \tau)$. Remark that $A_3(1, \tau)$ and $A_3(\sigma, \tau)$ are disjoint from each other.

Let $P_{i,4}$ denote the intersection point of $E_{i,3}$ and $A_3(1, \tau)$ and let $P_{i,5}$ be that of $E_{i,3}$ and $A_3(\sigma, \tau)$ for $i = 0, \infty$. Denote by $P_{i,6}$ the intersection point of $E_{i,3}$ and the strict transform to W_3 of $E_{i,1}$. Let ϕ_6 :

$\tilde{W} \to W_3$ be the blow-up at the six points $P_{i,j}$ with $i = 0, \infty$ and $j = 4, 5, 6$. Set $E_{i,j} = \phi_6^{-1}(P_{i,j})$. For $i = 0, \infty$ and $k = 1, 2, 3$, we denote by $\hat{E}_{i,k}$ the strict transform to $\tilde{W}$ of $E_{i,k}$. In the same way, $\hat{\Delta}_i$ and $\hat{\Gamma}_i$ denote respectively that of Δ_i and Γ_i. Let $A_6(\gamma, \tau)$ be the strict transform to $\tilde{W}$ of $A(\gamma, \tau)$. For simplicity, we denote the pull-back to $\tilde{W}$ of them by the same symbols. Thus we have $A_6(1, \tau) + E_{0,4} + E_{\infty,4} \sim A_6(\sigma, \tau) + E_{0,5} + E_{\infty,5} \sim 3\Delta_0 + 2\Gamma_0 - 2E_{0,1} - E_{0,2} - E_{0,3} - 2E_{\infty,1} - E_{\infty,2} - E_{\infty,3}$.

Now, we set

$$\tilde{B} = A_6(1, \tau) + A_6(\sigma, \tau) + \hat{\Gamma}_0 + \hat{\Gamma}_\infty + \hat{E}_{0,1} + \hat{E}_{0,3} + \hat{E}_{\infty,1} + \hat{E}_{\infty,3}.$$

Since $\tilde{B}$ is smooth and divisible by two in $\mathrm{Pic}(\tilde{W})$, we obtain a smooth projective surface $\tilde{X}$ by the finite double cover $\varpi : \tilde{X} \to \tilde{W}$ branched along $\tilde{B}$. Put $\tilde{\phi} = \phi_1 \circ \phi_2 \circ \phi_3 \circ \phi_6$. Then there exists the birational morphism $\mu : \tilde{X} \to \hat{X}$ with $\pi \circ \mu = \tilde{\phi} \circ \varpi$. We call $\tilde{X}$ the canonical resolution of singularities of $\hat{X}$. For simplicity, we put $\tilde{\pi} = \pi \circ \mu$ and $\tilde{f} = pr_1 \circ \tilde{\pi}$.

Let us consider $\tilde{f} : \tilde{X} \to \mathbf{P}^1$. We set $q(\gamma, \tau) = \tau(3\tau - 2)^3/(2\gamma\tau - \gamma)$ for two complex numbers γ and τ with $\gamma \neq 0$ and $\tau \neq 0, 1/2, 1$. If $\tau \neq 2/3$, then $\Gamma_{q(\gamma,\tau)}$ is tangent to $A(\gamma, \tau)$. We know from Lemma 2 that $\Gamma_{1/\gamma}$ does so. By restricting $pr_1 \circ \tilde{\phi} : \tilde{W} \to \mathbf{P}^1$ to $A_6(1, \tau)$ and to $A_6(\sigma, \tau)$, we see that Γ_q meets B transversely at six points for any $q \in \mathbf{P}^1 \setminus \{0, \infty, 1, 1/\sigma, q(1, \tau), q(\sigma, \tau)\}$ from the Riemann-Hurwitz formula. Therefore a general fibre of $\tilde{f} : \tilde{X} \to \mathbf{P}^1$ is a smooth projective curve of genus two. Furthermore, we remark that $q(1, \tau) - 1 = (3\tau - 1)^3(\tau - 1)/(2\tau - 1)$ and Γ_1 meets $A(\sigma, 1/3)$ transversely at three points. Exactly at two points Γ_1 meets B transversely if and only if σ and τ satisfy $q(\sigma, \tau) = 1$. When $q(1, \tau) \neq 1$ and $q(\sigma, \tau) \neq 1$, at four points Γ_1 meets B transversely. Thus Γ_1 meets B transversely at least at two points. In this way, we can check that $\Gamma_{1/\sigma}$, $\Gamma_{q(1,\tau)}$ and $\Gamma_{q(\sigma,\tau)}$ also do so. Hence the reducible fibres of $\tilde{f}$ are $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(\infty)$ only.

Let e_i denote the (-1)-curve on $\tilde{X}$ with $2e_i = \varpi^*\hat{\Gamma}_i$ for $i = 0, \infty$. Although e_i meets $\varpi^*\hat{E}_{i,2}$ at one point, e_i is disjoint from the other components of $\tilde{f}^{-1}(i)$. Additionally, $\varpi^*\hat{E}_{i,2}$ is not a (-2)-curve. Thus, after the contraction $\eta : \tilde{X} \to X$ of e_0 and e_∞, we obtain the relatively minimal model $f : X \to \mathbf{P}^1$ of $\tilde{f} : \tilde{X} \to \mathbf{P}^1$.

Proposition 3. *For two complex numbers σ and τ with $\sigma \neq 0, 1$ and $\tau \neq 0, 1/2, 1$, the fibration $f : X \to \mathbf{P}^1$ obtained as above is a relatively minimal fibration of genus two. The fibre $f^{-1}(\infty)$ is as in* [11, p. 172, 3–$\mathrm{II}^*_{3\text{–}0}$]. *If $\tau \neq 2/3$, then $f^{-1}(0)$ is also as in* [11, p. 172, 3–$\mathrm{II}^*_{3\text{–}0}$]. *However, $f^{-1}(0)$ is as in* [11, p. 155, 1–IV] *if $\tau = 2/3$. The other fibres of f are irreducible and reduced. Furthermore, X is a smooth rational surface with $\rho(X) = 12$.*

Proof. Consider $A_1(1, \tau) + A_1(\sigma, \tau) + E_{0,1} + E_{\infty,1}$ on W_1. For $i = 0, \infty$, the strict transform to W_1 of Γ_i is a (-1)-curve. The contraction of the two (-1)-curves translates singularities of the branch divisor into those as in [11, p. 155, 1–IV] and [11, p. 172, 3–$\mathrm{II}^*_{3\text{–}0}$]. Thus, it is enough to prove that $\tilde{X}$ is a rational surface with $\rho(\tilde{X}) = 14$.

Consider the projection map $pr'_1 : \Sigma'_0 \to \mathbf{P}^1$ onto the first factor. Let Γ'_1 be the pull-back to $\tilde{W}$ of the fibre given by $v = 1$ on Σ'_0. We remark that $\varpi^*\Gamma'_1$ is nef. Let us compute $\varpi^*\Gamma'_1.K_{\tilde{X}}$. Lemma 2 yields $\varpi^*\Gamma'_1 \sim \varpi^*(2\Delta_0 + \Gamma_0 - E_{0,1} - E_{0,2} - E_{\infty,1} - E_{\infty,2})$. We know

$$\tilde{B} \sim 6\Delta_0 + 6\Gamma_0 - 4(E_{0,1} + E_{0,2} + E_{\infty,1} + E_{\infty,2}) - 2\sum_{j=3}^{6}(E_{0,j} + E_{\infty,j}).$$

Hence $K_{\tilde{X}} \sim \varpi^*(K_{\tilde{W}} + \tilde{B}/2) \sim \varpi^*(\Delta_0 + \Gamma_0 - E_{0,1} - E_{0,2} - E_{\infty,1} - E_{\infty,2})$. For all positive integers n, we conclude that the n-th plurigenus of $\tilde{X}$ is zero from $\varpi^*\Gamma'_1.nK_{\tilde{X}} = -2n < 0$. The finite double cover ϖ also gives us $\chi(\tilde{X}) = 2\chi(\tilde{W}) + \tilde{B}.K_{\tilde{W}}/4 + \tilde{B}^2/8 = 1$ (e.g., [1, p. 237]), where $\chi(\tilde{X})$ and $\chi(\tilde{W})$ respectively denote the Euler characteristic of $\tilde{X}$ and $\tilde{W}$. This implies that the irregularity of $\tilde{X}$ is zero. Therefore $\tilde{X}$ is a rational surface by Castelnuovo's rationality criterion. Thus $b_1(\tilde{X}) = b_3(\tilde{X}) = 0$ and $b_2(\tilde{X}) = \rho(\tilde{X})$, where $b_n(\tilde{X})$ denotes the n-th Betti number of $\tilde{X}$. This and Noether's formula provide $\rho(\tilde{X}) = 10 - K_{\tilde{X}}^2$. So $\rho(\tilde{X}) = 14$ follows. □

Corollary 4. *Keep the notation and assumptions as above. For $i = 0, \infty$ and $j = 1, 3$, let $\Theta_{i,j}$ denote the (-2)-curve with $2\eta^*\Theta_{i,j} = \varpi^*\hat{E}_{i,j}$. Let $\Theta_{i,j}$ be the (-2)-curve which is identified with $\varpi^*E_{i,j}$ through η for $i = 0, \infty$ and $j = 4, 5, 6$. Put $\Theta_{i,2} = \eta(\varpi^*\hat{E}_{i,2})$ for $i = 0, \infty$. Then $\Theta_{0,2}$ is a (-3)-curve with $\Theta_{0,2}.\Theta_{0,3} = 1$ if $\tau = 2/3$ and an elliptic curve with $\Theta_{0,2}.\Theta_{0,1} = -\Theta_{0,2}^2 = 1$ otherwise. $\Theta_{\infty,2}$ is an elliptic curve with $\Theta_{\infty,2}.\Theta_{\infty,1} = -\Theta_{\infty,2}^2 = 1$ for every τ. For $i = 0, \infty$, $\Theta_{i,4}.\Theta_{i,3} = \Theta_{i,5}.\Theta_{i,3} = \Theta_{i,3}.\Theta_{i,6} = \Theta_{i,6}.\Theta_{i,1} = 1$. For the other pairs of irreducible components of the fibres, two components are* *disjoint from each other. In particular, the dual graph of the configuration of* $\Theta_{i,4}, \Theta_{i,5}, \Theta_{i,3}, \Theta_{i,6}$ *and* $\Theta_{i,1}$ *has the Dynkin diagram of type* D_5*. Furthermore, the irreducible decompositions of the two reducible fibres of* f *are as follows*: $f^{-1}(0) = 2\Theta_{0,2} + 6\Theta_{0,3} + 3\Theta_{0,4} + 3\Theta_{0,5} + 4\Theta_{0,6} + 2\Theta_{0,1}$ *if* $\tau = 2/3$ *and* $f^{-1}(0) = \Theta_{0,4} + \Theta_{0,5} + 2\Theta_{0,3} + 2\Theta_{0,6} + 2\Theta_{0,1} + 2\Theta_{0,2}$ *otherwise.* $f^{-1}(\infty) = \Theta_{\infty,4} + \Theta_{\infty,5} + 2\Theta_{\infty,3} + 2\Theta_{\infty,6} + 2\Theta_{\infty,1} + 2\Theta_{\infty,2}$ *for every* τ.

Remark 5. Let $f : X \to \mathbf{P}^1$ be a fibration with $\tau = 2/3$ as in Proposition 3. Then a non-existence result for sections can be derived from an easy calculation as follows: Suppose that f admits a section. We can regard it as a horizontal curve D_0 on X such that $D_0.f^{-1}(q) = 1$ for all $q \in \mathbf{P}^1$. However, $D_0.f^{-1}(0) \geq 2$ since multiplicities of the irreducible components of $f^{-1}(0)$ are at least two. We obtain a contradiction.

In order to prove that f has no sections for every τ, we need the following

Lemma 6. *Let* $f : X \to \mathbf{P}^1$ *be a fibration as in Proposition* 3*. If there exists a section of* f*, then* f *has at least two sections.*

Proof. Suppose that there exists a section of f, which can be regarded as a horizontal curve D_1 on X. Let $\tilde{D}_1$ be the strict transform by η of D_1. Then the image C by $\tilde{\pi}$ of $\tilde{D}_1$ is a section of $pr_1 : \Sigma_0 \to \mathbf{P}^1$. Here we recall the irreducible decomposition of B. In particular, B does not contain C as an irreducible component. Hence we obtain another section $\tilde{D}_2$ of $\tilde{f}$ as the other component $(\tilde{\pi}^*C - \tilde{D}_1)$ from the irreducible decomposition of $\tilde{\pi}^*C$. So we have another section of f as the image by η of $\tilde{D}_2$. □

To describe irreducible components of fibres of $f : X \to \mathbf{P}^1$ in $\mathrm{NS}(X)$, through a birational morphism $X \to \mathbf{P}^2$, we interpret $f : X \to \mathbf{P}^1$ as a pencil generated by plane curves of degree seven which have one triple point and ten double points at eleven base points.

Proposition 7. *Let* $f : X \to \mathbf{P}^1$ *be a fibration as in Proposition* 3*,* F *a general fibre of* f *and* $\Theta_{i,j}$ *the irreducible components of the reducible fibres as in Corollary* 4*. Then there exists a birational morphism* $\nu : X \to \mathbf{P}^2$ *such that the pull-back to* X *of a line* ℓ *on* $\mathbf{P}^2$ *and that of* (-1)*-curves* $e_h, h = 1, 2, \ldots, 11$ *contracted by* ν *satisfy the following*

(*a*) $F = 7\ell - 3e_1 - 2\sum_{h=2}^{11} e_h$, $\Theta_{0,4} = \ell - e_1 - e_8 - e_{10}$, $\Theta_{\infty,4} = \ell - e_1 - e_9 - e_{11}$,

$$\Theta_{0,1} = e_4 - e_2, \quad \Theta_{\infty,1} = e_5 - e_3,$$
$$\Theta_{0,6} = e_6 - e_4, \quad \Theta_{\infty,6} = e_7 - e_5,$$
$$\Theta_{0,3} = e_8 - e_6, \quad \Theta_{\infty,3} = e_9 - e_7,$$
$$\Theta_{0,5} = e_{10} - e_8, \quad \Theta_{\infty,5} = e_{11} - e_9$$

and $\Theta_{\infty,2} = -K_X + e_3$ *for every* τ.

(*b*) $\Theta_{0,2} = 2\ell - e_3 - e_5 - e_7 - e_8 - e_9 - e_{10} - e_{11}$ *if* $\tau = 2/3$ *and* $\Theta_{0,2} = -K_X + e_2$ *otherwise.*

Proof. Let e_1 denote the (-1)-curve with $2\eta^*e_1 = \varpi^*A_6(1, \tau)$. Since $\Gamma_1.A_6(1, \tau) = 3$ on $\tilde{W}$, we have $F.e_1 = 3$. The images by η of $\varpi^*\hat{\Delta}_0$ and $\varpi^*\hat{\Delta}_\infty$ are two disjoint (-1)-curves on X, which meet F at two points. Put $e_2 = \eta(\varpi^*\hat{\Delta}_0)$ and $e_3 = \eta(\varpi^*\hat{\Delta}_\infty)$. We also remark that $e_2.e_1 = e_3.e_1 = 0$, since $\hat{\Delta}_0.A_6(1, \tau) = \hat{\Delta}_\infty.A_6(1, \tau) = 0$.

We know $\varpi^*\Gamma_1'.K_{\tilde{X}} = -2$ in the proof of Proposition 3. By the adjunction formula we verify that $pr_1' : \Sigma_0' \to \mathbf{P}^1$ induces a ruling $\tilde{\xi} : \tilde{X} \to \mathbf{P}^1$ through $\psi \circ \tilde{\pi} : \tilde{X} \to \Sigma_0'$. The degenerate fibres of $\tilde{\xi}$ are $\tilde{\xi}^{-1}(0)$ and $\tilde{\xi}^{-1}(\infty)$ only, since the fibres of pr_1' except for $pr_1'^{-1}(0)$ and $pr_1'^{-1}(\infty)$ do not pass through the intersection points of $A'(1, \tau)$ and $A'(\sigma, \tau)$. Recall the definition of ψ in Lemma 2. Then $\tilde{\xi}^{-1}(i)$ contains e_i for each $i = 0, \infty$ as an irreducible component. Hence there is a unique ruling $\xi : X \to \mathbf{P}^1$ such that $\tilde{\xi} = \xi \circ \eta$. Furthermore, $\xi^{-1}(\infty)$ consists of $\Theta_{0,j}, j = 1, 3, 4, 5, 6$ and e_2 whose configuration is as in [8, Figure 2]. In a similar configuration as above, $\xi^{-1}(0)$ consists of $\Theta_{\infty,j}, j = 1, 3, 4, 5, 6$ and e_3.

Notice that $e_1.\Theta_{0,4} = e_1.\Theta_{\infty,4} = 1$ and the other components are disjoint from e_1. Let $\nu_1 : X \to \Sigma_1$ be the birational morphism contracting e_2, e_3 and eight (-2)-curves $\Theta_{i,1}, \Theta_{i,6}, \Theta_{i,3}, \Theta_{i,5}$ with $i = 0, \infty$, which are the strict transforms of eight (-1)-curves. Here Σ_1 is a relatively minimal model of $\xi : X \to \mathbf{P}^1$. We remark that the image of e_1 is the minimal section of Σ_1. Let $\nu_0 : \Sigma_1 \to \mathbf{P}^2$ be the contraction and put $\nu = \nu_0 \circ \nu_1$.

From the configuration of e_2, e_3 and eight (-2)-curves $\Theta_{i,1}, \Theta_{i,6}, \Theta_{i,3}, \Theta_{i,5}$ with $i = 0, \infty$, we can denote the pull-backs to X of the eight (-1)-curves by $e_h, h = 4, 5, \ldots, 11$ so that $e_4 = \Theta_{0,1} + e_2, e_5 = \Theta_{\infty,1} + e_3, e_6 = \Theta_{0,6} + e_4, e_7 = \Theta_{\infty,6} + e_5, e_8 = \Theta_{0,3} + e_6, e_9 = \Theta_{\infty,3} + e_7, e_{10} = \Theta_{0,5} + e_8$ and $e_{11} = \Theta_{\infty,5} + e_9$. We recall that the images by ν_1 of $\Theta_{0,4}$ and $\Theta_{\infty,4}$ are fibres of Σ_1. It follows from the configurations of $\xi^{-1}(0)$ and $\xi^{-1}(\infty)$ that $\Theta_{0,4} = \ell - e_1 - e_8 - e_{10}$ and $\Theta_{\infty,4} = \ell - e_1 - e_9 - e_{11}$, where ℓ denotes the pull-back to X of a line on $\mathbf{P}^2$.

Next we recall that $\varpi^*\hat{\Delta}_0 = \eta^* e_2 - e_\infty$ and $K_{\tilde{X}} \sim \varpi^*(K_{\tilde{W}} + \tilde{B}/2) \sim \varpi^*\hat{\Delta}_0 - \varpi^* E_{0,2} + \varpi^*\hat{\Gamma}_\infty$. Hence we have $\varpi^* E_{0,2} \sim -K_{\tilde{X}} + \eta^* e_2 + e_\infty \sim \eta^*(-K_X + e_2) - e_0$. Therefore we get $\Theta_{0,2} = -K_X + e_2$ from $\hat{E}_{0,2} = E_{0,2}$ when $\tau \neq 2/3$. In the same way, we see $\Theta_{\infty,2} = -K_X + e_3$ for every τ. Furthermore, we show $F \sim f^{-1}(\infty) = 2\Theta_{\infty,2} + 2\Theta_{\infty,1} + 2\Theta_{\infty,6} + 2\Theta_{\infty,3} + \Theta_{\infty,4} + \Theta_{\infty,5} \sim 7\ell - 3e_1 - 2\sum_{h=2}^{11} e_h$.

To complete the proof, it is enough to describe $\Theta_{0,2}$ by the $\mathbf{Z}$-linear combinations of ℓ and e_h's when $\tau = 2/3$. The description follows immediately from the others as in the condition (a) and the irreducible decomposition of $f^{-1}(0)$ as in Corollary 4. □

Corollary 8. *Put $D = e_1 - e_{11}$. Then*

$$\mathrm{NS}(X) \simeq \mathbf{Z}D \oplus \bigoplus_{j=1}^{6} \mathbf{Z}\Theta_{0,j} \oplus \bigoplus_{j=1}^{4} \mathbf{Z}\Theta_{\infty,j} \oplus \mathbf{Z}\Theta_{\infty,6}.$$

Proof. It is well-known that $\mathrm{NS}(X) \simeq \mathbf{Z}\ell \oplus \bigoplus_{h=1}^{11} \mathbf{Z}e_h$. Thus we only have to represent ℓ and e_h's as $\mathbf{Z}$-linear combinations of D and $\Theta_{i,j}$'s from Proposition 7. Suppose $\tau \neq 2/3$. Then D and the irreducible components of the reducible fibres except for $\Theta_{\infty,5}$ generate ℓ and e_1 as follows:

$$\begin{aligned}
\ell &= 7D + 16\Theta_{0,1} + 13\Theta_{0,2} + 22\Theta_{0,3} + 14\Theta_{0,4} \\
&\quad + 11\Theta_{0,5} + 19\Theta_{0,6} - 13\Theta_{\infty,1} - 16\Theta_{\infty,2} \\
&\quad - 7\Theta_{\infty,3} - 4\Theta_{\infty,4} - 10\Theta_{\infty,6}, \\
e_1 &= 3D + 6\Theta_{0,1} + 5\Theta_{0,2} + 8\Theta_{0,3} + 5\Theta_{0,4} + 4\Theta_{0,5} \\
&\quad + 7\Theta_{0,6} - 5\Theta_{\infty,1} - 6\Theta_{\infty,2} - 3\Theta_{\infty,3} - 2\Theta_{\infty,4} \\
&\quad - 4\Theta_{\infty,6}.
\end{aligned}$$

Furthermore, Proposition 7 immediately yields that

$$\begin{aligned}
&e_{11} = e_1 - D,\ e_9 = \ell - e_1 - e_{11} - \Theta_{\infty,4}, \\
&e_7 = e_9 - \Theta_{\infty,3},\ e_5 = e_7 - \Theta_{\infty,6},\ e_3 = e_5 - \Theta_{\infty,1}, \\
&e_2 = e_3 + \Theta_{0,2} - \Theta_{\infty,2},\ e_4 = e_2 + \Theta_{0,1}, \\
&e_6 = e_4 + \Theta_{0,6},\ e_8 = e_6 + \Theta_{0,3},\ e_{10} = e_8 + \Theta_{0,5}.
\end{aligned}$$

Hence, we see that e_h, $h = 2, 3, \ldots, 11$ can be the $\mathbf{Z}$-linear combinations inductively.

In the case of $\tau = 2/3$, we only have to replace $\varpi^* E_{0,2} = \Theta_{0,2}$, which holds if $\tau \neq 2/3$, with $\varpi^* E_{0,2} = \Theta_{0,2} + 2\Theta_{0,3} + \Theta_{0,4} + \Theta_{0,5} + \Theta_{0,6}$. Therefore D, $\Theta_{0,1}, \Theta_{0,2}, \ldots, \Theta_{0,6}$, $\Theta_{\infty,1}, \Theta_{\infty,2}, \Theta_{\infty,3}, \Theta_{\infty,4}$ and $\Theta_{\infty,6}$ form $\mathbf{Z}$-basis of $\mathrm{NS}(X)$ for every τ. □

3. Virtual Mordell-Weil groups. We shall introduce the theory of the virtual Mordell-Weil groups, which is used to complete the proof of Theorem 1. Let S be a smooth projective rational surface and $\varphi : S \to \mathbf{P}^1$ a relatively minimal fibration whose general fibre F is a projective curve of genus $g \geq 2$. We denote by n_q the number of irreducible components of $\varphi^{-1}(q)$ for any $q \in \mathbf{P}^1$. In [12, Definition 0.2], the *virtual* Mordell-Weil rank r of φ, *which does not necessarily admit a section*, is defined as $r = \rho(S) - 2 - \sum_{q \in \mathbf{P}^1}(n_q - 1)$. For example, when $\varphi : S \to \mathbf{P}^1$ is a fibration as in Proposition 3, we know that $\rho(S) = 12$ and $n_0 = n_\infty = 6$ with $n_q = 1$ for all $q \in \mathbf{P}^1 \setminus \{0, \infty\}$, which lead to $r = 0$.

Let V be the subgroup of $\mathrm{NS}(S)$ generated by the irreducible components of the fibres of φ. The primitive closure $\hat{V}$ is defined as $\hat{V} = V \otimes \mathbf{Q} \cap \mathrm{NS}(S)$. Let us observe that $\hat{V}/V$ can be regarded as the *virtual* Mordell-Weil group of φ with $r = 0$. Assume that there exists on S a divisor D with $D.F = 1$. When φ admits a section, we can regard it as the above D. Set $T_D = \mathbf{Z}D \oplus V \subset \mathrm{NS}(S)$. The primitive closure $\hat{T}_D$ is defined as $\hat{T}_D = T_D \otimes \mathbf{Q} \cap \mathrm{NS}(S)$. Then, for all D with $D.F = 1$, the quotient groups of $\hat{T}_D$ by T_D are isomorphic to each other and to $\hat{V}/V$ as follows:

Lemma 9. *The natural projection $T_D \to V$ induces an isomorphism $\hat{T}_D/T_D \simeq \hat{V}/V$ naturally.*

Proof. Let $\Theta_{q,j}$ be an irreducible component of a reducible fibre $\varphi^{-1}(q)$ for $j = 1, 2, \ldots, n_q$. Take any $E \in \hat{T}_D$. We have $E = \alpha D + \sum \beta_{q,j}\Theta_{q,j}$ for some $\alpha, \beta_{q,j} \in \mathbf{Q}$ by definition. In fact, $\alpha = E.F$ must be an integer. In particular, $\sum \beta_{q,j}\Theta_{q,j} = E - \alpha D \in \mathrm{NS}(S)$. Therefore, $\sum \beta_{q,j}\Theta_{q,j} \in \hat{V}$. In this way, while $D.F = 1$, the natural projection $\hat{T}_D \to \hat{V}$ of the primitive closures is well-defined. Consider the composition $\hat{T}_D \to \hat{V} \to \hat{V}/V$, which is a surjective homomorphism of groups. Its kernel is equal to T_D. So $\hat{T}_D/T_D \simeq \hat{V}/V$ follows. □

Via φ, we can regard S as a smooth projective curve of genus g defined over the rational function field $\mathbf{K} = \mathbf{C}(\mathbf{P}^1)$. We assume that it has a $\mathbf{K}$-rational point O. Let $\mathcal{J}_\varphi/\mathbf{K}$ be the Jacobian variety of the generic fibre $\mathcal{F}/\mathbf{K}$ of φ. The Mordell-Weil group of φ is the group of $\mathbf{K}$-rational points $\mathcal{J}_\varphi(\mathbf{K})$. It is a finitely generated abelian group. The rank of the group is equal to r. There is a natural one-to-one correspondence between the set of $\mathbf{K}$-rational points $\mathcal{F}(\mathbf{K})$ and the set of sections of φ. For $P \in \mathcal{F}(\mathbf{K})$, we denote by (P) the section corresponding to P which is regarded as a horizontal curve on S. In particular, (O) corresponding to the origin O of $\mathcal{J}_\varphi(\mathbf{K})$ is called the *zero section*. In [15, Theorem 3], we have the natural isomorphism of groups $\mathcal{J}_\varphi(\mathbf{K}) \simeq \mathrm{NS}(S)/T_{(O)}$. By definition, the torsion part $\mathcal{J}_\varphi(\mathbf{K})_{\mathrm{tor}}$ is isomorphic to $\hat{T}_{(O)}/T_{(O)}$. From Lemma 9, we conclude $\mathcal{J}_\varphi(\mathbf{K})_{\mathrm{tor}} \simeq \hat{V}/V$.

Let $f : X \to \mathbf{P}^1$ be a fibration as in Proposition 3. We recall $r = 0$. Furthermore, if f admits a section, then we see that the Mordell-Weil group of f is trivial from Corollary 8 and Lemma 9. Hence, we have the following

Lemma 10. *Let $f : X \to \mathbf{P}^1$ be a fibration as in Proposition* 3. *Suppose that f admits a section. Then it is the unique section of f.*

Proof of Theorem 1. Let $f : X \to \mathbf{P}^1$ be a fibration as in Proposition 3. We only have to show that f has no sections. The assertion follows from Lemmas 6 and 10. □

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