An algebraic proof of determinant formulas of Grothendieck polynomials

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Abstract: We give an algebraic proof of the determinant formulas for factorial Grothendieck polynomials obtained by Hudson–Ikeda–Matsumura–Naruse in [6] and by Hudson–Matsumura in [7].

Key words: Symmetric polynomials; Grothendieck polynomials; K-theory; Grassmannians; Schubert varieties.

1. Definition and Theorems. In [12] and [14], Lascoux and Schützenberger introduced (*double*) *Grothendieck polynomials* indexed by permutations as representatives of K-theory classes of structure sheaves of Schubert varieties in a full flag variety. In [4] and [5], Fomin and Kirillov introduced β-Grothendieck polynomials in the framework of Yang-Baxter equations together with their combinatorial formula and showed that they coincide with the ones defined by Lascoux and Schützenberger with the specialization $\beta = -1$. Let $x = (x_1, \ldots, x_d)$, $b = (b_1, b_2, \ldots)$ be sets of indetermiants. A *Grassmannian permutation* with descent at d corresponds to a *partition* λ of length at most d, *i.e.* a sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_d)$ such that $\lambda_i \geq \lambda_{i+1}$ for each $i = 1, \ldots, d-1$. For such permutation, Buch [3] gave a combinatorial expression of the corresponding Grothendieck polynomial $G_\lambda(x)$ as a generating series of *set-valued tableaux*, a generalization of semi-standard Young tableaux by allowing a filling of a box in the Young diagram to be a set of integers. In [18], McNamara gave an expression of *factorial* (double β-) Grothendieck polynomials $G_\lambda(x|b)$ also in terms of set-valued tableaux.

In this paper, we prove the following Jacobi–Trudi type determinant formulas for $G_\lambda(x|b)$. For each non-negative integer k and an integer m, let $G_m^{(k)}(x|b)$ be a function of x and b given by

$$G^{(k)}(u) := \sum_{m\in\mathbf{Z}} G_m^{(k)}(x|b)u^m$$

$$:= \frac{1}{1+\beta u^{-1}} \prod_{i=1}^{d} \frac{1+\beta x_i}{1-x_i u} \prod_{j=1}^{k} (1+(u+\beta)b_j),$$

where β is a formal variable of degree -1 and $\dfrac{1}{1+\beta u^{-1}}$ is expanded as $\sum_{s\geq 0}(-1)^s\beta^s u^{-s}$. We use the generalized binomial coefficients $\binom{n}{i}$ given by $(1+x)^n = \sum_{i\geq 0}\binom{n}{i}x^i$ for $n \in \mathbf{Z}$ with the convention that $\binom{n}{i} = 0$ for all integers $i < 0$.

Theorem 1.1. *For each partition λ of length at most d, we have*

$$G_\lambda(x|b) = \det\left(\sum_{s\geq 0}\binom{i-d}{s}\beta^s G^{(\lambda_i+d-i)}_{\lambda_i+j-i+s}(x|b)\right)_{1\leq i,j\leq d}.$$

Theorem 1.2. *We have*

$$G_\lambda(x|b) = \det\left(\sum_{s\geq 0}\binom{i-j}{s}\beta^s G^{(\lambda_i+d-i)}_{\lambda_i+j-i+s}(x|b)\right)_{1\leq i,j\leq d}.$$

In particular, we have

$$G_{(k,0,\ldots,0)}(x|b) = G_k^{(k+d-1)}(x|b).$$

Theorems 1.1 and 1.2 were originally obtained in the context of degeneracy loci formulas for flag bundles by Hudson–Matsumura in [7] and Hudson–Ikeda–Matsumura–Naruse in [6] respectively. The proof in this paper is purely algebraic, generalizing Macdonald's argument in [16, (3.6)] for Jacobi–Trudi formula of Schur polynomials. It is based on the following "bi-alternant" formula of $G_\lambda(x|b)$ described by Ikeda–Naruse in [8]:

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$$(1)\quad G_\lambda(x|b) = \frac{\det\left([x_j|b]^{\lambda_i+d-i}(1+\beta x_j)^{i-1}\right)_{1\le i,j\le d}}{\prod_{1\le i<j\le d}(x_i-x_j)}.$$

Here we denote $x\oplus y := x+y+\beta xy$ and $[y|b]^k := (y\oplus b_1)\cdots(y\oplus b_k)$ for any variable x,y. Note that the Grothendieck polynomial $G_\lambda(x)$ given in [3] coincides with $G_\lambda(x|b)$ by setting $\beta=-1$ and $b_i=0$.

Determinant formulas different from the ones in Theorems 1.1 and 1.2 have been also obtained by Lenart in [15] (*cf.* [2], [13]), by Kirillov in [10] and [11], and by Yeliussizov [22]. Each entry of these previously known determinant formulas is given as a finite linear combination of elementary/complete symmetric polynomials, while in our formula it is given as a possibly infinite linear combination of Grothendieck polynomials associated to one row partitions. A combinatorial proof of Theorem 1.2 has been also obtained in [17] for the non-factorial case, as well as an analogous determinant formula for skew flagged Grothendieck polynomials, special cases of which arise as the Grothendieck polynomials associated to 321-avoiding permutations [1] and vexillary permutations.

It is also worth mentioning that in [3] Buch obtained the Littlewood-Richardson rule for the structure constants of Grothendieck polynomials $G_\lambda(x)$, and hence the Schubert structure constants of the K-theory of Grassmannians (see also the paper [9] by Ikeda-Shimazaki for another proof). For the equivariant K-theory of Grassmannians (or equivalently for $G_\lambda(x|b)$), the structure constants were determined by Pechenik and Yong in [20] by introducing a new combinatorial object called *genomic tableaux*. Motegi–Sakai [19] identified Grothendieck polynomials with the wave functions arising in the five vertex models and obtained a variant of the Cauchy identity. Using this framework of integrable systems, Wheeler–Zinn-Justin [21] recently obtained another equivariant Littlewood-Richardson rule for factorial Grothendieck polynomials.

2. Proof of Theorem 1.1. By (1), it suffices to show the identity

$$\frac{\det\left([x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-1}\right)_{1\le i,j\le d}}{\prod_{1\le i<j\le d}(x_i-x_j)} = \det\left(\sum_{s\ge 0}\binom{i-d}{s}\beta^s G^{(a_i+d-i)}_{a_i+j-i+s}(x|b)\right)_{1\le i,j\le d},$$

for each $(a_1,\ldots,a_d)\in\mathbf{Z}^d$ such that $a_i+d-i\ge 0$. For each $j=1,\ldots,d$, we let

$$E^{(j)}(u) := \sum_{p=0}^{d-1} e_p^{(j)}(x)u^p := \prod_{\substack{1\le i\le d\\ i\ne j}}(1+x_iu).$$

We denote $\bar{y} := \dfrac{-y}{1+\beta y}$. Since $1+(u+\beta)y = \dfrac{1-\bar{y}u}{1+\beta\bar{y}}$, we have

$$G^{(k)}(u) = \frac{1}{1+\beta u^{-1}}\prod_{i=1}^d\frac{1+\beta x_i}{1-x_iu}\prod_{\ell=1}^k\frac{1-\bar{b}_\ell u}{1+\beta\bar{b}_\ell}.$$

Consider the identity

$$\begin{aligned}&G^{(k)}(u)E^{(j)}(-u)\\ &= \frac{1}{1+\beta u^{-1}}\frac{1}{1-x_ju}\prod_{i=1}^d(1+\beta x_i)\prod_{\ell=1}^k\frac{1-\bar{b}_\ell u}{1+\beta\bar{b}_\ell}.\end{aligned}$$

By comparing the coefficient of $u^m, m\ge k$ in (2) we obtain

$$\begin{aligned}&\sum_{p=0}^{d-1}G^{(k)}_{m-p}(x|b)(-1)^p e_p^{(j)}(x)\\ &= x_j^{m-k}\frac{\prod_{\ell=1}^k(x_j-\bar{b}_\ell)}{\prod_{\ell=1}^k(1+\beta\bar{b}_\ell)}\prod_{\substack{1\le i\le d\\ i\ne j}}(1+\beta x_i).\end{aligned}$$

Since $\dfrac{y-\bar{b}}{1+\beta\bar{b}} = y\oplus b$, we have

$$\begin{aligned}(2)\quad &\sum_{p=0}^{d-1}G^{(k)}_{m-p}(x|b)(-1)^p e_p^{(j)}(x)\\ &= x_j^{m-k}[x_j|b]^k\prod_{\substack{1\le i\le d\\ i\ne j}}(1+\beta x_i),\quad (m\ge k).\end{aligned}$$

Consider the matrices

$$H := \left(\sum_{s\ge 0}\binom{i-d}{s}\beta^s G^{(a_i+d-i)}_{a_i+j-i+s}(x|b)\right)_{1\le i,j\le d}$$

and

$$M := \left((-1)^{d-i}e^{(j)}_{d-i}(x)\right)_{1\le i,j\le d}.$$

By using (2), we find that the (i,j)-entry of HM is

$$(HM)_{ij} = [x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-d-1}\prod_{1\le t\le d}(1+\beta x_t).$$

By taking the determinant of HM, the factor $\prod_{1\le j\le d}(1+\beta x_j)^{-d}\prod_{1\le t\le d}(1+\beta x_t)^d$ which turns to be 1 comes out, and therefore we obtain

$$\det H\det M = \det\left([x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-1}\right)_{1\le i,j\le d}.$$

By dividing by $\det M$, we obtain the desired identity since $\det M = \prod_{1\le i<j\le d}(x_i-x_j)$ (see

[16, p. 42]). □

3. Proof of Theorem 1.2. By (1), it suffices to show the identity

$$\frac{\det\left([x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-1}\right)_{1\le i,j\le d}}{\prod_{1\le i<j\le d}(x_i-x_j)} = \det\left(\sum_{s\ge 0}\binom{i-j}{s}\beta^s G^{(a_i+d-i)}_{a_i+j-i+s}(x|b)\right)_{1\le i,j\le d}$$

for each $(a_1,\dots,a_d)\in \mathbf{Z}^d$ such that $a_i+d-i\ge 0$. For each $j=1,\dots,d$, let

$$\overline{E}^{(j)}(u) := \sum_{p=0}^{d-1} e_p^{(j)}(-\bar{x})u^p := \prod_{\substack{1\le i\le d\\ i\neq j}}(1-\bar{x}_i u).$$

Since $1+(u+\beta)y = \dfrac{1-\bar{y}u}{1+\beta\bar{y}}$, we have the identity

$$\text{(3)}\qquad G^{(k)}(u)\overline{E}^{(j)}(-u-\beta) = \frac{1}{1+\beta u^{-1}}\frac{1+\beta x_j}{1-x_j u}\prod_{1\le \ell\le k}\frac{1-\bar{b}_\ell u}{1+\beta\bar{b}_\ell}.$$

By comparing the coefficient of $u^m, m\ge k$ in (3) we obtain

$$\text{(4)}\qquad \sum_{p=0}^{d-1}\sum_{s=0}^{p}\binom{p}{s}\beta^s G^{(k)}_{m-p+s}(x|b)(-1)^p e_p^{(j)}(-\bar{x}) = x_j^{m-k}\prod_{1\le\ell\le k}\frac{x_j-\bar{b}_\ell}{1+\beta\bar{b}_\ell} = x_j^{m-k}[x_j|b]^k$$

where the last equality follows from the identity $\dfrac{x-\bar{y}}{1+\beta\bar{y}} = x\oplus y$ for any variable x, y.

Consider the matrices

$$H' := \left(\sum_{s\ge 0}\binom{i-j}{s}\beta^s G^{(a_i+d-i)}_{a_i+j-i+s}(x|b)\right)_{1\le i,j\le d}$$

and

$$\overline{M} := \left((-1)^{d-i}e_{d-i}^{(j)}(-\bar{x})\right)_{1\le i,j\le d}.$$

We write the (i,j)-entry of the product $H'\overline{M}$ as

$$(H'\overline{M})_{ij} = \sum_{p=0}^{d-1}\sum_{s\ge 0}\binom{i-d+p}{s}\beta^s \times G^{(a_i+d-i)}_{a_i+d-i+s-p}(x|b)(-1)^p e_p^{(j)}(-\bar{x}).$$

By writing $\binom{i-d+p}{s} = \sum_{\ell\ge 0}\binom{i-d}{\ell}\binom{p}{s-\ell}$ using a well-known identity of binomial coefficients and then applying (4), we obtain

$$(H'\overline{M})_{ij} = [x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-1}(1+\beta x_j)^{1-d}.$$

By taking the determinant of $H'\overline{M}$, we have

$$\det H'\det\overline{M} = \prod_{1\le j\le d}(1+\beta x_j)^{1-d} \times \det\left([x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-1}\right)_{1\le i,j\le d}.$$

Since we have (see [16, p. 42])

$$\det\overline{M} = \prod_{1\le i<j\le d}(\bar{x}_j-\bar{x}_i) = \prod_{1\le i<j\le d}\frac{x_i-x_j}{(1+\beta x_i)(1+\beta x_j)} = \prod_{1\le i\le d}(1+\beta x_i)^{1-d}\prod_{1\le i<j\le d}(x_i-x_j),$$

we obtain the desired identity. □

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