Borel structures coming from various topologies on B($\mathcal{H}$)

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Abstract: Although there exist different types of (well-known) locally convex topologies on $\mathbf{B}(\mathcal{H})$, the notion of measurability on the set of operator valued functions $f:\Omega \to \mathbf{B}(\mathcal{H})$ is unique when $\mathcal{H}$ is separable (see [1]). In this current discussion we observe that unlike the separable case, in the non-separable case we have to face different types of measurability. Moreover the algebraic operations "*addition and product*" are not compatible with the set of operator valued measurable functions.

Key words: von Neumann algebras; operator valued functions; σ-algebras; measurability.

Introduction. Let $\mathcal{H}$ be a Hilbert space and $\mathbf{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. In the literature, there are some well-known locally convex topologies on $\mathbf{B}(\mathcal{H})$ which are given in the following diagram. For definitions and details see [3].

$$\begin{array}{lccccc} \text{(0.1) Arens-Mackey} \supset & \sigma\text{-strong}^* & \supset & \sigma\text{-strong} & \supset & \sigma\text{-weak} \\ & \bigcup & & \bigcup & & \bigcup \\ & \text{strong}^* & \supset & \text{strong} & \supset & \text{weak} \end{array}$$

where $\supset$ means that the right-hand side is coarser than the left-hand side. In the sequel, the notation τ runs over these seven topologies. We just recall the Arens-Mackey topology which is less classical than the six other ones. This locally convex topology is given as the uniform convergence topology on $\sigma(\mathbf{B}(\mathcal{H})_*, \mathbf{B}(\mathcal{H}))$-compact convex subsets of $\mathbf{B}(\mathcal{H})_*$. The following two questions will be our concerns:

1) Determine the diagram of the sigma algebras generated by these seven topologies.

It is important to answer this question because, to any sigma algebra on $\mathbf{B}(\mathcal{H})$, a particular type of operator valued measurable functions is corresponded. Let Ω be a measurable space. Naturally, we say that an operator valued function $f:\Omega \to \mathbf{B}(\mathcal{H})$ is τ-measurable provided that $f^{-1}(O)$ is measurable for any open set O in the topology τ. In the classical case (when $\dim\mathcal{H} = 1$), the set of complex valued measurable functions forms a complex $*$-algebra which is also closed under the point-wise limit. In the general case, compatibility of these two algebraic and topological structures on the set of operator valued τ-measurable functions will be our second challenge. Indeed the major point is the τ-measurability of the product of two operator valued τ-measurable functions.

2) Is the product of two τ-measurable functions τ-measurable as well?

Separable case. When $\mathcal{H}$ is a separable Hilbert space, these two questions have been fully studied in [1]. There, it was proved that the σ-algebra generated by all these seven topologies coincide. Also, the set of all measurable functions $f:\Omega \to \mathbf{B}(\mathcal{H})$ forms an $*$-algebra and is closed under the point-wise product.

Here by an expository discussion, we follow these two problems when $\mathcal{H}$ is a non-separable Hilbert space. We will find that:

- In spite of the separable case, two different types of sigma algebras are implemented by the diagram of topologies (0.1).
- Neither addition nor product operations is compatible with the set of operator valued τ-measurable functions $f:\Omega \to \mathbf{B}(\mathcal{H})$.

1. Diagram of sigma algebras generated by topologies. We first emphasize that throughout this discussion, $\mathcal{H}$ is a non-separable Hilbert space and $\mathcal{E} = \{e_i\}_{i\in I}$ is fixed as an orthonormal basis for $\mathcal{H}$. We also denote $\mathrm{Fin}(\mathcal{E})$, by the family of all finite subsets of $\mathcal{E}$. Authors believe that, a complete and exact analysis concerning relations between sigma algebras generated by topologies τ's, because no attempt on this subject has been done up to now, is much more difficult than the separable case.

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Let us denote $\mathcal{M}_\tau$, by the sigma algebra generated by τ. The elements of $\mathcal{M}_\tau$ are also called τ-measurable sets. By a simple argument, we first observe that the diagram of the sigma algebras $\mathcal{M}_\tau$'s is reduced to

$$\mathcal{M}_{s^*} \supseteq \mathcal{M}_s \supseteq \mathcal{M}_w. \tag{1.1}$$

To see this, it is enough to combine the following two facts:

- On bounded parts of $\mathbf{B}(\mathcal{H})$, the topologies given in each column of the diagram (0.1) are the same. The same result holds for the Arens-Mackey topology and the σ-strong* topology ([3], Theorem III.5.7).
- The norm closed ball $\mathbf{B}(\mathcal{H})_{\|\cdot\|\leq n}$, centered at 0 with radius n, is closed under topologies τ's. Any arbitrary subset E in $\mathbf{B}(\mathcal{H})$ can be represented by
$$E = \bigcup_1^\infty (E \cap \mathbf{B}(\mathcal{H})_{\|\cdot\|\leq n}).$$

One may now conclude that $\mathcal{M}_{\sigma\text{-}w} = \mathcal{M}_w$, $\mathcal{M}_{\sigma\text{-}s} = \mathcal{M}_s$ and $\mathcal{M}_{s^*} = \mathcal{M}_{\sigma\text{-}s^*} = \mathcal{M}_{a.m}$ (where $\mathcal{M}_{a.m}$ is the sigma algebra generated by the Arens-Mackey topology).

Let us denote $\mathcal{B}_\tau$, by the family of all sub-basic neighborhoods that determine the topology τ. We also denote $\mathcal{B}_\tau$, by the sigma algebra generated by $\mathcal{B}_\tau$. To have a complete evaluation, we have to determine whether the relations between sigma algebras given in the diagram (1.1) are proper or not. We recall in the separable case, the key point which makes the subject move forward well is $\mathcal{M}_\tau = \mathcal{B}_\tau$ (see [1]). We will show that it is no longer valid in the non-separable case and based on this point, some new (complicated) challenges are raised. In the rest of this discussion, we will observe that:

(1) The sigma algebra $\mathcal{B}_\tau$ is properly contained in $\mathcal{M}_\tau$, which makes us face two diagrams $\mathcal{M}_\tau$'s and $\mathcal{B}_\tau$'s.
(2) We first simplify the diagram of relations between $\mathcal{B}_\tau$'s and then show that all $\mathcal{B}_\tau$'s takes place in $\mathcal{M}_w$, the smallest sigma algebra between $\mathcal{M}_\tau$'s.
(3) By some calculations, we will find the diagram of the sigma algebra $\mathcal{B}_\tau$'s, in comparison to $\mathcal{M}_\tau$'s, is much more secretive.

Let us recall a fact which will be used several times in the sequel.

Remark 1.1. Let Ω be a non-empty set and Γ be a subset of 2^Ω. Let us denote $\sigma(\Gamma)$, by the sigma algebra generated by Γ. For a given A in $\sigma(\Gamma)$, one may find a sequence $A_1, A_2, \cdots$ in Γ such that A is contained in the sigma algebra generated by the sequence $\{A_n\}_{n=1}^\infty$ (just check that $\cup\{\sigma(\Delta) : \Delta \subseteq \Gamma, \Delta \text{ is countable}\}$ forms a sigma algebra).

Proposition 1.2. *Let A be a proper subset of $\mathbf{B}(\mathcal{H})$.*

(1) *Assume A is a w-measurable set in $\mathcal{B}_w$. There exists a countable set E in $\mathcal{E}$ such that $A = E_w^\perp + A$ where,*
$$E_w^\perp = \{x \in \mathbf{B}(\mathcal{H}) : \langle xe, f\rangle = 0 \textit{ for all } e, f \in E\}.$$
(2) *Assume A is a s-measurable set in $\mathcal{B}_s$. There exists a countable set E in $\mathcal{E}$ such that $A = E_s^\perp + A$ where,*
$$E_s^\perp = \{x \in \mathbf{B}(\mathcal{H}) : xe = 0 \textit{ for all } e \in E\}.$$
(3) *Assume A is a s^*-measurable set in $\mathcal{B}_{s^*}$. There exists a countable set E in $\mathcal{E}$ such that $A = E_{s^*}^\perp + A$ where,*
$$E_{s^*}^\perp = \{x \in \mathbf{B}(\mathcal{H}) : x^*e = xe = 0 \textit{ for all } e \in E\}.$$
(4) *Assume A is an a.m-measurable (Arens-Mackey measurable) set in $\mathcal{B}_{a.m}$. There exists a countable set E in $\mathcal{E}$ such that $A = E_{a.m}^\perp + A$ where,*
$$E_{a.m}^\perp = \{x \in \mathbf{B}(\mathcal{H}) : x^*e = xe = 0 \textit{ for all } e \in E\}.$$

Proof. For a given τ-measurable set A in $\mathcal{B}_\tau$, there exists a sequence of sub-basic neighborhoods $\{A_n\}$ (for the topology τ) such that A is generated by A_n's (see Remark 1.1). Indeed by a countably many operations "*union and intersection*" on the union $\{A_n\} \bigcup \{A_n^c\}$, one may construct A. Therefore it is enough to prove these assertions when A or A^c is itself a sub-basic neighborhood. At first, assume that A is a sub-basic neighborhood in the weak operator topology. There exist ζ and η in $\mathcal{H}$ and $x_0 \in \mathbf{B}(\mathcal{H})$ with
$$A = \{x \in \mathbf{B}(\mathcal{H}) : |\langle x\zeta, \eta\rangle - \langle x_0\zeta, \eta\rangle| < \epsilon\}.$$
Let E be a countable subset in $\mathcal{E}$ such that both ζ and η are generated by E. One may directly check that $E_w^\perp$ works for both A and A^c, that is, $E_w^\perp + A = A$ and $E_w^\perp + A^c = A^c$. Except the Arens-Mackey topology, others are obtained by a similar argument as well as the weak operator topology. As for the

last one, let us consider the sub-basic neighborhood

$$A = \{x \in \mathbf{B}(\mathcal{H}) : \sup_{\phi\in K} |\phi(x) - \phi(x_0)| < \epsilon\},$$

where $x_0 \in \mathbf{B}(\mathcal{H})$ and K is a weakly compact convex subset of the predual $\mathbf{B}(\mathcal{H})_*$. To any element F of $\mathrm{Fin}(\mathcal{E})$, a finite subset of $\mathcal{E}$, we correspond to the finite rank projection whose range is linearly spanned by F, say p_F. Clearly $\{p_F\}_{\mathrm{Fin}(\mathcal{E})}$ forms an increasing net of projections which is strongly convergent to the identity operator. A deep result due to Akemann ([3], Theorem III.5.4) says that

$$\lim_F \sup_{\phi\in K} \|(1-p_F)\phi(1-p_F)\| = 0.$$

It implies that there exists (at most) a countable subset E in $\mathcal{E}$ such that

$$\sup_{\phi\in K} \|(1-p_E)\phi(1-p_E)\| = 0, \tag{1.2}$$

where p_E is the orthogonal projection whose range is generated by E. We check that E works for this assertion. To do this, assume $ye = y^*e = 0$ for all $e \in E$. Equivalently, $yp_E = y^*p_E = 0$ which implies that $y = (1-p_E)y(1-p_E)$. We now apply (1.2) to conclude $\phi(y) = 0$ for all ϕ in K which finishes the proof. □

Let A be a τ-measurable set in $\mathcal{B}_\tau$. By the previous result, there exists a countable subset E of $\mathcal{E}$, called *a refiner set of* A, such that $A + E_\tau^\perp = A$.

Corollary 1.3. *Every τ-measurable set A in $\mathcal{B}_\tau$ is unbounded.*

Proof. Let E be a refiner set of A. Since $\mathcal{H}$ is non-separable Hilbert space, then $E_\tau^\perp$ is unbounded. It implies A is unbounded too. □

Corollary 1.4. *The sigma algebra $\mathcal{B}_\tau$ is properly contained in $\mathcal{M}_\tau$.*

Proof. Norm closed balls are contained in $\mathcal{M}_\tau$. The previous corollary finishes the proof. □

Proposition 1.5. *Concerning the sigma algebras $\mathcal{B}_\tau$'s, we have that*:

(1) $\mathcal{B}_w = \mathcal{B}_{\sigma\text{-}w} \subseteq \mathcal{B}_{a.m}$,
(2) $\mathcal{B}_s = \mathcal{B}_{\sigma\text{-}s}$ *and*
(3) $\mathcal{B}_{s^*} = \mathcal{B}_{\sigma\text{-}s^*}$.

Proof. We first show that $\mathcal{B}_{\sigma\text{-}w} \subseteq \mathcal{B}_w$ which implies $\mathcal{B}_w = \mathcal{B}_{\sigma\text{-}w}$. Let us consider the following sub-basic neighborhood in the σ-weak operator topology

$$\{x \in \mathbf{B}(\mathcal{H}) : |\phi(x) - \phi(x_0)| < \epsilon\},$$

where ϕ is a normal functional on $\mathbf{B}(\mathcal{H})$ and x_0 is in $\mathbf{B}(\mathcal{H})$. There exist two square summable sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in $\mathcal{H}$ with $\phi(x) = \sum\langle x\zeta_n, \eta_n\rangle$. Then

$$\begin{aligned}\phi(x) - \phi(x_0) &= \sum_1^\infty \langle (x-x_0)\zeta_n, \eta_n\rangle \\ &= \lim_{n\to\infty} \underbrace{\sum_{j=1}^n \langle x\zeta_j, \eta_j\rangle}_{\phi_n(x)} - \left(\sum_1^\infty \langle x_0\zeta_n, \eta_n\rangle\right).\end{aligned}$$

Let us consider $\phi_n : (\mathbf{B}(\mathcal{H}), \mathcal{B}_w) \to \mathbf{C}$ given by $\phi_n(x) = \sum_{k=1}^n \langle x\zeta_k, \eta_k\rangle$. Obviously, ϕ_n's are all measurable functions. Since ϕ is the point-wise limit of the sequence $\{\phi_n\}$, then ϕ is measurable with respect to the sigma algebra $\mathcal{B}_w$ too. Therefore, the sub-basic open set $\{x \in \mathbf{B}(\mathcal{H}) : |\phi(x) - \phi(x_0)| < \epsilon\}$ is contained in $\mathcal{B}_w$. By a similar argument, one may show that $\mathcal{B}_s = \mathcal{B}_{\sigma\text{-}s}$ and $\mathcal{B}_{s^*} = \mathcal{B}_{\sigma\text{-}s^*}$.

It remains to prove $\mathcal{B}_w$ is properly contained in $\mathcal{B}_{a.m}$. To do this, we make an example of a sub-basic neighborhood in the Arens-Mackey topology which is not in $\mathcal{B}_w$. Let us fix $e \in \mathcal{E}$ and consider

$$K_e := \{\omega_{\eta,e} \in \mathbf{B}(\mathcal{H})_* : \eta \text{ is in the unit ball of } \mathcal{H}\},$$

where $\omega_{\eta,e}(x) = \langle x\eta, e\rangle$. Consider the map $F_e : \mathcal{H}_1 \to \mathbf{B}(\mathcal{H})_*$ which maps $\eta \to \omega_{\eta,e}$, where $\mathcal{H}_1$ denotes the unit ball of $\mathcal{H}$. One may directly check that F_e is weak-weak continuous. Then K_e, the range of F_e, is weakly compact.

Since K_e is also convex, then

$$O_e = \{x \in \mathbf{B}(\mathcal{H}) : |\phi(x)| < 1 \text{ for all } \phi \in K_e\}$$

forms a sub-basic neighborhood containing 0, in the Arens-Mackey topology. We apply the first item of Proposition 1.2 and show that O_e is no longer in $\mathcal{B}_w$. If O_e is in $\mathcal{B}_w$, then there exists a refiner set $E \subseteq \mathcal{E}$ with

$$E_w^\perp = \{x \in \mathbf{B}(\mathcal{H}) : \langle xf, g\rangle = 0 : f, g \in E\} \subseteq O_e.$$

Let us select $e' \in \mathcal{E} - E$. The rank one operator $x = 2e \otimes e'$ is in $E_w^\perp$ and so should be contained in O_e ($0 \in O_e$) which is not true since $|\omega_{e',e}(x)| = 2$. □

Remark 1.6. In the previous proposition, as for weakly compactness of K_e, we have replaced the proof with the one suggested by the referee since it was really much easier.

Let us consider $A = \{x \in \mathbf{B}(\mathcal{H}) : \|(x - x_0)\zeta\|^2 < \epsilon\}$ which is a sub-basic neighborhood in the strong operator topology.

$$A = \bigcup_{n=1}^\infty \left\{ y \in \mathbf{B}(\mathcal{H}) : \|(y-x)\zeta\|^2 \le \epsilon\left(1 - \frac{1}{n}\right)\right\}$$

$$= \bigcup_{n=1}^{\infty} \underbrace{\left(\bigcap_{F \in \mathrm{Fin}(\mathcal{E})} \left\{ y \in \mathbf{B}(\mathcal{H}) : \sum_{e \in F} |\langle (y-x)\zeta, e\rangle|^2 \le \epsilon\left(1 - \frac{1}{n}\right) \right\} \right)}_{\text{closed set in the weak operator topology}}.$$

If one follows a similar argument for strong* operator topology and the Arens-Mackey topology, then the following facts will be obtained:

- Any sub-basic neighborhood in the strong operator topology forms a F_σ-set in $\mathcal{M}_w$.
- Any sub-basic neighborhood in the strong* operator topology forms a F_σ-set in $\mathcal{M}_w$.
- Any sub-basic neighborhood in the Arens-Mackey topology forms a F_σ-set in $\mathcal{M}_{\sigma\text{-}w}$.

Based on these three points, we infer the following fact.

Proposition 1.7. *The sigma algebras $\mathcal{B}_\tau$'s are all contained in M_w.*

We end this section with two unsolved problems:

Problem 1.8. (1) Let $b(\mathcal{H})$ be the set of all closed balls in $\mathbf{B}(\mathcal{H})$. Is $\mathcal{M}_w$ generated by the union $\mathcal{B}_w \bigcup b(\mathcal{H})$?

(2) Let $\{e_j\}_{j\in J}$ be a proper subset of $\mathcal{E}$ whose cardinal is larger than continuum and then consider $O_s = \bigcup_{j\in J}\{x \in \mathbf{B}(\mathcal{H}) : \|xe_j\| < 1\}$ which forms an open set in the strong operator topology. Is O_s a w-measurable set?

2. Sigma algebras generated by the base of topologies. In this section, we focus on the sigma algebras $\mathcal{B}_\tau$'s. We saw that discussion on this sigma algebras is reduced to $\mathcal{B}_s, \mathcal{B}_{s^*}$ and $\mathcal{B}_w \subseteq \mathcal{B}_{a.m}$. We first follow this question, what do we know more concerning relations between these sigma algebras? Secondly, we show that the algebraic operations, addition and product, do not fit on the set of operator valued τ-measurable functions.

Let $\{\phi_\tau\}$ be the family of semi-norms that determines the locally convex topology τ. By a τ- sub-basic neighborhood centered at 0, we mean a subset of the form of $\{x \in \mathbf{B}(\mathcal{H}) : \phi_\tau(x) < \epsilon\}$. Let us denote $\mathcal{B}^0_\tau$, by the sigma algebra generated by all τ- sub-basic neighborhoods centered at 0.

Remark 2.1. As for the sigma algebras $\mathcal{B}^0_\tau$'s, we have that any τ- sub-basic neighborhood centered at 0 is invariant under any rotation ($A = e^{i\theta}A$, for any arbitrary angle θ). Therefore, any element of the sigma algebra $\mathcal{B}^0_\tau$ is also invariant under any rotation. It implies that $\mathcal{B}^0_\tau$ is properly contained in $\mathcal{B}_\tau$.

Proposition 2.2. *We have that*:

(1) $\mathcal{B}_w \cap \mathcal{B}^0_s = \{\emptyset, \mathbf{B}(\mathcal{H})\}$,
(2) $\mathcal{B}_w \cap \mathcal{B}^0_{s^*} = \{\emptyset, \mathbf{B}(\mathcal{H})\}$ *and*
(3) $\mathcal{B}_s \cap \mathcal{B}^0_{s^*} = \{\emptyset, \mathbf{B}(\mathcal{H})\}$.

Proof. (1) Let A be in the intersection $\mathcal{B}_w \cap \mathcal{B}^0_s$ which is neither empty set nor $\mathbf{B}(\mathcal{H})$. We may assume A contains 0, otherwise A^c is considered.

i) Since A is contained in $\mathcal{B}^0_s$, then there exists a sequence $\{N_n\}$ of sub-basic neighborhoods centered at 0 in the strong operator topology such that A is generated by N_n's (see Remark 1.1). Based on definition of N_n's, there are a sequence $\{\zeta_n\}$ of vectors in $\mathcal{H}$ and a sequence of positive numbers $\{\epsilon_n\}$ with $N_n = \{x \in \mathbf{B}(\mathcal{H}) : \|x\zeta_n\| < \epsilon_n\}$. We put

$$\Gamma_A = \{\gamma(x) : x \in A\},$$

where $\gamma(x)$ is the sequence $\{\|x\zeta_n\|\}$. Notice that

$$x \in A \Longleftrightarrow \gamma(x) \in \Gamma(A).$$

Since A is properly contained in $\mathbf{B}(\mathcal{H})$ then, there exists an operator y such that $\gamma(y)$ is no longer in Γ_A.

ii) Since A is a w-measurable set in $\mathcal{B}_w$ then, there is a refiner set $E = \{e_n\}$ in $\mathcal{E}$ with $A = E^\perp_w + A$. Any countable set in $\mathcal{E}$ containing E also forms a refiner set of A, then we may also assume that $y\zeta_n$'s are all in the closed subspace generated by E.

Let us select a sequence $\{f_n\} \subseteq \mathcal{E}$ such that $\langle f_n, e_m\rangle = 0$ for all n and m ($\mathcal{H}$ is non-separable), and then consider the operator $q = \sum f_n \otimes e_n$. As for the operator $\tilde{y} = qy$, we have that:

- A direct calculation shows $\gamma(\tilde{y}) = \gamma(y)$. Since A does not contain y, then item i) implies that $\tilde{y}$ is not in A too.
- $\tilde{y}$ is clearly contained in $E^\perp_w$. Since A contains 0, then $A = E^\perp_w + A$ forces $\tilde{y}$ should be in A.

This is a contradiction and so the intersection $\mathcal{B}_w \cap \mathcal{B}^0_s$ should be trivial.

(2) To prove this case, one may exactly repeat the proof given in (1), when some notations in the proof are changed as follow:

$$\begin{cases} N_n = \{x \in \mathbf{B}(\mathcal{H}) : \|x\zeta_n\|^2 + \|x^*\zeta_n\|^2 < \epsilon_n\} \text{ and} \\ \gamma(x) = \{\|x\zeta_n\|^2 + \|x^*\zeta_n\|^2\}. \\ \text{The closed linear span of the refiner set} \\ \text{contains all sequences } \{y\zeta_n\},\ \{y^*\zeta_n\} \text{ and } \{\zeta_n\}, \\ \tilde{y} = qy + yq^*. \end{cases}$$

(3) To obtain the last one, again we may exactly repeat the proof given in (1), and the list of

notations should be changed as follow:

$$\begin{cases} N_n = \{x \in \mathbf{B}(\mathcal{H}) : \|x\zeta_n\|^2 + \|x^*\zeta_n\|^2 < \epsilon_n\} \text{ and} \\ \gamma(x) = \{\|x\zeta_n\|^2 + \|x^*\zeta_n\|^2\}. \\ \text{The closed linear span of the refiner set} \\ \text{contains all sequnces } \{y\zeta_n\},\ \{y^*\zeta_n\} \text{ and } \{\zeta_n\}. \\ \tilde{y} = y^*q^* + yp \text{ where } p = \sum e_n \otimes g_n \text{ and the} \\ \text{sequence } \{g_n\} \subset \mathcal{E} \text{ is selected so that} \\ \langle g_n, f_m\rangle = \langle g_n, e_m\rangle = 0 \text{ for all } n, m. \end{cases}$$

□

Finally by some examples we show how the addition and product of two τ-measurable functions may be no longer τ-measurable. We need to recall a fact from a classical set theory. Let X be a measurable space whose cardinal is larger than continuum. Let us consider the cartesian product $X \times X$ equipped with the product sigma algebra. Assume that $S \subseteq X \times X$ is a measurable set. Then the family of sections $S_x = \{y \in X : (x, y) \in X\}$ contains at most continuum of distinct sets and consequently the diagonal $D = \{(x, x) : x \in X\}$ is no longer a measurable set (see [2], p. 231 Problem 3.10.44).

Example 2.3. Let $\Omega_j = (\mathbf{B}(\mathcal{H}), \mathcal{M}_\tau)$ for $j = 1, 2$. We consider the cartesian product space $\Omega = \Omega_1 \times \Omega_2$ equipped with the product sigma algebra.

(1) Let us consider functions f and g on Ω given by $f(x, y) = x$ and $g(x, y) = -y$ which are clearly τ-measurable. We have then

$$(f + g)^{-1}(0) = \{(x, x) : x \in \mathbf{B}(\mathcal{H})\}.$$

The cardinal number of $\mathbf{B}(\mathcal{H})$ is larger than continuum ($\mathcal{H}$ is non-separable). Therefore, the diagonal $(f + g)^{-1}(0)$ is not measurable in Ω which implies that $f + g$ is not τ-measurable.

(2) In this example, we assume that the cardinal number $\{e_i\}_{i\in I}$ is 2^c. The projections $f(x, y) = x$ and $g(x, y) = y$ on Ω are clearly τ-measurable functions. We now verify that the product $f \cdot g$ is not τ-measurable. To do this, we put

$$S = (f \cdot g)^{-1}(id_{\mathcal{H}}) = \{(x, y) \in \Omega : xy = id_{\mathcal{H}}\},$$

and check that the family of sections $S_x = \{y \in \mathbf{B}(\mathcal{H}) : (x, y) \in S\}$ contains 2^c of distinct sets. Let I_0 consist of those subsets in I which are infinite and countable. Then I and I_0 have the same cardinal. Let J be the subset in I_0 and $\mathcal{H}_J$ be the subspace generated by $\{e_j : j \in J\}$. Let v_J be a bilateral shift on the Hilbert space $\mathcal{H}_J$ and consider the operator $x_J = v_J \oplus id_{\mathcal{H}_J^\perp}$ in $\mathbf{B}(\mathcal{H})$. We have then (x_J, y_J) is in S where $y_J = v_J^* \oplus id_{\mathcal{H}_J^\perp}$. Obviously, $S_{x_{J_1}}$ and $S_{x_{J_2}}$ are not the same when J_1 and J_2 are different sets in I_0.

(3) Let us consider the measurable space $\Omega_w = (\mathbf{B}(\mathcal{H}), \mathcal{B}_w)$ and the inclusion mapping $\iota : \Omega_w \to (\mathbf{B}(\mathcal{H}), \mathcal{B}_w)$. We check that ι^2 is not measurable. Let us fix $e \in \mathcal{E}$. If ι^2 is measurable function, then $A = (\iota^2)^{-1}(N_e)$ is a measurable set, where $N_e = \{x \in \mathbf{B}(\mathcal{H}) : |\langle xe, e\rangle| < 1\}$. Let E be a $\mathcal{B}_w$-refiner set for A containing e. We select $f \in \mathcal{E} - E$ and then consider the rank one operators $a = 2e \otimes f$ and $b = f \otimes e$. One may directly check that $a \in E_w^\perp$ and $b \in A$. Therefore, $a + b \in E_w^\perp + A = A$. But we have that

$$|\langle \iota^2(a + b)e, e\rangle| = 2,$$

which is a contradiction. This means that the set of measurable functions $\{f : \Omega_w \to (\mathbf{B}(\mathcal{H}), \mathcal{B}_w)\}$ is not closed under product.

(4) If the sigma algebra $\mathcal{B}_w$ is replaced by any of the sigma algebras $\mathcal{B}_s, \mathcal{B}_{s^*}$ or $\mathcal{B}_{a.m}$ in the Example (2) above, then the same result is obtained, i.e., the product dose not work well again. To prove them, one may exactly repeat the proof when some notions are changed. The list of changes are given as follow:

changes when $\mathcal{B}_w$ is replaced by

$$\mathcal{B}_s : N_e = \{x \in \mathbf{B}(\mathcal{H}) : \|xe\| < 1\}$$

changes when $\mathcal{B}_w$ is replaced by

$$\mathcal{B}_{s^*} : \begin{cases} N_e = \{x \in \mathbf{B}(\mathcal{H}) : \|xe\|^2 + \|x^*e\|^2 < 1\} \\ a = 2f \otimes f \end{cases}$$

changes when $\mathcal{B}_w$ is replaced by

$$\mathcal{B}_{a.m} : \begin{cases} N_e = \{x \in \mathbf{B}(\mathcal{H}) : \sup_{\phi \in K_e} |\phi(x)| < 1\} \\ a = 2f \otimes f \end{cases}$$

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References

[1] G. A. Bagheri-Bardi, Operator-valued measurable functions, Bull. Belg. Math. Soc. Simon Stevin **22** (2015), no. 1, 159–163.

[2] V. I. Bogachev, *Measure theory. Vol. I*, Springer, Berlin, 2007.

[3] M. Takesaki, *Theory of operator algebras. I*, Springer, New York, 1979.