On the topology of arrangements of a cubic and its inflectional tangents

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Abstract: A k-Artal arrangement is a reducible algebraic curve composed of a smooth cubic and k inflectional tangents. By studying the topological properties of their subarrangements, we prove that for $k = 3, 4, 5, 6$, there exist Zariski pairs of k-Artal arrangements. These Zariski pairs can be distinguished in a geometric way by the number of collinear triples in the set of singular points of the arrangement contained in the cubic.

Key words: Subarrangement; Zariski pair; k-Artal arrangement.

1. Introduction. In this article, we continue to study Zariski pairs for reducible plane curves based on the idea used in [3]. A pair $(\mathcal{B}^1, \mathcal{B}^2)$ of reduced plane curves in $\mathbf{P}^2$ is said to be a Zariski pair if (i) both $\mathcal{B}^1$ and $\mathcal{B}^2$ have the same combinatorics and (ii) $(\mathbf{P}^2, \mathcal{B}^1)$ is *not* homeomorphic to $(\mathbf{P}^2, \mathcal{B}^2)$ (see [2] for details about Zariski pairs). As we have seen in [2], the study of Zariski pairs, roughly speaking, consists of two steps:

(i) How to construct (or find) plane curves with the same combinatorics but having *some* different properties.

(ii) How to distinguish the topology of $(\mathbf{P}^2, \mathcal{B}^1)$ and $(\mathbf{P}^2, \mathcal{B}^2)$.

As for the second step, various tools such as fundamental groups, Alexander invariants, braid monodromies, existence/non-existence of Galois covers and so on have been used. In [3], the first and last authors considered another elementary method in order to study Zariski k-plets for arrangements of reduced plane curves and showed its effectiveness by giving some new examples. In this article, we study the topology of arrangements of a smooth cubic and its inflectional tangents along the same line.

1.1. Subarrangements. We here reformulate our idea in [3] more precisely. Let $\mathcal{B}_o$ be a (possibly empty) reduced plane curve $\mathcal{B}_o$. We define $\underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o}$ to be the set of the reduced plane curves of the form $\mathcal{B}_o + \mathcal{B}$, where $\mathcal{B}$ is a reduced curve with no common component with $\mathcal{B}_o$.

Let $\mathcal{B} = \mathcal{B}_1 + \cdots + \mathcal{B}_r$ denote the irreducible decomposition of $\mathcal{B}$. For a subset $\mathcal{I}$ of the power set $2^{\{1,\dots,r\}}$ of $\{1, \dots, r\}$, which does not contain the empty set $\emptyset$, we define the subset $\underline{\mathrm{Sub}}_{\mathcal{I}}(\mathcal{B}_o, \mathcal{B})$ of $\underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o}$ by:

$$\underline{\mathrm{Sub}}_{\mathcal{I}}(\mathcal{B}_o, \mathcal{B}) := \left\{ \mathcal{B}_o + \sum_{i \in I} \mathcal{B}_i \;\middle|\; I \in \mathcal{I} \right\}.$$

For $\mathcal{I} = 2^{\{1,\dots,r\}} \setminus \emptyset$, we denote $\underline{\mathrm{Sub}}(\mathcal{B}_o, \mathcal{B}) = \underline{\mathrm{Sub}}_{\mathcal{I}}(\mathcal{B}_o, \mathcal{B})$.

Let A be a set and suppose that a map

$$\Phi_{\mathcal{B}_o} : \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o} \to A$$

with the following property is given: for $\mathcal{B}_o + \mathcal{B}^1, \mathcal{B}_o + \mathcal{B}^2 \in \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o}$, if there exists a homeomorphism $h : (\mathbf{P}^2, \mathcal{B}_o + \mathcal{B}^1) \to (\mathbf{P}^2, \mathcal{B}_o + \mathcal{B}^2)$ with $h(\mathcal{B}_o) = \mathcal{B}_o$, then $\Phi_{\mathcal{B}_o}(\mathcal{B}_o + \mathcal{B}^1) = \Phi_{\mathcal{B}_o}(\mathcal{B}_o + \mathcal{B}^2)$.

We denote by $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}}$ the restriction of $\Phi_{\mathcal{B}_o}$ to $\underline{\mathrm{Sub}}(\mathcal{B}_o, \mathcal{B})$. Note that if there exists a homeomorphism $h : (\mathbf{P}^2, \mathcal{B}_o + \mathcal{B}^1) \to (\mathbf{P}^2, \mathcal{B}_o + \mathcal{B}^2)$ for $\mathcal{B}_o + \mathcal{B}^1, \mathcal{B}_o + \mathcal{B}^2 \in \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o}$ with $h(\mathcal{B}_o) = \mathcal{B}_o$, then we have the induced map $h_\natural : \underline{\mathrm{Sub}}(\mathcal{B}_o, \mathcal{B}^1) \to \underline{\mathrm{Sub}}(\mathcal{B}_o, \mathcal{B}^2)$ such that $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}^1} = \tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}^2} \circ h_\natural$:

$$\begin{tikzcd}
\underline{\mathrm{Sub}}(\mathcal{B}_o, \mathcal{B}^1) \arrow[d, "h_\natural"'] \arrow[dr, "\tilde{\Phi}_{\mathcal{B}_o,\mathcal{B}^1}"] & \\
\underline{\mathrm{Sub}}(\mathcal{B}_o, \mathcal{B}^2) \arrow[r, "\tilde{\Phi}_{\mathcal{B}_o,\mathcal{B}^2}"'] & A
\end{tikzcd}$$

Remark 1.1. In §2 we give four explicit examples for $\Phi_{\mathcal{B}_o}$ and $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}}$ allowing to distinguish

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the k-Artal arrangements (see §1.2 for the definition), using the Alexander polynomial, the existence of D_6-covers, the splitting numbers and the linking set.

If $\mathcal{D}_o+\mathcal{D}^1, \mathcal{D}_o+\mathcal{D}^2 \in \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{D}_o}$ have the same combinatorics, then any homeomorphism $h:(\mathcal{T}^1, \mathcal{D}_o+\mathcal{D}^1) \to (\mathcal{T}^2, \mathcal{D}_o+\mathcal{D}^2)$ with $h(\mathcal{D}_o)=\mathcal{D}_o$ induces a map $h_\natural : \underline{\mathrm{Sub}}(\mathcal{D}_o, \mathcal{D}^1) \to \underline{\mathrm{Sub}}(\mathcal{D}_o, \mathcal{D}^2)$, where $\mathcal{T}^i$ is a tubular neighborhood of $\mathcal{D}_o+\mathcal{D}^i$ for $i=1,2$. Let $(\mathcal{D}_o+\mathcal{D}^1, \mathcal{D}_o+\mathcal{D}^2)$ be a Zariski pair of curves in $\underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{D}_o}$ such that

- it is distinguished by $\Phi_{\mathcal{D}_o} : \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{D}_o} \to A$, i.e., any homeomorphism $h:(\mathcal{T}^1, \mathcal{D}_o+\mathcal{D}^1) \to (\mathcal{T}^2, \mathcal{D}_o+\mathcal{D}^2)$ necessarily satisfies $h(\mathcal{D}_o)=\mathcal{D}_o$ and $\Phi_{\mathcal{D}_o}(\mathcal{D}_o+\mathcal{D}^1) \neq \Phi_{\mathcal{D}_o}(\mathcal{D}_o+\mathcal{D}^2)$, and
- the combinatorial type of $\mathcal{D}_o+\mathcal{D}^1$ and $\mathcal{D}_o+\mathcal{D}^2$ is $\underline{\mathrm{C}}$.

Assuming the existence of such a Zariski pair for the combinatorial type $\underline{\mathrm{C}}$, we construct Zariski pairs with glued combinatorics. We first note that the following proposition is immediate:

Proposition 1.2. *Choose $\mathcal{B}_o+\mathcal{B}^1, \mathcal{B}_o+\mathcal{B}^2 \in \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o}$ with the same combinatorial type. Let $\underline{\mathrm{Sub}}_{\underline{\mathrm{C}}}(\mathcal{B}_o, \mathcal{B}^j)$ $(j=1,2)$ be the sets of subarrangements of $\mathcal{B}_o+\mathcal{B}^j$ having the combinatorial type $\underline{\mathrm{C}}$ $(j=1,2)$, respectively. If*

(i) *any homeomorphism $h:(\mathcal{T}^1, \mathcal{B}_o+\mathcal{B}^1) \to (\mathcal{T}^2, \mathcal{B}_o+\mathcal{B}^2)$ necessarily satisfies $h(\mathcal{B}_o)=\mathcal{B}_o$, where $\mathcal{T}^i$ is a tubular neighborhood of $\mathcal{B}_o+\mathcal{B}^i$ for $i=1,2$, and*

(ii) *for some elements $a_1 \in A$,*

$$\sharp(\tilde{\Phi}_{\mathcal{B}_o,\mathcal{B}^1}^{-1}(a_1) \cap \underline{\mathrm{Sub}}_{\underline{\mathrm{C}}}(\mathcal{B}_o, \mathcal{B}^1)) \neq \sharp(\tilde{\Phi}_{\mathcal{B}_o,\mathcal{B}^2}^{-1}(a_1) \cap \underline{\mathrm{Sub}}_{\underline{\mathrm{C}}}(\mathcal{B}_o, \mathcal{B}^2)),$$

then $(\mathcal{B}_o+\mathcal{B}^1, \mathcal{B}_o+\mathcal{B}^2)$ is a Zariski pair.

Remark 1.3. If for all automorphism σ of the combinatorics of $\mathcal{B}_o+\mathcal{B}^j$, $\sigma(\mathcal{B}_o)=\mathcal{B}_o$ then hypothesis (i) of Proposition 1.2 is always verified. In particular, it is the case if $\deg(\mathcal{B}_o) \neq \deg(\mathcal{B}_i)$, for $i=1,\dots,r$.

1.2. Artal arrangements. In this article, we apply Proposition 1.2 to distinguish Zariski pairs formed by *Artal arrangements*. These curves are defined as follows:

Let E be a smooth cubic, let P_i $(1 \le i \le 9)$ be its 9 inflection points and let L_{P_i} be the tangent lines at P_i $(1 \le i \le 9)$, respectively.

Definition 1.4. Choose a subset $I \subset \{1,\dots,9\}$. We call an arrangement $\mathcal{C} = E + \sum_{i\in I} L_{P_i}$ an Artal arrangement for I. In particular, if $k=\sharp(I)$, we call $\mathcal{C}$ a k-Artal arrangement.

The idea is to apply Proposition 1.2 to the case when $\mathcal{B}_o = E$ and $\mathcal{B} = \sum_{i\in I} L_{P_i}$. Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two k-Artal arrangements. Note that if there exists a homeomorphism $h:(\mathbf{P}^2, \mathcal{C}^1) \to (\mathbf{P}^2, \mathcal{C}^2)$, $h(E)=E$ always holds. In [1], E. Artal gave an example of a Zariski pair for 3-Artal arrangements. Based on this example, we make use of our method to find other examples of Zariski pairs of k-Artal arrangements and obtain the following

Theorem 1.5. *There exist Zariski pairs for k-Artal arrangements for $k=4,5,6$.*

Remark 1.6. Note that the case of $k=5$ is considered in [4], where it is shown that there exists a Zariski pair for 5-Artal arrangements.

2. Some explicit examples for $\Phi_{\mathcal{B}_o}$. We here introduce four examples for $\Phi_{\mathcal{B}_o}$. The last two were recently considered by the second author and J.-B. Meilhan [4] and the third author [6], respectively.

2.1. D_{2p}-covers. For terminologies and notation, we use those introduced in [2, §3], freely.

Let D_{2p} be the dihedral group of order $2p$. Let $\mathrm{Cov}_b(\mathbf{P}^2, 2\mathcal{B}, D_{2p})$ be the set of isomorphism classes of D_{2p}-covers branched at $2\mathcal{B}$.

We now define $\Phi_{\mathcal{B}_o}^{D_{2p}} : \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o} \to \{0,1\}$ as follows:

$$\Phi_{\mathcal{B}_o}^{D_{2p}}(\mathcal{B}_o+\mathcal{B}) = \begin{cases} 1 \text{ if } \mathrm{Cov}_b(\mathbf{P}^2, 2(\mathcal{B}_o+\mathcal{B}), D_{2p}) \neq \emptyset \\ 0 \text{ if } \mathrm{Cov}_b(\mathbf{P}^2, 2(\mathcal{B}_o+\mathcal{B}), D_{2p}) = \emptyset. \end{cases}$$

Note that $\Phi_{\mathcal{B}_o}^{D_{2p}}$ satisfies the required condition described in the Introduction. Thus, we define the map $\tilde{\Phi}_{\mathcal{B}_o,\mathcal{B}}^{D_6}$ as the restriction of $\Phi_{\mathcal{B}_o}^{D_6}$ to $\underline{\mathrm{Sub}}(\mathcal{B}_o, \mathcal{B})$.

2.2. Alexander polynomials. For the Alexander polynomials of reduced plane curves, see [2, §2]. Let $\Delta : \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o} \to \mathbf{C}[t]$ be the map assigning to a curve of $\underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o}$ its Alexander polynomial. We define the map $\Phi_{\mathcal{B}_o}^{\mathrm{Alex}} : \underline{\mathrm{Curve}}_{\mathrm{red}}^{\mathcal{B}_o} \to \{0,1\}$ by:

$$\Phi_{\mathcal{B}_o}^{\mathrm{Alex}}(\mathcal{B}_o+\mathcal{B}') = \begin{cases} 1 & \text{if } \Delta(\mathcal{B}_o+\mathcal{B}') \neq 1 \\ 0 & \text{if } \Delta(\mathcal{B}_o+\mathcal{B}') = 1. \end{cases}$$

As previously, we define $\tilde{\Phi}_{\mathcal{B}_o,\mathcal{B}}^{\mathrm{Alex}}$ as the restriction of $\Phi_{\mathcal{B}_o}^{\mathrm{Alex}}$ to $\underline{\mathrm{Sub}}(\mathcal{B}_o, \mathcal{B})$.

2.3. Splitting numbers. Let $\mathcal{B}_o+\mathcal{B}$ be a reduced curves such that $\mathcal{B}_o$ is smooth, and let m be the degree of $\mathcal{B}$. Let $\pi_{\mathcal{B}} : X \to \mathbf{P}^2$ be the unique cover branched over $\mathcal{B}$, corresponding to the surjection of $\pi_1(\mathbf{P}^2 \setminus \mathcal{B}) \to \mathbf{Z}/m\mathbf{Z}$ sending all meridians of the $\mathcal{B}_i$ to 1. The *splitting number* of $\mathcal{B}_o$ for $\pi_{\mathcal{B}}$, denoted by $s_{\pi_{\mathcal{B}}}(\mathcal{B}_o)$ is the number of irreducible components of the pull-back $\pi_{\mathcal{B}}^*\mathcal{B}_o$ of $\mathcal{B}_o$ by $\pi_{\mathcal{B}}$ (see [6] for the general definition). By [6, Proposition 1.3], the application:

$$\Phi^{\mathrm{split}}_{\mathcal{B}_o} : \left\{ \begin{array}{ccc} \underline{\mathrm{Curve}}^{\mathcal{B}_o}_{\mathrm{red}} & \longrightarrow & \mathbf{N}^* \\ \mathcal{B}_o + \mathcal{B} & \longmapsto & s_{\pi_{\mathcal{B}}}(\mathcal{B}_o) \end{array} \right. ,$$

verify the condition of Proposition 1.2, where $\mathbf{N}^*$ is the set of integers more than 0. We can then define the map $\tilde{\Phi}^{\mathrm{split}}_{\mathcal{B}_o,\mathcal{B}} : \underline{\mathrm{Sub}}(\mathcal{B}_o,\mathcal{B}) \to \mathbf{N}^*$ as the restriction of $\Phi^{\mathrm{split}}_{\mathcal{B}_o}$ to $\underline{\mathrm{Sub}}(\mathcal{B}_o,\mathcal{B})$.

2.4. Linking set. Let $\mathcal{B}_o$ be a non-empty curve, with smooth irreducible components. A cycle of $\mathcal{B}_o$ is a S^1 embedded in $\mathcal{B}_o$. For $\mathcal{B}_o + \mathcal{B} \in \underline{\mathrm{Curve}}^{\mathcal{B}_o}_{\mathrm{red}}$, we define the *linking set* of $\mathcal{B}_o$, denoted by $\mathrm{lks}_{\mathcal{B}}(\mathcal{B}_o)$, as the set of classes in $\mathbf{H}_1(\mathbf{P}^2 \setminus \mathcal{B})/\mathrm{Ind}_{\mathcal{B}_o}$ of the cycles of $\mathcal{B}_o$ which do not intersect $\mathcal{B}$, where $\mathrm{Ind}_{\mathcal{B}_o}$ is the subgroup of $\mathrm{H}_1(\mathbf{P}^2 \setminus \mathcal{B})$ generated by the meridians in $\mathcal{B}_o$ around the points of $\mathcal{B}_o \cap \mathcal{B}$. This definition is weaker than [4, Definition 3.9]. By [4, Theorem 3.13], the map defined by:

$$\Phi^{\mathrm{lks}}_{\mathcal{B}_o} : \left\{ \begin{array}{ccc} \underline{\mathrm{Curve}}^{\mathcal{B}_o}_{\mathrm{red}} & \longrightarrow & \mathbf{N}^* \cup \{\infty\} \\ \mathcal{B}_o + \mathcal{B} & \longmapsto & \sharp\mathrm{lks}_{\mathcal{B}}(\mathcal{B}_o) \end{array} \right.$$

verify the condition of Proposition 1.2. We can thus define the map $\tilde{\Phi}^{\mathrm{lks}}_{\mathcal{B}_o,\mathcal{B}}$ as the restriction of $\Phi^{\mathrm{lks}}_{\mathcal{B}_o}$ to $\underline{\mathrm{Sub}}(\mathcal{B}_o,\mathcal{B})$.

3. The geometry of inflection points of a smooth cubic. Let $E \subset \mathbf{P}^2$ be a smooth cubic curve and let $O \in E$ be an inflection point of E. In this section we consider the elliptic curve (E, O). The following facts are well-known:

(a) The set of inflection points of E can be identified with $(\mathbf{Z}/3\mathbf{Z})^{\oplus 2} \subset E$, the subgroup of three torsion points of E. This identification comes from the cubic group law with neutral element, a flex.

(b) Let P, Q, R be distinct inflection points of E. Then P, Q, R are collinear if and only if $P + Q + R = O \in (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$.

From the above facts we can study the geometry of inflection points and the following proposition follows:

Proposition 3.1. *Let E be a cubic curve and $\{P_1, \dots, P_k\} \in E$ be a set of distinct inflection points of E. Let n be the number of triples $\{P_{i_1}, P_{i_2}, P_{i_3}\} \subset \{P_1, \dots, P_k\}$ such that they are collinear. Then the possible values of n for $k = 3, \dots, 9$ are as in the following table*:

Table

k	3	4	5	6	7	8	9
n	$0,1$	$0,1$	$1,2$	$2,3$	5	8	12

4. Proof of the main theorem.

4.1. The case of 3-Artal arrangements. Using the four invariants introduced in §2, we can prove the original result of E. Artal. Let $I = \{i_1, i_2, i_3\} \subset \{1, \dots, 9\}$ and $\mathcal{L}_I = \sum_{i \in I} L_{P_i}$.

Theorem 4.1. *For a 3-Artal arrangement $\mathcal{C} = E + \sum_{i\in I} L_{P_i} = E + \mathcal{L}_I$, we have*:

(a) $\tilde{\Phi}^{D_6}_{E,\mathcal{L}_I}(\mathcal{C}) = \left\{ \begin{array}{ll} 1 & \textit{if } P_{i_1}, P_{i_2}, P_{i_3} \textit{ are collinear} \\ 0 & \textit{if } P_{i_1}, P_{i_2}, P_{i_3} \textit{ are not collinear} \end{array} \right.$

(b) $\tilde{\Phi}^{\mathrm{Alex}}_{E,\mathcal{L}_I}(\mathcal{C}) = \left\{ \begin{array}{ll} 1 & \textit{if } P_{i_1}, P_{i_2}, P_{i_3} \textit{ are collinear} \\ 0 & \textit{if } P_{i_1}, P_{i_2}, P_{i_3} \textit{ are not collinear} \end{array} \right.$

(c) $\tilde{\Phi}^{\mathrm{split}}_{E,\mathcal{L}_I}(\mathcal{C}) = \left\{ \begin{array}{ll} 3 & \textit{if } P_{i_1}, P_{i_2}, P_{i_3} \textit{ are collinear} \\ 1 & \textit{if } P_{i_1}, P_{i_2}, P_{i_3} \textit{ are not collinear} \end{array} \right.$

(d) $\tilde{\Phi}^{\mathrm{lks}}_{E,\mathcal{L}_I}(\mathcal{C}) = \left\{ \begin{array}{ll} 1 & \textit{if } P_{i_1}, P_{i_2}, P_{i_3} \textit{ are collinear} \\ 3 & \textit{if } P_{i_1}, P_{i_2}, P_{i_3} \textit{ are not collinear} \end{array} \right.$

Proof. (a) This is the result of the last author ([7]).

(b) This is the result of E. Artal ([1]).

(c) By [6, Theorem 2.7], we obtain $\Phi^{\mathrm{split}}_{E}(\mathcal{C}) = 3$ if the three tangent points are collinear, and $\Phi^{\mathrm{split}}_{E}(\mathcal{C}) = 1$ otherwise.

(d) Using the same arguments as in [5], we can prove that, in the case of 3-Artal arrangements, $\tilde{\Phi}^{\mathrm{split}}_{E,\mathcal{L}_I}(\mathcal{C}) = \frac{3}{\sharp\mathrm{lks}_{\mathcal{L}_I}(E)}$. Using the previous point we obtain the result. □

Remark 4.2. It is also possible to consider $\tilde{\Phi}^{\mathrm{lks}}_{\mathcal{L}_J,E}(E)$. But in this case, we have no method to compute it in the general case. But, if E is the cubic defined by $x^3 - xz^2 - y^2z = 0$, the computation done in [4] implies the result.

Corollary 4.3. *Choose $\{i_1, i_2, i_3, i_4\} \subset \{1, \dots, 9\}$ such that $P_{i_1}, P_{i_2}, P_{i_3}$ are collinear, while $P_{i_1}, P_{i_2}, P_{i_4}$ are not collinear. Put $\mathcal{C}_1 = E + L_{i_1} + L_{i_2} + L_{i_3}$ and $\mathcal{C}_2 = E + L_{i_1} + L_{i_2} + L_{i_4}$. Then $(\mathcal{C}_1, \mathcal{C}_2)$ is a Zariski pair.*

4.2. The other cases. Choose a subset J of $\{1, \dots, 9\}$ such that $4 \le \sharp J \le 6$ and let

$$\mathcal{C} := E + \mathcal{L}_J,\ \mathcal{L}_J = \sum_{j \in J} L_{P_j},$$

be a k-Artal arrangement. To distinguish these arrangements in a geometric way (as the collinearity in the case of 3-Artal arrangements), let us introduce the type of a k-Artal arrangement.

Definition 4.4. For $k=4,5,6$, we say an arrangement of the form $\mathcal{C}=E+L_{P_1}+\cdots+L_{P_k}$ to be of Type I if the number n of collinear triples in $\{P_1,\dots,P_k\}$ is $n=k-3$, while we say $\mathcal{C}$ to be of Type II if the number n of collinear triples in $\{P_1,\dots,P_k\}$ is $n=k-4$.

Theorem 4.5. *Let $\mathcal{C}_1$ be an arrangement of Type I and $\mathcal{C}_2$ be an arrangement of Type II. Then $(\mathbf{P}^2,\mathcal{C}_1)$ and $(\mathbf{P}^2,\mathcal{C}_2)$ are not homeomorphic as pairs.*

Furthermore if $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same combinatorics, $\mathcal{C}_1,\mathcal{C}_2$ form a Zariski pair.

Proof. Let $\mathcal{C}$ be a k-Artal arrangement ($k=4,5,6$). We denote by $\underline{\mathrm{Sub}}(E,\mathcal{L}_J)_3$ the set of 3-Artal arrangements contained in $\underline{\mathrm{Sub}}(E,\mathcal{L}_J)$. Let $\Phi^{D_6}_{\mathcal{C},3}$, $\Phi^{\mathrm{Alex}}_{\mathcal{C},3}$, $\Phi^{\mathrm{split}}_{\mathcal{C},3}$ and $\Phi^{\mathrm{lks}}_{\mathcal{C},3}$ be the restrictions of $\tilde{\Phi}^{D_6}_{E,\mathcal{L}_J}$, $\tilde{\Phi}^{\mathrm{Alex}}_{E,\mathcal{L}_J}$, $\tilde{\Phi}^{\mathrm{split}}_{E,\mathcal{L}_J}$ and $\tilde{\Phi}^{\mathrm{lks}}_{E,\mathcal{L}_J}$ to $\underline{\mathrm{Sub}}(E,\mathcal{L}_J)_3$, respectively. Then by Theorem 4.1, we have

$$
\begin{aligned}
&\sharp({\Phi^{D_6}_{\mathcal{C},3}}^{-1}(1))\\
&\quad=\sharp({\Phi^{\mathrm{Alex}}_{\mathcal{C},3}}^{-1}(1))\\
&\quad=\sharp({\Phi^{\mathrm{split}}_{\mathcal{C},3}}^{-1}(3))\\
&\quad=\sharp({\Phi^{\mathrm{lks}}_{\mathcal{C},3}}^{-1}(1))\\
&\quad=\begin{cases} k-3 & \text{if } \mathcal{C} \text{ is Type I}\\ k-4 & \text{if } \mathcal{C} \text{ is Type II.}\end{cases}
\end{aligned}
$$

If a homeomorhism $h:(\mathbf{P}^2,\mathcal{C}_1)\to(\mathbf{P}^2,\mathcal{C}_2)$ exists, it follows that $h_\natural(\underline{\mathrm{Sub}}(\mathcal{C}_1)_3)=\underline{\mathrm{Sub}}(\mathcal{C}_2)_3$. This contradicts the above values. Hence our statements follow. $\square$

Remark 4.6. Here are two remarks:

(i) If the j-invariant of the cubic E is not equal to 0, no three inflectional tangent lines are concurrent. Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ have only double points as their singularities and their combinatorics are the same.

(ii) We note that for $k=1,2,7,8,9$ it can be proved that there do not exist Zariski pairs consisting of k-Artal arrangements with only double points as their singularities.

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