Value distribution of L-functions concerning shared values and certain differential polynomials

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Abstract: In this paper, we study a uniqueness question of meromorphic functions of certain differential polynomials that share a nonzero finite value or have the same fixed points with the same of L-functions. The results in this paper extend the corresponding results from Steuding [12, p. 152], Li [7], Fang [1] and Yang-Hua [14].

Key words: Nevanlinna theory; L-functions; differential polynomials; shared values; uniqueness theorems.

1. Introduction and main results. L-functions, with the Riemann zeta function as a prototype, are important objects in number theory, and value distribution of L-functions has been studied extensively, which can be found, for example in Steuding [12]. Value distribution of L-functions concerns distribution of zeros of L-functions L and, more generally, the c-points of L, i.e., the roots of the equation $L(s) = c$, or the points in the pre-image $L^{-1} = \{s \in \mathbf{C} : L(s) = c\}$, where and in what follows, s denotes a complex variable in the complex plane $\mathbf{C}$ and c denotes a value in the extended complex plane $\mathbf{C} \cup \{\infty\}$. L-functions can be analytically continued as meromorphic functions in $\mathbf{C}$. It is well-known that a nonconstant meromorphic function in $\mathbf{C}$ is completely determined by five such pre-images (cf. [2,10,15,17]), which is a famous theorem due to Nevanlinna and often referred to as Nevanlinna's uniqueness theorem. Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbf{C} \cup \{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in $\mathbf{C}$. Moreover, f and g are said to share a value c CM (counting multiplicities) if they share the value c and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities. Throughout the paper, an L-function always means an L-function L in the Selberg class, which includes the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an L-function is defined to be a Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ satisfying the following axioms (cf. [11,12]): (i) Ramanujan hypothesis. $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$. (ii) Analytic continuation. There is a nonnegative integer k such that $(s-1)^k L(s)$ is an entire function of finite order. (iii) Functional equation. L satisfies a functional equation of type $\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})}$, where $\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$ with positive real numbers Q, λ_j and complex numbers ν_j, ω with $Re\nu_j \geq 0$ and $|\omega| = 1$. (iv) Euler product hypothesis. $L(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$ with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$, where the product is taken over all prime numbers p.

We first recall the following result due to Steuding [12], which actually holds without the Euler product hypothesis:

Theorem A ([12, p. 152]). *If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $L_1 = L_2$.*

Later on, Li [7] proved the following result to deal with a question posed by Chung-Chun Yang (cf. [7]):

Theorem B ([7]). *Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a nonconstant L-function L share a CM and b IM, then $L = f$.*

In 1997, Lahiri [4] posed the following question: What can be said about the relationship between

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two meromorphic functions f and g, when two differential polynomials, generated by f and g respectively, share some nonzero finite value? In this direction, Fang [1] and Yang-Hua [14] respectively proved the following results:

Theorem C ([1]). *Let f and g be two nonconstant entire functions, and let n and k be two positive integers such that $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_l e^{cz}$, $g(z) = c_2 e^{-cz}$, where C_1, C_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.*

Theorem D ([14]). *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_l e^{cz}$, $g(z) = c_2 e^{-cz}$, where C_1, C_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Regarding Theorems A–D, one may ask, what can be said about the relationship between a meromorphic function f and an L-function L, if $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM or that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ have the same fixed points, where n and k are positive integers? In this direction, we will prove the following two results respectively:

Theorem 1.1. *Let f be a nonconstant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k+6$. If $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.*

Theorem 1.2. *Let f be a nonconstant meromorphic function, let L be an L-function, and let n and k be two positive integers with $n > 3k+6$. If $(f^n)^{(k)}(z) - z$ and $(L^n)^{(k)}(z) - z$ share 0 CM, then $f = tL$ for a constant t satisfying $t^n = 1$.*

To prove Theorems 1.1 and 1.2 in the present paper, we will apply Nevanlinna theory, which can be found in [2,6,15,17]. In addition, we will use the lower order $\mu(f)$ and the order $\rho(f)$ of a meromorphic function f, which can be found, for example in [2,6,17], are in turn defined as follows:

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

We also need the following two definitions:

Definition 1.1 ([5, Definition 1]). Let p be a positive integer and $a \in \mathbf{C} \bigcup \{\infty\}$. Next we denote by $N_{p)}(r, \frac{1}{f-a})$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, and denote by $N_{(p}(r, \frac{1}{f-a})$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p. We denote by $\overline{N}_{p)}(r, \frac{1}{f-a})$ and $\overline{N}_{(p}(r, \frac{1}{f-a})$ the reduced forms of $N_{p)}(r, \frac{1}{f-a})$ and $N_{(p}(r, \frac{1}{f-a})$ respectively. Here $N_{p)}(r, \frac{1}{f-\infty})$, $\overline{N}_{p)}(r, \frac{1}{f-\infty})$, $N_{(p}(r, \frac{1}{f-\infty})$ and $\overline{N}_{(p}(r, \frac{1}{f-\infty})$ mean $N_{p)}(r,f)$, $\overline{N}_{p)}(r,f)$, $N_{(p}(r,f)$ and $\overline{N}_{(p}(r,f)$ respectively.

Definition 1.2. Let a be an any value in the extended complex plane and let k be an arbitrary nonnegative integer. We define

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

where

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Remark 1.1. By Definition 1.2 we have

$$0 \leq \delta_k(a,f) \leq \delta_{k-1}(a,f) \leq \delta_1(a,f) \leq \Theta(a,f) \leq 1.$$

2. Preliminaries. In this section, we will give the following lemmas that play an important role in proving the main results in this paper:

Lemma 2.1 ([2, Theorem 3.2] and [17, Theorem 4.3]). *Let f be a nonconstant meromorphic function, let $k \geq 1$ be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) \\ &\quad - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) \\ &\quad - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f), \end{aligned}$$

where $N_0(r, \frac{1}{f^{(k+1)}})$ is the counting function of those zeros of $f^{(k+1)}$ in $|z| < r$ which are not zeros of $f(f^{(k)} - c)$ in $|z| < r$.

Lemma 2.2 ([8, Lemma 2.5]). *Let F and G be two nonconstant meromorphic functions such* *that $F^{(k)}-P$ and $G^{(k)}-P$ share 0 CM, where $k\geq 1$ is a positive integer, $P\not\equiv 0$ is a polynomial. If*

$$(k+2)\Theta(\infty,F)+2\Theta(\infty,G)+\Theta(0,F)+\Theta(0,G)+\delta_{k+1}(0,F)+\delta_{k+1}(0,G)>k+7$$

and

$$(k+2)\Theta(\infty,G)+2\Theta(\infty,F)+\Theta(0,G)+\Theta(0,F)+\delta_{k+1}(0,G)+\delta_{k+1}(0,F)>k+7,$$

then either $F^{(k)}G^{(k)}=P^2$ or $F=G$.

Lemma 2.3 ([15, Theorem 1.24]). *Suppose that f is a nonconstant meromorphic function in the complex plane and k is a positive integer. Then*

$$N\left(r,\frac{1}{f^{(k)}}\right)\leq N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+O(\log T(r,f)+\log r),$$

as $r\to\infty$, outside of a possible exceptional set of finite linear measure.

Lemma 2.4 ([18, Lemma 6]). *Let f_1 and f_2 be two nonconstant meromorphic functions satisfying $\overline{N}(r,f_j)+\overline{N}(r,\frac{1}{f_j})=S(r),(j=1,2)$. Then, either $\overline{N}_0(r,1;f_1,f_2)=S(r)$ or there exist two integers p and q satisfying $|p|+|q|>0$, such that $f_1^pf_2^q=1$, where $\overline{N}_0(r,1;f_1,f_2)$ denotes the reduced counting function of the common 1-points of f_1 and f_2 in $|z|<r$, $T(r)=T(r,f_1)+T(r,f_2)$ and $S(r)=o(T(r))$, as $r\notin E$ and $r\to\infty$. Here $E\subset(0,+\infty)$ is a subset of finite linear measure.*

Lemma 2.5 ([3]). *Let f be a transcendental meromorphic function in $\mathbf{C}$. Then, for each $K>1$, there exists a set $M(K)\subset(0,+\infty)$ of the lower logarithmic density at most $d(K)=1-(2e^{K-1}-1)^{-1}>0$, that is*

$$\underline{\log\text{dens}}M(K)=\liminf_{r\to\infty}\frac{1}{\log r}\int_{M(K)\cap[1,r]}\frac{dt}{t}\leq d(K),$$

such that, for every positive integer k,

$$\limsup_{\substack{r\to\infty\\ r\notin M(K)}}\frac{T(r,f)}{T(r,f^{(k)})}\leq 3eK.$$

Lemma 2.6 ([16, proof of Lemma 1]). *Let f be a nonconstant meromorphic function, let $k\geq 1$ be a positive integer, and let $\varphi\not\equiv 0,\infty$ be a small function of f, i.e., $T(r,\varphi)=S(r,f)$. Then*

$$T(r,f)\leq\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-\varphi}\right)-N\left(r,\frac{1}{\left(\frac{f^{(k)}}{\varphi}\right)'}\right)+S(r,f).$$

3. Proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. First of all, we denote by d the degree of L. Then $d=2\sum_{j=1}^{K}\lambda_j>0$ (cf. [12, p. 113]), where K and λ_j are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of L-functions. Therefore, by Steuding [12, p. 150] we have

$$T(r,L)=\frac{d}{\pi}r\log r+O(r). \tag{3.1}$$

Noting that an L-function at most has one pole $z=1$ in the complex plane, we deduce by Lemmas 2.1 and Valiron-Mokhonko lemma (cf. [9]) that

$$\begin{aligned}T(r,L^n)&=nT(r,L)+O(1)\\&\leq\overline{N}(r,L^n)+N_{k+1}\left(r,\frac{1}{L^n}\right)+\overline{N}\left(r,\frac{1}{(L^n)^{(k)}-1}\right)\\&\quad-N_0\left(r,\frac{1}{(L^n)^{(k+1)}}\right)+O(\log r)\\&\leq\overline{N}(r,L)+(k+1)\overline{N}\left(r,\frac{1}{L}\right)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}-1}\right)\\&\quad+O(\log r)\\&\leq(k+1)T(r,L)+T(r,(f^n)^{(k)})+O(\log r),\end{aligned}$$

i.e.,

$$(n-k-1)T(r,L)\leq T(r,(f^n)^{(k)})+O(\log r). \tag{3.2}$$

By (3.1) we see that L is a transcendental meromorphic function. Combining this with (3.2), Theorem 1.5 [15] and the assumption $n>3k+6$, we deduce that $(f^n)^{(k)}$, and so f is a transcendental meromorphic function. Now we let

$$\Delta_1=(k+2)\Theta(\infty,f^n)+2\Theta(\infty,L^n)+\Theta(0,f^n)+\Theta(0,L^n)+\delta_{k+1}(0,f^n)+\delta_{k+1}(0,L^n) \tag{3.3}$$

and

$$\Delta_2=(k+2)\Theta(\infty,L^n)+2\Theta(\infty,f^n)+\Theta(0,L^n)+\Theta(0,f^n)+\delta_{k+1}(0,L^n)+\delta_{k+1}(0,f^n). \tag{3.4}$$

By Valiron-Mokhonko lemma we have

$$\Theta(\infty, f^n) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f^n)}{T(r, f^n)} = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{nT(r, f) + O(1)} \geq 1 - \frac{1}{n}, \tag{3.5}$$

$$\delta_{k+1}(0, f^n) = 1 - \limsup_{r\to\infty} \frac{N_{k+1}\left(r, \frac{1}{f^n}\right)}{T(r, f^n)} = 1 - \limsup_{r\to\infty} \frac{(k+1)\overline{N}\left(r, \frac{1}{f}\right)}{nT(r, f) + O(1)} \geq 1 - \frac{k+1}{n} \tag{3.6}$$

and

$$\Theta(0, f^n) \geq 1 - \frac{1}{n}, \quad \Theta(0, L^n) \geq 1 - \frac{1}{n}, \quad \delta_{k+1}(0, L^n) \geq 1 - \frac{k+1}{n}. \tag{3.7}$$

Noting that an L-function at most has one pole $z = 1$ in the complex plane, we have by (3.1) that

$$\Theta(\infty, L^n) = 1. \tag{3.8}$$

By (3.3), (3.5)–(3.8) we have

$$\Delta_1 \geq k + 8 - \frac{3k+6}{n}, \quad \Delta_2 \geq k + 8 - \frac{2k+6}{n}. \tag{3.9}$$

By (3.9) and the assumption $n > 3k + 6$ we have $\Delta_1 > k + 7$ and $\Delta_2 > k + 7$. This together with (3.3), (3.4) and Lemma 2.2 gives $(f^n)^{(k)}(L^n)^{(k)} = 1$ or $f^n = L^n$. We consider the following two cases:

Case 1. Suppose that $(f^n)^{(k)}(L^n)^{(k)} = 1$. First of all, we prove that 0 is a Picard exceptional value of f and L. Indeed, suppose that $z_0 \in \mathbf{C}$ is a zero of f with multiplicity m. Then, by the assumption $(f^n)^{(k)}(L^n)^{(k)} = 1$ we can find that $z_0 = 1$ is a pole of L with multiplicity, say p, such that $mn - k = np + k$, and so $(m - p)n = 2k$, and so we have $n \leq 2k$, which contradicts the assumption $n > 3k + 6$. Similarly, we can prove that 0 is a Picard exceptional value of L. On the other hand, by (3.1), Valiron-Mokhonko lemma, the assumption $(f^n)^{(k)}(L^n)^{(k)} = 1$, a result from Whittaker [13, p. 82] and the definition of the order of a meromorphic function we have

$$\rho(f) = \rho(f^n) = \rho((f^n)^{(k)}) = \rho((L^n)^{(k)}) = \rho(L^n) = \rho(L) = 1. \tag{3.10}$$

Noting that L has at most one pole $z = 1$ in the complex plane, we have by (3.10), Lemma 2.3 and $(f^n)^{(k)}(L^n)^{(k)} = 1$ that

$$(n + k)\overline{N}(r, f) \leq N\left(r, \frac{1}{(L^n)^{(k)}}\right) \leq N\left(r, \frac{1}{L^n}\right) + k\overline{N}(r, L^n) + O(\log r) = O(\log r). \tag{3.11}$$

Therefore,

$$\overline{N}(r, f) + \overline{N}(r, L) \leq O(\log r). \tag{3.12}$$

Now we set

$$f_1 = \frac{(f^n)^{(k)}}{(L^n)^{(k)}}, \quad f_2 = \frac{(f^n)^{(k)} - 1}{(L^n)^{(k)} - 1}. \tag{3.13}$$

By (3.13) and the assumption that f and L are transcendental meromorphic functions, we have $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$. Suppose that one of f_1 and f_2 is a nonzero constant. Then, by (3.13) we see that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share ∞ CM. Combining this with $(f^n)^{(k)}(L^n)^{(k)} = 1$ we deduce that ∞ is a Picard exceptional value of f and L. Next we suppose that f_1 and f_2 are nonconstant meromorphic functions. We set

$$F_1 = (f^n)^{(k)}, \quad G_1 = (L^n)^{(k)}. \tag{3.14}$$

Then, by (3.13) and (3.14) we have

$$F_1 = \frac{f_1(1 - f_2)}{f_1 - f_2}, \quad G_1 = \frac{1 - f_2}{f_1 - f_2}. \tag{3.15}$$

By (3.15) we can find that there exists a subset $I \subset (0, +\infty)$ with infinite linear measure such that $S(r) = o(T(r))$ and

$$T(r, F_1) \leq 2(T(r, f_1) + T(r, f_2)) + S(r) \leq 8T(r, F_1) + S(r) \tag{3.16}$$

or

$$T(r, G_1) \leq 2(T(r, f_1) + T(r, f_2)) + S(r) \leq 8T(r, G_1) + S(r), \tag{3.17}$$

as $r \in I$ and $r \to \infty$, where $T(r) = T(r, f_1) + T(r, f_2)$. Without loss of generality, we suppose that (3.16) holds. Then we have $S(r) = S(r, F_1)$, as $r \in I$ and $r \to \infty$. By $(f^n)^{(k)}(L^n)^{(k)} = 1$ we see that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 and -1 CM. Noting that 0 is a Picard exceptional value of f and L, we deduce by (3.10) and Lemma 2.3 that

$$N\left(r, \frac{1}{(f^n)^{(k)}}\right) \leq k\overline{N}(r, f) + O(\log r). \tag{3.18}$$

By (3.11), (3.12) and (3.18) we have

$$N\left(r, \frac{1}{(f^n)^{(k)}}\right) + N\left(r, \frac{1}{(L^n)^{(k)}}\right) \leq O(\log r). \tag{3.19}$$

Noting that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ are transcendental meromorphic functions such that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM, we deduce by (3.12), (3.13) and (3.19) that

$$\overline{N}\left(r,\frac{1}{f_j}\right)+\overline{N}(r,f_j)=o(T(r)),\ (j=1,2), \tag{3.20}$$

as $r\in I$ and $r\to\infty$. Noting that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share -1 CM, we deduce by (3.12), (3.14), (3.16), (3.18) and the second fundamental theorem that

$$\begin{aligned} T(r,F_1) &\le \overline{N}(r,F_1)+\overline{N}\left(r,\frac{1}{F_1}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{F_1+1}\right)+o(T(r,F_1))\\ &\le \overline{N}\left(r,\frac{1}{F_1+1}\right)+O(\log r)+o(T(r,F_1))\\ &\le \overline{N}_0(r,1;f_1,f_2)+o(T(r,F_1)), \end{aligned} \tag{3.21}$$

as $r\in I$ and $r\to\infty$. By (3.16) and (3.21) we have

$$T(r,f_1)+T(r,f_2)\le \overline{N}_0(r,1;f_1,f_2)+o(T(r)). \tag{3.22}$$

By (3.13), (3.14), (3.20), (3.22) and Lemma 2.4 we find that there exist two relatively prime integers s and t satisfying $|s|+|t|>0$, such that $f_1^s f_2^t=1$. Combining this with (3.13) and (3.14), we have

$$\left(\frac{F_1}{G_1}\right)^s\left(\frac{F_1-1}{G_1-1}\right)^t=1. \tag{3.23}$$

By (3.23) we consider the following two subcases:

Subcase 1.1. Suppose that $st<0$, say $s>0$ and $t<0$, say $t=-t_1$, where t_1 is some positive integer. Then, (3.23) can be rewritten as

$$\left(\frac{F_1}{G_1}\right)^s=\left(\frac{F_1-1}{G_1-1}\right)^{t_1}. \tag{3.24}$$

Let $z_1\in\mathbf{C}$ be a pole of F_1 of multiplicity $p_1\ge 1$. Then, by $F_1G_1=1$ we can see that z_1 is a zero of G_1 of multiplicity p_1. Therefore, by (3.24) we deduce that $2s=t_1=-t$. Combining this with the assumption that s and t are two relatively prime integers, we have $s=1$ and $t=-t_1=-2$. Therefore, (3.24) can be rewritten as $F_1(G_1-1)^2=G_1(F_1-1)^2$, this is equivalent to the obtained result $F_1G_1=1$. Next we can deduce a contradiction by using the other method. Indeed, by (3.19) and the fact that L, and so $L^{(k)}$ have at most one pole $z=1$ in the complex plane, we have

$$(L^n)^{(k)}(z)=\frac{P_1(z)}{(z-1)^{p_2}}e^{A_1z+B_1}, \tag{3.25}$$

where P_1 is a nonzero polynomial, $p_2\ge 0$ is an integer, $A_1\ne 0$ and B_1 are constants. By (3.25), Lemma 2.5 and Hayman [2, p. 7] we deduce that there exists a subset $I\subset(0,+\infty)$ with logarithmic measure logmeas $\mathrm{I}=\int_{\mathrm{I}}\frac{dt}{t}=\infty$ such that for some given sufficiently large positive number $K>1$, we have

$$\begin{aligned} T(r,L) &\le 3eKT(r,(L^n)^{(k)})\\ &=\frac{3eK|A_1|r}{\pi}(1+o(1))+O(\log r), \end{aligned} \tag{3.26}$$

as $r\in I$ and $r\to\infty$. By (3.1) and (3.26) we have a contradiction.

Subcase 1.2. Suppose that $st=0$ or $st>0$. Then, by (3.23) we can see that F_1 and G_1 share ∞ CM. This together with (3.14) and the assumption $(f^n)^{(k)}(L^n)^{(k)}=1$ implies that ∞ is a Picard exceptional value of f and L. Combining this with the obtained result that 0 is a Picard exceptional value of f and L, we have

$$L(z)=e^{A_2z+B_2}, \tag{3.27}$$

where $A_2\ne 0$ and B_2 are constants. By (3.27) and Hayman [2, p. 7] we have

$$T(r,L)=T(r,e^{A_2z+B_2})=\frac{|A_2|r}{\pi}(1+o(1)), \tag{3.28}$$

which contradicts (3.1).

Case 2. Suppose that $f^n=L^n$. Then, we have $f=tL$, where t is a constant satisfying $t^n=1$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. First of all, in the same manner as in the beginning of the proof of Theorem 1.1 we have (3.1). Now we let $z_2\in\mathbf{C}$ be a zero of L with multiplicity p_2. Then z_2 is a zero of L^n with multiplicity np_2, and so z_2 is a zero of $\left(\frac{(L^n)^{(k)}}{z}\right)'$ with multiplicity np_2-k-2 at least. Again let z_3 be a zero of $\frac{(L^n)^{(k)}}{z}-1$ with multiplicity p_3. Then, z_3 is a zero of $\left(\frac{(L^n)^{(k)}}{z}\right)'$ with multiplicity p_3-1. Then, by (3.1), Lemma 2.6 and the value sharing assumption we have

$$T(r,L^n)\le N\left(r,\frac{1}{L^n}\right)+N\left(r,\frac{1}{\frac{(L^n)^{(k)}}{z}-1}\right) \tag{3.29}$$

$$- N\left(r, \frac{1}{\left(\frac{(L^n)^{(k)}}{z}\right)'}\right) + O(\log r)$$
$$\leq (k+2)\overline{N}\left(r, \frac{1}{L}\right) + \overline{N}\left(r, \frac{1}{\frac{(L^n)^{(k)}}{z} - 1}\right)$$
$$- N_0\left(r, \frac{1}{\left(\frac{(L^n)^{(k)}}{z}\right)'}\right) + O(\log r)$$
$$\leq (k+2)T(r, L) + \overline{N}\left(r, \frac{1}{\frac{(f^n)^{(k)}}{z} - 1}\right)$$
$$+ O(\log r)$$
$$\leq (k+2)T(r, L) + T(r, (f^n)^{(k)}) + O(\log r),$$

where $N_0\left(r, \frac{1}{\left(\frac{(L^n)^{(k)}}{z}\right)'}\right)$ is the counting function of those zeros of $\left(\frac{(L^n)^{(k)}}{z}\right)'$ in $|z| < r$ that are not zeros of $\frac{(L^n)^{(k)}}{z}$ in $|z| < r$. By Valiron-Mokhonko lemma we have $T(r, L^n) = nT(r, L) + O(1)$. This together with (3.29) gives

$$(3.30) \quad (n - k - 2)T(r, L) \leq T(r, (f^n)^{(k)}) + O(\log r).$$

By (3.30) and the assumption $n > 3k + 6$, we deduce that $(f^n)^{(k)}$, and so f is a transcendental meromorphic function. Next in the same manner as in the proof of Theorem 1.1 we have $(f^n)^{(k)}(L^n)^{(k)} = z^2$ or $f^n = L^n$ by Lemma 2.2. We consider the following two cases:

Case 1. Suppose that $(f^n)^{(k)}(L^n)^{(k)} = z^2$. Then, $F_2G_2 = 1$, where

$$(3.31) \qquad F_2 = \frac{(f^n)^{(k)}}{z}, \quad G_2 = \frac{(g^n)^{(k)}}{z}.$$

Next, in the same manner as in Case 1 of the proof of Theorem 1.1 we can get a contradiction.

Case 2. Suppose that $f^n = L^n$. Then we get the conclusion of Theorem 1.2. This completes the proof of Theorem 1.2. □

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